MATHEMATICS 300 — FALL 2009

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

(September 17, 2009)

1 Homework assignment No. 3, due on Thursday, September 24

1. Prove the following: If a and b are positive real numbers, then

$$\frac{9a}{b} + \frac{b}{a} \geq 6\,.$$

2. Prove the following: If x is a real number and $x^3 + 5x > 0$ then $x > 0$. (In symbolic language: $(\forall x \in \mathbb{R})(x^3 + 5x > 0 \Rightarrow x > 0)\,$.)

3. Prove the following: If a, b, c and d are real numbers, then

$$ac + bd \leq \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}\,.$$

HINT: you may want to use the arithmetic-geometric inequality for two numbers.
(NOTE: the symbol $\sqrt{\ }$ means "nonnegative square root of". So, for example, the number 16 has two square roots, namely, 4 and -4, but $\sqrt{16}$ stands for the one of the two roots which is nonnegative, so that $\sqrt{16} = 4$.)
(ANOTHER NOTE: if you think of a pair (x, y) of real numbers as a "two-dimensional vector", then if you have two vectors $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$, the *dot product* $\mathbf{v} \cdot \mathbf{w}$ of $\mathbf{v}$ and $\mathbf{w}$ is the number $ac + bd$. Also, the *length* $||\mathbf{v}||$ of a vector $\mathbf{v} = (a, b)$ is the number $\sqrt{a^2 + b^2}$. Then what you are asked to prove in this problem is that, if you have two two-dimensional vectors $\mathbf{v}$, $\mathbf{w}$, then $\mathbf{v} \cdot \mathbf{w} \leq ||\mathbf{v}||\,||\mathbf{w}||$, that is, “”the dot product of two vectors is less than or equal to the product of the lengths of the vectors.")

4. Book, page 37, problem 5, parts (b) and (c).

5. Book, pages 38-39, problem 11, part (b).

6. Prove the following: If x, y, t are real numbers such that $0 < t < 1$, then

$$\Big(tx + (1-t)y\Big)^2 \leq tx^2 + (1-t)y^2\,.$$

HINT: you may want to use the arithmetic-geometric inequality for two numbers.

(NOTE: A function f, defined on an interval I, is said to be *convex* if the inequality $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$ holds for all $x, y \in I$ and all $t \in \mathbb{R}$ such that $0 < t < 1$. So the result that you are asked to prove in this problem says that, if f is the function—defined on all of $\mathbb{R}$—given by $f(x) = x^2$ for $x \in \mathbb{R}$, then f is convex.)