MATHEMATICS 300 — FALL 2009

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

(October 1 2009)

1 Homework assignment no. 5, due on Thursday October 9

NEW RULES!!!! *In this assignment, the rules of the game are slightly changed, as follows. If you are asked to prove something, the answer could be "It cannot be proved because it isn't true," followed by a proof that the statement isn't true. For example, if you are asked to prove that the equation* $x^2 + 1 = 0$ *has a real solution (i.e., that* $(\exists x \in \mathbb{R})x^2 + 1 = 0$*), the answer should be "I cannot prove this because it isn't true. Here is a proof that the statement isn't true, i.e., that the equation* $x^2 + 1 = 0$ *does not have a real solution: suppose there exists an* $x \in \mathbb{R}$ *such that* $x^2 + 1 = 0$*; pick one such* x *and fix it from now on; then* $x^2 \geq 0$*, because the square of any real number is* ≥ 0*; so* $x^2 + 1 > 0$*, because* $1 > 0$*, so* $x^2 + 1 > x^2 \geq 0$*; but* $x^2 + 1 = 0$*, so* $0 > 0$*; on the other hand,* $\sim 0 > 0$*; so we got a contradiction. Hence* $\sim (\exists x \in \mathbb{R})x^2 + 1 = 0$*."*

1. (The concepts of "rational number" and "irrational number" are defined as follows:

   DEFINITION. A **rational number** is a real number r for which there exist integers m, n such that $n \neq 0$ and $r = \frac{m}{n}$.

   DEFINITION. An **irrational number** is a real number r which is not a rational number.

   In addition, we use the following notation: the symbol $\mathbb{Q}$ denotes the set of all rational numbers, so the expression "$x \in \mathbb{Q}$" says that "x is a rational number." Then, if you want to say that "x is an irrational number", you can say "$x \in \mathbb{R} \wedge x \notin \mathbb{Q}$".)

   Translate each of the following statements into formal language and prove it. Use only quantifiers over the reals, that is, $(\forall x \in \mathbb{R})$ or $(\exists x \in \mathbb{R})$. Do not use quantifiers such as $(\forall x \in \mathbb{Q})$. For example, the translation into formal language of the first statement below is the following sentence: $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})((x \in \mathbb{Q} \wedge y \in \mathbb{Q}) \Rightarrow x + y \in \mathbb{Q})$.

   i. If x, y are rational numbers then $x + y$ is rational.

ii. If x, y are rational numbers then $x \cdot y$ is rational.

iii. If x, y are rational numbers then $x - y$ is rational.

iv. If x, y are rational numbers and $y \neq 0$ then $\frac{x}{y}$ is rational.

v. If x is a rational number and y is irrational then $x + y$ is irrational.

vi. If x is a rational number and y is irrational then $x \cdot y$ is irrational.

vii. If x, y are irrational numbers then $x + y$ is irrational.

viii. If x, y are irrational numbers then $x \cdot y$ is irrational.

2. The "maximum" of two real numbers a, b is the largest of the two numbers. We use $a \uparrow b$ to denote the maximum of a and b. The precise mathematical definition is as follows:

   DEFINITION. Let a, b be real numbers. The **maximum** of a and b is the real number $a \uparrow b$ given by the following formulas:

$$\begin{aligned} a \uparrow b &= a \qquad \text{if} \quad a \geq b \\ a \uparrow b &= b \qquad \text{if} \quad b \geq a\,. \end{aligned}$$

   Prove the following statements:

   i. $(\forall a, b, c \in \mathbb{R})(a \uparrow b) \uparrow c = a \uparrow (b \uparrow c)$ (associative law of $\uparrow$),

   ii. $(\forall a, b \in \mathbb{R}) a \uparrow b = b \uparrow a$ (commutative law of $\uparrow$),

   iii. $(\forall a, b, c \in \mathbb{R}) a \uparrow (b + c) = (a \uparrow b) + (a \uparrow c)$ (distributive law of $\uparrow$ with respect to $+$).

   iv. $(\forall a, b \in \mathbb{R}) a \uparrow b = \frac{a+b+|a-b|}{2}$ (formula showing how to obtain $\uparrow$ from $|\cdots|$).

   v. $(\forall a \in \mathbb{R})|a| = a \uparrow (-a)$ (formula showing how to obtain $|\cdots|$ from $\uparrow$).

3. Book, page 65, problem 10.

4. Prove each of the following:

   i. $(\exists! x \in \mathbb{R}) x^3 = 2x$,

   ii. $(\exists! x \in \mathbb{Z}) x^3 = 2x$.

   iii. $(\exists! x \in \mathbb{Z}) x^3 = x$.