MATHEMATICS 300 — SPRING 2010

Introduction to Mathematical Reasoning

H. J. Sussmann

INSTRUCTOR'S NOTES

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1 Square root of 2

The square root of 2, also known as Pythagoras' constant, is the positive real number that, when multiplied by itself, gives the number 2. We write $\sqrt{2}$ to denote this number. (Remember that, if x is a real number and $x \geq 0$ then $\sqrt{x}$ denotes the nonnegative real number y such that $y^2 = x$.)

Geometrically the square root of 2 is the length of a diagonal across a square with sides of one unit of length; this follows from the Pythagorean theorem. It was probably the first number known to be irrational. Its numerical value truncated to 65 decimal places is:

1.41421356237309504880168872420969807856967187537694807317667973799...

The quick approximation $\frac{99}{70}$ for the square root of two is most frequently used. Despite having a denominator of only 70, it differs from the correct value by less than 1/10,000.

The discovery of the existence of irrational numbers is usually attributed to the Pythagorean Hippasus of Metapontum (5th century BCE), who produced a (most likely geometrical) proof of the irrationality of the square root of 2. According to one legend, Pythagoras believed in the absoluteness of numbers, and could not accept the existence of irrational numbers. He could not disprove their existence through logic, but his beliefs would not accept the existence of irrational numbers and so he sentenced Hippasus to death by drowning. Other legends report that Hippasus was drowned by some Pythagoreans, or merely expelled from their circle.

2 Homework assignment No. 3, due on Thursday February 11

1. In this problem, $P(x)$ is an unknown sentence involving the variable x. (For example, $P(x)$ could be "x is a cow", or "x is a real number", or "$x > 3$".) You are allowed to use the equivalence

$$(\exists! x)P(x) \Longleftrightarrow \Big((\exists x)P(x) \wedge (\forall y)(\forall z)(P(y) \wedge P(z) \Longrightarrow y = z)\Big).$$

Prove that

$$(\exists! x)P(x) \Longleftrightarrow (\exists x)\Big(P(x) \wedge (\forall y)(P(y) \Longrightarrow y = x)\Big)\,.$$

2. Prove or disprove: if x and y are irrational numbers then $x+y$ is irrational. (That is, $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})((x \notin \mathbb{Q} \wedge y \notin \mathbb{Q}) \Longrightarrow x + y \notin \mathbb{Q})$. Recall that "$a \notin A$" means the same as "$\sim a \in A$", that is, "$a$ does not belong to A.")

3. Prove that $\sqrt{6}$ is irrational.

4. Prove that $\sqrt{12}$ is irrational.

5. Prove that $\sqrt[3]{2}$ is irrational.

6. Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

7. Prove that $\sqrt{2} + \sqrt[3]{2}$ is irrational.