MATHEMATICS 300 — SPRING 2010

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

(February 14, 2010)

1 Homework assignment No. 4, due on Thursday February 18

Problem 1. Prove or disprove: If p is a prime number then $p+5$ is not prime. (That is, $(\forall p \in \mathbb{N})(p \text{ is prime} \Longrightarrow (\sim p+5 \text{ is prime}))$.)

Problem 2. Prove or disprove: If p is a prime number then $p+7$ is not prime. (That is, $(\forall p \in \mathbb{N})(p \text{ is prime} \Longrightarrow (\sim p+7 \text{ is prime}))$.)

Problem 3. Prove or disprove: If x, y are real numbers such that x is rational and $x+y$ is rational then y is rational. (That is, in formal language, $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})((x \in \mathbb{Q} \wedge x+y \in \mathbb{Q}) \Longrightarrow y \in \mathbb{Q})$.)

Problem 4. Prove or disprove: If x, y are real numbers such that x is rational and xy is rational then y is rational. (That is, in formal language, $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})((x \in \mathbb{Q} \wedge xy \in \mathbb{Q}) \Longrightarrow y \in \mathbb{Q})$.)

Problem 5. Prove or disprove: If n is a natural number such that (i) n is the product $p_1p_2p_3p_4$ of four distinct primes p_1, p_2, p_3, p_4 and (ii) $n \leq 1,000$, then n is even.

Problem 6. In this problem, $P(x)$ is an unknown sentence involving the variable x and U is a set. (For example, $P(x)$ could be "x is a cow", and U could be the set of all animals. With this interpretation of $P(x)$ and U, the sentence "$\sim (\forall x \in U)P(x)$" says "not all animals are cows", and the sentence "$(\exists x \in U)(\sim P(x))$" says "some animals aren't cows", so the sentence "$\Big(\sim (\forall x \in U)P(x)\Big) \Longleftrightarrow (\exists x \in U)(\sim P(x))$" says "not all animals are cows if and only if some anumals aren't cows".)

Prove, using the logical rules, that

$$\Big(\sim (\forall x \in U)P(x)\Big) \Longleftrightarrow (\exists x \in U)(\sim P(x))\,.$$

I am giving you part of the proof (slightly more than half), and asking you to do the rest, by filling in the space marked by the big box (and also filling the numbers in the small boxes).

1. Assume $\sim (\forall x \in U)P(x)$. [Assumption]
2. Assume $\sim (\exists x \in U)(\sim P(x))$. [Assumption]
3. Let $x \in U$ be arbitrary. [Introduction of a constant]
4. Assume $\sim P(x)$. [Assumption]
5. $(\exists x \in U)(\sim P(x))$. [Rule $\exists_{prove}$, from 4.]
6. $(\sim (\exists x \in U)(\sim P(x))) \wedge (\exists x \in U)(\sim P(x))$. [Rule $\wedge_{prove}$, from 2 and 5.]
7. $P(x)$. [Proof by contradiction rule, since the statement of 6 is a contradiction.]
8. $(\forall x \in U)P(x)$. [Rule $\forall_{prove}$, from 3 and 7]
9. $(\sim (\forall x \in U)P(x)) \wedge (\forall x \in U)P(x)$. [Rule $\wedge_{prove}$, fr. 1,8]
10. $(\exists x \in U)(\sim P(x))$. [Proof by contradiction rule, since the statement of 9 is a contradiction.]
11. $\Big(\sim (\forall x \in U)P(x)\Big) \Longrightarrow (\exists x \in U)(\sim P(x))$. [Rule $\Longrightarrow_{prove}$, from 1 & 10]
12. Now assume $(\exists x \in U)(\sim P(x))$. [Assumption]

You put the missing steps here. (It should be about 5-6 steps.)

☐ $(\exists x \in U)(\sim P(x)) \Longrightarrow \Big(\sim (\forall x \in U)P(x)\Big)$. [Rule $\Longrightarrow_{prove}$, from 12 & ☐]

☐ $\Big(\sim (\forall x \in U)P(x)\Big) \Longleftrightarrow (\exists x \in U)(\sim P(x))$. [Rule $\Longleftrightarrow_{prove}$, from 11 & ☐]

Problem 7. In this problem, $P(x)$ and $Q(x)$ are unknown sentences involving the variable x and U is a set. Prove, using the logical rules, that

$$(\forall x \in U)(P(x) \Longrightarrow Q(x)) \Longrightarrow \Big((\forall x \in U)P(x) \Longrightarrow (\forall x \in U)Q(x)\Big).$$

(NOTE: This proof should take about 9 steps.)