MATHEMATICS 300 — SPRING 2010

Introduction to Mathematical Reasoning

H. J. Sussmann

INSTRUCTOR'S NOTES

(February 18, 2010)

1 Functions

Given two sets A, B, a *function* from A to B is a rule that assigns to every member x of A a member $f(x)$ of B. We write

$$f : A \mapsto B$$

to indicate that f is a function from A to B. If $f : A \mapsto B$, then the set A is the *domain* of the function f.

Example 1. Let A be the set of all people, and let B be the set of all women. Then we can define a function $f : A \mapsto B$ (the "mother function") by stipulating that

$$f(x) = x'\text{s} \quad \text{mother} \qquad \text{for} \quad x \in A\,.$$

Example 2. Let A be the set of all real numbers. Then we can define a function $f : A \mapsto A$ (the "squaring" function) by stipulating that

$$f(x) = x^2 \qquad \text{for} \quad x \in A\,.$$

2 Convex functions

Let I be an interval of the real line. A function $f : I \mapsto \mathbb{R}$ is *convex* if the following is true: whenever x_1, x_2 are in I, $t \in \mathbb{R}$, and $0 \leq t \leq 1$, the inequality

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)$$

holds. (That is, $(\forall x_1 \in I)(\forall x_2 \in I)(\forall t \in \mathbb{R})(0 \leq t \leq 1 \Longrightarrow f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2))\,.$)

Example 3. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be the squaring function, defined by

$$f(x) = x^2 \qquad \text{for} \quad x \in \mathbb{R}\,.$$

Then f is convex.

Proof. We want to prove that

$$(\forall x_1, x_2, t \in \mathbb{R})(0 \le t \le 1 \Longrightarrow f((1-t)x_1 + tx_2) \le (1-t)f(x_1) + tf(x_2))\,,$$

that is, that

$$(\forall x_1, x_2, t \in \mathbb{R})\Big(0 \le t \le 1 \Longrightarrow ((1-t)x_1 + tx_2)^2 \le (1-t)x_1^2 + tx_2^2\Big)\,.$$

Let x_1, x_2, t be arbitrary real numbers. We want to prove that

$$0 \le t \le 1 \Longrightarrow \Big((1-t)x_1 + tx_2\Big)^2 \le (1-t)x_1^2 + tx_2^2\,.$$

Assume that $0 \le t \le 1$. We want to prove that

$$\Big((1-t)x_1 + tx_2\Big)^2 \le (1-t)x_1^2 + tx_2^2\,.$$

We have

$$\Big((1-t)x_1 + tx_2\Big)^2 = (1-t)^2x_1^2 + t^2x_2^2 + 2(1-t)tx_1x_2\,.$$

On the other hand, the arithmetic-geometric inequality for two numbers tells us that

$$2x_1x_2 \le x_1^2 + x_2^2\,.$$

So

$$\begin{aligned}\Big((1-t)x_1 + tx_2\Big)^2 &= (1-t)^2x_1^2 + t^2x_2^2 + 2(1-t)tx_1x_2 \\ &\le (1-t)^2x_1^2 + t^2x_2^2 + (1-t)t(x_1^2 + x_2^2) \\ &= ((1-t)^2 + (1-t)t)x_1^2 + (t^2 + (1-t)t)x_2^2 \\ &= (1-t)((1-t) + t)x_1^2 + t(t + (1-t))x_2^2 \\ &= (1-t)x_1^2 + tx_2^2\,,\end{aligned}$$

that is,

$$\Big((1-t)x_1 + tx_2\Big)^2 \le (1-t)x_1^2 + tx_2^2\,,$$

which is precisely the inequality we wanted to prove.

Example 4. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be the absolute value function, defined by

$$f(x) = |x| \qquad \text{for} \quad x \in \mathbb{R}\,.$$

Then f is convex.

Proof. We want to prove that

$$(\forall x_1, x_2, t \in \mathbb{R})\Big(0 \leq t \leq 1 \Longrightarrow f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2)\Big)\,,$$

that is, that

$$(\forall x_1, x_2, t \in \mathbb{R})\Big(0 \leq t \leq 1 \Longrightarrow |(1-t)x_1 + tx_2| \leq (1-t)|x_1| + t|x_2|\Big)\,.$$

Let x_1, x_2, t be arbitrary real numbers. We want to prove that

$$0 \leq t \leq 1 \Longrightarrow |(1-t)x_1 + tx_2| \leq (1-t)|x_1| + t|x_2|\,.$$

Assume that $0 \leq t \leq 1$. We want to prove that

$$|(1-t)x_1 + tx_2| \leq (1-t)|x_1| + t|x_2|\,.$$

It follows from the triangle inequality that

$$|(1-t)x_1 + tx_2| \leq |(1-t)x_1| + |tx_2|\,.$$

On the other hand,

$$\begin{aligned} |(1-t)x_1| &= |1-t| \cdot |x_1| \\ &= (1-t) \cdot |x_1|\,, \end{aligned}$$

using the identity $|ab| = |a| \cdot |b|$ and the fact that $1-t \geq 0$ (because $t \leq 1$), and

$$\begin{aligned} |tx_2| &= |t| \cdot |x_2| \\ &= t|x_2|\,, \end{aligned}$$

using the identity $|ab| = |a| \cdot |b|$ and the fact that $t \geq 0$. Therefore

$$|(1-t)x_1 + tx_2| \leq (1-t)|x_1| + t|x_2|\,,$$

which is precisely the inequality we wanted to prove.

3 Homework assignment No. 5, due on Thursday February 25

Problem 1. Prove that if x is a real number such that $x > 20$ or $x < 2$ then $|x - 12| > 7$.

Problem 2. Let $\mathbb{R}_+$ be the set of all positive real numbers (that is, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$). Let $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be the function given by

$$f(x) = \frac{1}{x} \quad \text{for } \; x \in \mathbb{R}_+ \,.$$

Prove that f is convex.

Problem 3. Prove that sum of two convex functions is convex. (Precisely: if I is an interval, and $f : I \mapsto \mathbb{R}$, $g : I \mapsto \mathbb{R}$ are convex functions, then $f + g$ is a convex function.)

Problem 4. A *linear function* is a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that, for some real numbers a, b,

$$f(x) = ax + b \qquad \text{for every} \quad x \in \mathbb{R}\,.$$

Prove that every linear function is convex.

Problem 5. Prove or disprove: the product of two convex functions is convex.

Problem 6. In this problem, $P(x)$ and $Q(x)$ are unknown sentences involving the variable x and U is a set. Prove, using the logical rules, that

$$(\forall x \in U)(P(x) \wedge Q(x)) \Longleftrightarrow \Big((\forall x \in U)P(x) \wedge (\forall x \in U)Q(x)\Big)\,.$$

Problem 7. In this problem, $P(x)$ and $Q(x)$ are unknown sentences involving the variable x and U is a set. Show that it cannot be proved, using the logical rules, that

$$(\forall x \in U)(P(x) \vee Q(x)) \Longleftrightarrow \Big((\forall x \in U)P(x) \vee (\forall x \in U)Q(x)\Big)\,.$$