MATHEMATICS 300 — FALL 2009

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

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1 An example of a proof

Here is a theorem with its proof. The theorem is extremely simple—you might even call it stupid, but the point is that, before we write proofs that are difficult and challenging, we have to get started by doing some proofs that are very simple.

> Think of what we are doing here as similar to the beginnig of a language course. In a Spanish course for beginners, you would not start with readings of Cervantes or Quevedo. You would start by learning how to say very stupid things, such as "the book is on the table" (*el libro está sobre la mesa*). In this course, we are starting with very stupid proofs, because the goal is to show you what a well-written proof looks like.

Actually, I am going to give you *eight* ways of writing the proof.

THEOREM. *For every positive real number x there exists a positive real number y such that $0 < y < x$.*

> NOTE: An expression such as "$a < b < c$" is an abbreviation of "$a < b \wedge b < c$."

FIRST PROOF, WITH LOTS OF COMMENTS:
COMMENT: We first translate the statement we want to prove into formal language.

We want to prove that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R})0 < y < x)\,, \tag{1}$$

that is, that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R})(0 < y \wedge y < x))\,. \tag{2}$$

COMMENT: Since we are going to be talking about these sentences later, we give them a name (in this case, numbers) so we can just refer to them by their name, instead of have to repeat them.

COMMENT: Notice that I didn't just write Sentence (1) or Sentence (2). What I did write was "We want to prove that", followed by (1), and then I wrote "that is", followed by (2). If I hadn't written those words, I would have **asserted** *that (1) and (2) are true, which is something I cannot do, because I do not know yet that (1) and (2) are true. I will only know that when I have proved them.*

Since (2) is a universal sentence, of the form "$(\forall x \in \mathbb{R})B(x)$", we will pick an arbitrary value ξ of x and prove $B(\xi)$.

Let $\xi \in \mathbb{R}$ be arbitrary.

COMMENT: Now I have started a new proof, a "proof within a proof", or a "subproof". I have used **indentation** *to indicate that we are in a subproof. In this subproof, we have a number ξ, "given" to us. (You should think that an imaginary being called "the CAT"–"creator of arbitrary things"–has actually given us a real number ξ, but this ξ is inside a sealed envelope, and we do not know which number it is, so whatever we say had better be true for all possible numbers, because ξ could be any one of them.)*

We want to prove that

$$\xi > 0 \Rightarrow (\exists y \in \mathbb{R})(0 < y \wedge y < \xi)\,. \tag{3}$$

Since (3) is an implication, we will prove it by assuming the premiss and proving the conclusion.

Assume that $\xi > 0$.

COMMENT: Now I have started a "proof within the subproof where I was before", or a "subsubproof", or "level 2 subproof". I have used **extra indentation** *to indicate that we are in a subsubproof. In this subproof, we have the number ξ, "given" to us, and we have the extra information that ξ is positive.*

COMMENT: What is the justification for assuming that $\xi > 0$? The answer is that **you do not need any justification; you can always assume anything you want***; for example you could assume that pigs can fly, if you want to. But if, assuming that pigs can fly, you prove something else—for example, that pigs are very dangerous—then all you can do with that* *information is get out of your subproof to the previous level proof with the implication "pigs can fly$\Rightarrow$ pigs are very dangerous", and you will not be able to conclude that pigs are very dangerous unless you prove that pigs can fly.*

We want to prove that

$$(\exists y \in \mathbb{R})(0 < y \wedge y < \xi)\,. \tag{4}$$

Since (4) is an existential sentence, of the form $(\exists y \in \mathbb{R})C(y)$, we will prove it by exhibiting a value η of y for which $C(\eta)$ is true.

Let $\eta = \frac{\xi}{5}$.

COMMENT: Notice the extra indentation. We have started a "subsubsub-proof", or "level 3 subproof". In this level 3 subproof, we have

- *the number ξ, given to us by the CAT,*
- *the information that $\xi > 0$,*
- *the number η, picked by us, and known to be equal to $\frac{\xi}{5}$.*

COMMENT: Why do I take $\frac{\xi}{5}$? Because in class a student suggested $\frac{\xi}{2}$, and then when I did the problem I used $\frac{\xi}{3}$. So I am taking $\frac{\xi}{5}$ just because I want to do it differently. You could take η to be ξ multplied by any number which is positive and less than one.

Then

$$0 < \eta\,, \tag{5}$$

because η is the product of the positive numbers $\frac{1}{5}$ and ξ. And

$$\eta < \xi\,, \tag{6}$$

because $\frac{1}{5} < 1$.

Therefore

$$0 < \eta \wedge \eta < \xi\,, \tag{7}$$

COMMENT. Here we have applied the rule for proving an "and" sentence, which later in the course we will call **Rule** $\wedge_{prove}$. *This (extremely stupid) rule says that, if you have sentences A and B, then you can go to the sentence $A \wedge B$.*

Hence

$$0 < \eta < \xi \,, \tag{8}$$

because "$0 < \eta < \xi$" is just an abbreviation for "$0 < \eta \wedge \eta < \xi$".

COMMENT. We are now going to get out from our level 3 subproof, and move down to level 2. You will see that the indentation is smaller.

Since we have exhibited a value η of the variable y for which (8) holds, and $\eta \in \mathbb{R}$, we can conclude that

$$(\exists y \in \mathbb{R}) 0 < y < \xi \,. \tag{9}$$

COMMENT. We have applied a **rule for proving existential sentences**, *called the* **witness rule**, *or* **Rule** $\exists_{prove}$*: if you have proved $C(\eta)$ for some particular value η of the variable y, and $\eta \in U$, then you can go to the sentence $(\exists y \in U)C(y)$.*

COMMENT. We are now going to get out from our level 2 subproof, and move down to level 1. Once again, you will see that the indentation is smaller.

Since we have proved (9) assuming that $\xi > 0$, it follows that

$$\xi > 0 \Rightarrow (\exists y \in \mathbb{R}) 0 < y < \xi \,. \tag{10}$$

COMMENT. Here we have applied the **Rule for proving an implication**, *that we will call later* **Rule** $\Rightarrow_{prove}$*. (This is also called the* **deduction rule**.*) The rule says: if you start a subproof with "Assume A", and you prove B, then you may go back to your original proof level with $A \Rightarrow B$.*

COMMENT. Now, finally, we are going to get out from our level 1 subproof, and move down to level 0. Once again, you will see that the indentation is smaller.

Since we have proved (10) for an arbitrary value $\xi \in \mathbb{R}$ of the variable x. we can conclude that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R}) 0 < y < x) \,. \tag{11}$$

COMMENT. We have now safely landed back on Level 0, where we started. And we have proved exactly what we said we were going to prove, i.e., Sentence (1). So WE ARE DONE!!!

QED

COMMENT. "QED" is the acronym for the Latin phrase "quod erat demonstrandum", that is, "which is what was to be proved." This is a very common way to indicate the end of a proof.

SECOND PROOF, SAME AS THE FIRST BUT WITHOUT THE COMMENTS:

We want to prove that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R}) 0 < y < x)\,, \tag{12}$$

that is, that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R})(0 < y \wedge y < x))\,. \tag{13}$$

Since (13) is a universal sentence, of the form "$(\forall x \in \mathbb{R})B(x)$", we will pick an arbitrary value ξ of x and prove $B(\xi)$.

Let $\xi \in \mathbb{R}$ be arbitrary.

We want to prove that

$$\xi > 0 \Rightarrow (\exists y \in \mathbb{R})(0 < y \wedge y < \xi))\,. \tag{14}$$

Since (14) is an implication, we will prove it by assuming the premiss and proving the conclusion.

Assume that $\xi > 0$.
We want to prove that

$$(\exists y \in \mathbb{R})(0 < y \wedge y < \xi)\,. \tag{15}$$

Since (15) is an existential sentence, of the form $(\exists y \in \mathbb{R})C(y)$, we will prove it by exhibiting a value η of y for which $C(\eta)$ is true.

Let $\eta = \frac{\xi}{5}$.

Then

$$0 < \eta\,, \tag{16}$$

because η is the product of the positive numbers $\frac{1}{5}$ and ξ. And

$$\eta < \xi\,, \tag{17}$$

because $\frac{1}{5} < 1$.

Therefore

$$0 < \eta \wedge \eta < \xi\,, \tag{18}$$

Hence

$$0 < \eta < \xi\,, \tag{19}$$

because "$0 < \eta < \xi$" is just an abbreviation for "$0 < \eta \wedge \eta < \xi$".

Since we have exhibited a value η of the variable y for which (19) holds, and $\eta \in \mathbb{R}$, we can conclude that

$$(\exists y \in \mathbb{R}) 0 < y < \xi\,. \tag{20}$$

Since we have proved (20) assuming that $\xi > 0$, it follows that

$$\xi > 0 \Rightarrow (\exists y \in \mathbb{R}) 0 < y < \xi\,. \tag{21}$$

Since we have proved (21) for an arbitrary value $\xi \in \mathbb{R}$ of the variable x. we can conclude that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R}) 0 < y < x)\,. \tag{22}$$

QED

THIRD PROOF, SAME AS THE SECOND ONE BUT MORE CONCISE.

We want to prove that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R}) 0 < y < x)\,.$$

Let $\xi \in \mathbb{R}$ be arbitrary.

Assume that $\xi > 0$.

Let $\eta = \frac{\xi}{5}$.

Then $0 < \eta$ (because η is the product of the positive numbers $\frac{1}{5}$ and ξ), and $\eta < \xi$ (because $\frac{1}{5} < 1$). Therefore $0 < \eta < \xi$.

So we can conclude that $(\exists y \in \mathbb{R}) 0 < y < \xi$.

Since this was proved assuming that $\xi > 0$, it follows that

$$\xi > 0 \Rightarrow (\exists y \in \mathbb{R}) 0 < y < \xi\,. \tag{23}$$

Since we have proved (23) for an arbitrary value $\xi \in \mathbb{R}$ of the variable x. we can conclude that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R}) 0 < y < x)\,. \tag{24}$$

QED

FOURTH PROOF, EVEN MORE CONCISE.

We want to prove that $(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R})0 < y < x)$.

Let ξ be an arbitrary positive real number.

Let $\eta = \frac{\xi}{5}$.

Then $0 < \eta$ and $\eta < \xi$, so $0 < \eta < \xi$.

Therefore $(\exists y \in \mathbb{R})0 < y < \xi$.

Since ξ was an arbitrary positive real number, it follows that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R})0 < y < x). \qquad \text{QED}$$

FIFTH PROOF, EVEN SHORTER.

Let ξ be an arbitrary positive real number.

Let $\eta = \frac{\xi}{5}$.

Then $0 < \eta$ and $\eta < \xi$, so $0 < \eta < \xi$.

Therefore $(\exists y \in \mathbb{R})0 < y < \xi$.

Since ξ was an arbitrary positive real number, it follows that

$$(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R})0 < y < x). \qquad \text{QED}$$

SIXTH PROOF, SAME AS THE FIFTH PROOF, BUT WITH FEWER DISPLAYS: Let ξ be an arbitrary positive real number. Let $\eta = \frac{\xi}{5}$. Then $0 < \eta$ and $\eta < \xi$, so $0 < \eta < \xi$. Therefore $(\exists y \in \mathbb{R})0 < y < \xi$.

Since x was an arbitrary positive real number, it follows that $(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R})0 < y < x)$. QED

SEVENTH PROOF, SAME AS THE SIXTH PROOF, BUT WITHOUT USING DIFFERENT NAMES FOR PARTICULAR VALUES OF THE VARIABLES: Let x be an arbitrary positive real number. Let $y = \frac{x}{5}$. Then $0 < y$ and $y < x$, so $0 < y < x$. Therefore $(\exists y \in \mathbb{R})0 < y < x$.

Since x was an arbitrary positive real number, it follows that $(\forall x \in \mathbb{R})(x > 0 \Rightarrow (\exists y \in \mathbb{R})0 < y < x)$. QED

EIGHTH PROOF.

For any positive x, take $y = \frac{x}{5}$. This clearly works. QED

2 Another example of a proof

THEOREM. *Let x be a positive real number. Then $x + \frac{1}{x} \geq 2$.*

PROOF. We want to prove that

$$(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big). \tag{25}$$

Let $\xi \in \mathbb{R}$ be arbitrary. We want to prove that

$$\xi > 0 \Rightarrow \xi + \frac{1}{\xi} \geq 2. \tag{26}$$

Assume $\xi > 0$.
We want to prove that

$$\xi + \frac{1}{\xi} \geq 2. \tag{27}$$

If we could prove that

$$\xi^2 + 1 \geq 2\xi, \tag{28}$$

then (27) would follow, since multiplication of both sides of (28) by $\frac{1}{\xi}$ (which is possible, because $\frac{1}{\xi} > 0$, since $\xi > 0$), would yield (27). So we set out to prove (28).

Clearly, (28) will follow if we prove

$$\xi^2 - 2\xi + 1 \geq 0, \tag{29}$$

because once we have (29) we can get (28) by adding 2ξ to both sides.

To prove (29), we observe that

$$\xi^2 - 2\xi + 1 = (\xi - 1)^2, \tag{30}$$

and $(\xi - 1)^2 \geq 0$, because the square of any real number is nonnegative. So (29) holds.

Therefore, (28) is true as well.

Hence (27) is true.

So we have proved (27) under the assumption that $\xi > 0$. Hence (26) holds.

Since (26) is true for an abitrary real number ξ, it follows that (25) holds, completing our proof.

ANOTHER WAY OF WRITING THE SAME PROOF. We want to prove that

$$(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big)\,. \tag{31}$$

Let $x \in \mathbb{R}$ be arbitrary. We want to prove that

$$x > 0 \Rightarrow x + \frac{1}{x} \geq 2\,. \tag{32}$$

Assume $x > 0$.
We want to prove that

$$x + \frac{1}{x} \geq 2\,. \tag{33}$$

If we could prove that

$$x^2 + 1 \geq 2x\,, \tag{34}$$

then (33) would follow, since multiplication of both sides of (34) by $\frac{1}{x}$ (which is possible, because $\frac{1}{x} > 0$, since $x > 0$), would yield (33). So we set out to prove (34).

Clearly, (34) will follow if we prove

$$x^2 - 2x + 1 \geq 0\,, \tag{35}$$

because once we have (35) we can get (34) by adding $2x$ to both sides.

To prove (35), we observe that

$$x^2 - 2x + 1 = (x-1)^2\,, \tag{36}$$

and $(x-1)^2 \geq 0$, because the square of any real number is nonnegative.
So (35) holds.
Therefore, (34) is true as well.
Hence (33) is true.
So we have proved (33) under the assumption that $x > 0$. Hence (32) holds.
Since (32) is true for an abitrary real number x, it follows that (31) holds, completing our proof.

A THIRD WAY OF WRITING THE PROOF. We want to prove that

$$(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big)\,. \tag{37}$$

Let $\xi \in \mathbb{R}$ be an arbitrary positive number.
Since the square of any real number is nonnegative, we have

$$(\xi - 1)^2 \geq 0\,. \tag{38}$$

On the other hand,

$$(\xi - 1)^2 = \xi^2 - 2\xi + 1\,. \tag{39}$$

Therefore

$$\xi^2 - 2\xi + 1 \geq 0\,. \tag{40}$$

Then

$$\xi^2 + 1 \geq 2\xi\,. \tag{41}$$

Multiplication of both sides of (41) by $\frac{1}{\xi}$ (which is possible, because $\frac{1}{\xi} > 0$, since $\xi > 0$), yields

$$\xi + \frac{1}{\xi} \geq 2\,. \tag{42}$$

Since (42) is true for an abitrary positive real number ξ, it follows that (37) holds, completing our proof.

A FOURTH PROOF (BY CONTRADICTION). We want to prove that

$$(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big)\,. \tag{43}$$

Let $\xi \in \mathbb{R}$ be arbitrary. We want to prove that

$$\xi > 0 \Rightarrow \xi + \frac{1}{\xi} \geq 2\,. \tag{44}$$

Assume $\xi > 0$.
We want to prove that

$$\xi + \frac{1}{\xi} \geq 2\,. \tag{45}$$

Assume that (45) wasn't true, i.e., that

$$\xi + \frac{1}{\xi} < 2\,. \tag{46}$$

Multiplication of both sides of (46) by ξ (which is possible, because $\xi > 0$), yields

$$\xi^2 + 1 < 2\xi\,. \tag{47}$$

If we subtract 2ξ from both sides, we find

$$\xi^2 - 2\xi + 1 < 0\,. \tag{48}$$

But

$$\xi^2 - 2\xi + 1 = (\xi - 1)^2\,. \tag{49}$$

So

$$(\xi - 1)^2 < 0\,. \tag{50}$$

But $(\xi - 1)^2 \geq 0$, because the square of any real number is nonnegative. So we have proved that

$$\sim (\xi - 1)^2 \geq 0 \wedge (\xi - 1)^2 \geq 0\,, \tag{51}$$

which is a contradiction.

So assuming that (45) isn't true has led us to a contradiction, and we can conclude that (45) is true, i.e., that $\xi + \frac{1}{\xi} \geq 2$.

Since this was proved assuming that $\xi > 0$, it follows that $\xi > 0 \Rightarrow \xi + \frac{1}{\xi} \geq 2$. Since ξ was an arbitrary real number, it follows that

$$(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big)\,,$$

as desired.

A FIFTH PROOF (JUST ANOTHER WAY OF WRITING THE FOURTH PROOF). We want to prove that

$$(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big)\,. \tag{52}$$

Let $\xi \in \mathbb{R}$ be arbitrary such that $\xi > 0$. We want to prove that

$$\xi + \frac{1}{\xi} \geq 2\,. \tag{53}$$

Assume that (53) wasn't true. Then $\xi + \frac{1}{\xi} < 2$. Multiplication of both sides by ξ (which is possible, because $\xi > 0$), yields $\xi^2+1 < 2\xi$. If we subtract 2ξ from both sides, we find $\xi^2 - 2\xi + 1 < 0$. But $\xi^2 - 2\xi + 1 = (\xi - 1)^2$. So

$$(\xi - 1)^2 < 0\,. \tag{54}$$

But $(\xi - 1)^2 \geq 0$, because the square of any real number is nonnegative. So we have proved that

$$\sim (\xi - 1)^2 < 0 \wedge (\xi - 1)^2 \geq 0\,, \tag{55}$$

which is a contradiction.

So assuming that (53) isn't true has led us to a contradiction, and we can conclude that (53) is true, i.e., that $\xi + \frac{1}{\xi} \geq 2$.

Since this was proved for an arbitrary positive real number ξ, it follows that $(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big)$, as desired.

AN EXAMPLE OF HOW **NOT** *TO WRITE THE PROOF. (THIS WOULD GET A ZERO, NO "PARTIAL CREDIT".)* We want to prove that

$$(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big)\,. \tag{56}$$

Let $\xi \in \mathbb{R}$ be arbitrary such that $\xi > 0$. We want to prove that

$$\xi + \frac{1}{\xi} \geq 2\,. \tag{57}$$

Multiplication of both sides by ξ (which is possible, because $\xi > 0$), yields

$$\xi^2 + 1 \geq 2\xi\,.$$

If we subtract 2ξ from both sides, we find

$$\xi^2 - 2\xi + 1 < 0\,.$$

But $\xi^2 - 2\xi + 1 = (\xi - 1)^2$. So

$$(\xi - 1)^2 \geq 0\,. \tag{58}$$

And "$(\xi-1)^2 \geq 0$" is true, because the square of any real number is nonnegative. So (57) is true.

Since this was proved for arbitrary positive ξ, we have shown that (56) holds.

3 A few more examples

THEOREM. *Let a and b be positive real numbers. Then $\frac{a}{b} + \frac{b}{a} \geq 2$.*

PROOF. We want to prove that

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})\Big((a > 0 \wedge b > 0) \Rightarrow \frac{a}{b} + \frac{b}{a} \geq 2\Big)\,. \tag{59}$$

Let a_*, b_* be arbitrary positive real numbers. Let $\xi = \frac{a_*}{b_*}$.

A previous theorem says that $(\forall x \in \mathbb{R})\Big(x > 0 \Rightarrow x + \frac{1}{x} \geq 2\Big)$. Hence $\xi + \frac{1}{\xi} \geq 2$. But $\xi = \frac{a_*}{b_*}$, and $\frac{1}{\xi} = \frac{b_*}{a_*}$.
So $\frac{a_*}{b_*} + \frac{b_*}{a_*} \geq 2$.

Since a_* and b_* were arbitrary positive real numbers, we get (59). QED

THEOREM (A VERSION OF THE ARITHMETIC-GEOMETRIC INEQUALITY FOR TWO NUMBERS.) *Let a and b be real numbers. Then*

$$ab \leq \frac{a^2 + b^2}{2}. \tag{60}$$

PROOF. We want to prove that

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})\Big(ab \leq \frac{a^2 + b^2}{2}\Big)\,. \tag{61}$$

Let a, b be arbitrary real numbers. Then $(a - b)^2 \geq 0$, because the square of any real number is nonnegative. Since $(a-b)^2 = a^2 - 2ab + b^2$, we can conclude that

$$a^2 - 2ab + b^2 \geq 0\,. \tag{62}$$

Hence

$$a^2 + b^2 \geq 2ab\,. \tag{63}$$

Therefore

$$\frac{a^2 + b^2}{2} \geq ab\,. \tag{64}$$

So

$$ab \leq \frac{a^2 + b^2}{2}\,. \tag{65}$$

Since (65) has been proved for arbitrary real numbers a, b, we conclude that (61) holds. QED

THEOREM (THE ARITHMETIC-GEOMETRIC INEQUALITY FOR TWO NUMBERS.) *Let a and b be positive real numbers. Then*

$$\sqrt{ab} \leq \frac{a+b}{2}. \tag{66}$$

PROOF. We want to prove that

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})\Big((a > 0 \land b > 0) \Rightarrow \sqrt{ab} \leq \frac{a+b}{2}\Big). \tag{67}$$

Let a_*, b_* be arbitrary positive real numbers. Let

$$\alpha = \sqrt{a_*}\,, \qquad \beta = \sqrt{b_*}\,. \tag{68}$$

The previous theorem says that

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})\Big(ab \leq \frac{a^2+b^2}{2}\Big). \tag{69}$$

Applying this to $a = \alpha$, $b = \beta$, we find

$$\alpha\beta \leq \frac{\alpha^2+\beta^2}{2}. \tag{70}$$

But $\alpha = \sqrt{a_*}$ and $\beta = \sqrt{b_*}$. So $\alpha^2 = a_*$, $\beta^2 = b_*$ and $\alpha\beta = \sqrt{a_* b_*}$. Hence (70) says that

$$\sqrt{a_* b_*} \leq \frac{a_* + b_*}{2}. \tag{71}$$

Since (71) has been proved for arbitrary positive real numbers a_*, b_*, we conclude that (67) holds. QED