MATHEMATICS 300 — FALL 2009

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

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1 Some basic principles for writing proofs

Here are some important principles that you are strongly urged to obey when you write proofs. Some of these principles are *optional*, that is, you do not have to do things as the principle says, but it is recommended that you do it. Other principles are *absolute musts*: if you violate them, then your proof is not valid.

PRINCIPLE NO. 1. Your proof should consists of clear **steps**. I should be able to read the proof and see Step 1, Step 2, Step 3, and so on. Some people like to make the step structure clear by actually numbering the steps. For example, you could write this:

PROOF THAT $2+2=4$**, USING THE DEFINITIONS OF** 2**,** 3**,** 4**, WHICH SAY THAT** $2=1+1$**,** $3=2+1$**,** $4=3+1$**:**

1. $2=1+1$, by the definition of 2.
2. $2+2=2+2$, because $(\forall x)x=x$.
3. $2+2=2+(1+1)$, substituing equals for equals, from **1** and **2**.
4. $2+(1+1)=(2+1)+1$, by the associative law of addition.
5. $2+1=3$, by the definition of 3.
6. $(2+1)+1=(2+1)+1$, because $(\forall x)x=x$.
7. $(2+1)+1=3+1$, substituing equals for equals, from **5** and **6**.
8. $3+1=4$, by the definition of 4.
9. $2+2=4$, from **3, 4, 7** and **8**. QED

However, most people do not number the steps, and that is fine with me, as long as I can see what the steps are.

PRINCIPLE NO. 2. Each step must assert something, and the assertion the step makes must be clear and precise.

You should always ask about each step, and be able to answer, the following question: **What does this step assert?** *This is an absolute must. If it's not clear what your steps are asserting, then your proof is invalid.*

REMARK. "To assert" something means "to say that it is true." For example, if a prosecutor says "Wendell Smith killed Alice Jones", just like that, then she is asserting that Wendell Smith killed Alice Jones. But if the prosecutor says "I am going to prove that Wendell Smith killed Alice Jones", then she is not asserting that Wendell Smith killed Alice Jones; she is just asserting that she in going to prove that Wendell Smith killed Alice Jones.

The assertion made in a step could be

1. The statement of a mathematical fact. (For example: "$2 + 2 = 4$".)

2. The statement of your intention to do something. (For example: "I am going to prove that $2 + 2 = 4$."

3. The statement that you are starting a subproof, in which you introduce an assumption or declare a value for some variable.. (For example: "Assume that $x > 0$." You should think of this as meaning: "We now enter an imaginary world in which it is true that $x > 0$, and set out to find out what happens in that world." Another example: "Let x be an arbitrary real number." You should think of this as meaning: "We now enter an imaginary world in which we have been given a real number, called x, whose value is not known to us, because it is written on a piece of paper inside a sealed envelope." Another example: "Let $x = \frac{1+\sqrt{5}}{2}$.")

PRINCIPLE NO. 3. In each step it must be clear what the various objects the step talks about are.

You should always ask about each step, and be able to answer, the following question: **Who are the things that the step talks about?** *This is an absolute must. If it's not clear who your steps are talking about, then your proof is invalid.*

For example, if in a step you say "$x^2 > 3$", then you should be able to answer the question "who is x?". And the asnwer should be that x is some object that was introduced before. (For example: if in a step you say "$x^2 > 3$", then there should be a previous step in which you said something like "Let x be arbitrary", or "let $x = 6$.")

REMARK. The question to be asked is "who is x?", not "what is x?" We must ask *which specific object x is, not what kind of an object x is.* For example, if you write "$x^2 > 3$", and I askyou "what is x", and you answer "x is a real number", that does not answer the question.

If you violate this principle in your exam, and write, for example, "$x^2 > 3$" without telling me who x is, then I will give you a grade of "x", and when you come and ask me "what is this x?" I will answer "It's a grade." That won't satisfy you, will it? Similarly, "x is a real number" is not a satisfactory answer to the question "who is x?"

PRINCIPLE NO. 4. In each step it must be clear under what assumptions you are working.

> You should always ask about each step, and be able to answer, the following question: **What are the assumptions and known facts at this point?** *This is an absolute must. If it's not clear under what assumptions you are working and what facts are know at a particular point, then your proof is invalid.*

For example, suppose that in a step you say "$x^2 > 3$". Then you should be able to answer, first of all, "who is x?" Suppose that in a previous step you wrote "Let x be an arbitrary real number." Then you question "who is x?" is answered. But that's not enough. *It's got to be clear what, if anything, you are assuming at this point.* This makes a tremendous difference. For example, if you know nothing whatsoever about x, then the statement that "$x^2 > 3$" does not follow, so you are not entitled to asserting it. If, on the other hand, you know, for example, that $x > 2$ (either because you have proved that $x > 2$, or because you are working under the assumption that $x > 2$) then you are entitled to say that $x^2 > 3$.

PRINCIPLE NO. 5. In each step, if a mathematical assertion is made, it must be clear why it is true, based on what we know at that point.

> You should always ask about each step that makes a mathematical assertion: **Why is this true? How does it follow?** *This is an absolute must. If it's not clear why what you are asserting is true, then your proof is invalid.*

PRINCIPLE NO. 6. The proof must move step by step by step from things that we know to be true to things that follow, and end up with the desired conclusion. **You cannot start from the conclusion because you don't know that the conclusion is true until you have proved it. And for the same reason, you cannot bring in the conclusion before the end and use it.** (You may object as follows: don't we "bring in the conclusion in the middle of our proof when we do a proof by contradiction?" The answer is this: in a proof by contradiction, we do not assert that the conclusion is true. We *assume* (that, imagine) that the *negation* of the conclusion is true, i.e., that the conclusion is false, and explore an imaginary world in which the conclusion is false, in order to show that this imaginary world is impossible, so the conclusion cannot be false, and therefore must be true.)

Of course, **you could start by announcing that you are going to prove the conclusion**, but when you do this you do not know yet that the conclusion is true, so you cannot use it.

2 An example: analysis of a proof.

Here is an example of a proof written by a student. (Not a real student, mind you, but a composite of several students.) I will first show you the proof, and then analyze it. (Let me anticipate the conclusion: this proof gets a zero.)

PROOF THAT $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})((a > 0 \wedge b > 0) \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6)$ **:**

Step 1. We want to prove that $\frac{9a}{b} + \frac{b}{a} \geq 6$.

Step 2. Multiplying both sides by ab, we get

$$9a^2 + b^2 \geq 6ab\,.$$

Step 3. So $9a^2 + b^2 - 6ab \geq 0\,.$

Step 4. But $9a^2 + b^2 - 6ab = (3a - b)^2\,.$

Step 5. So $(3a - b)^2 \geq 0\,.$ This is true, so we have proved that $\frac{9a}{b} + \frac{b}{a} \geq 6$.

ANALYSIS OF THIS "PROOF".

1. Let's apply Principle No. 3 to Step 1. Who are a and b? We do not know, because the values of a and b have not been declared before. For Step 1 to make sense, there should have been a previous step saying: "Let a, b be arbitrary real numbers." So **Principle No. 3 is violated in Step 1.**

2. Now let us apply Principles No. 2, 4, and 5. In Step 1, it is clear what is being asserted: the author is saying that he/she is going to prove that $\frac{9a}{b} + \frac{b}{a} \geq 6$. The author is **not** asserting that "$\frac{9a}{b} + \frac{b}{a} \geq 6$." But then, in Step 2, we get into trouble. Exactly what is the author asserting here? Is the author asserting that $9a^2 + b^2 \geq 6ab$ is true?

   a. If the author is saying that $9a^2 + b^2 \geq 6ab$ is true (let's call that "Possibility (a)"), then **Principle 5 is violated**. (It would seem that "$9a^2 + b^2 \geq 6ab$" is true because $\frac{9a}{b} + \frac{b}{a} \geq 6$. But **we do not know that** $\frac{9a}{b} + \frac{b}{a} \geq 6$ **is true.** All we know is that the author announced that he/she is going to prove it. That doesn't mean it's been proved, so we cannot use it! So Principle 5 is violated.)

   b. Another possibility is that all the author meant to say is that "if you manipulate the inequality of Step 1 in some way then you get the inequality of Step 2" (let's call this "Possibility (b)"), without asserting that either one is true. Then the statement asserted in Step 2 would be O.K. (except for a mistake that I will discuss later). But then we get into trouble in Step 3, because in Step 3 the author is clearly asserting that $9a^2 + b^2 - 6ab \geq 0$, and this could only follow if we know that the inequality of Step 2 is true, but we don't know that, if you interpret Step 2 according to Possibility (b).

3. Summarizing the above discussion, **Principle No. 2 is violated in Step 2, because it's not clear exactly what that step asserts**. Furthermore, **Principle No. 5 is violated in Step 2** (if we interpret Step 2 as in Possibility (a)), **or in Step 3** (if we interpret Step 2 as in Possibility (b)).

4. Furthermore, **the manipulation of the inequality is not justified**, because the author seems to be using the principle that "you can multiply both sides of an inequality by a number". But this is not true! (For example, if you multiply the inequality $6 \geq 4$ by -1 you get the inequality $-6 \geq -4$, which is false.)

5. In Steps 1, 2, 3 the author has gradually moved from saying "I am going to prove such-and-such thing" to asserting things as true. So, ultimately, **the author is violating Principle No. 6, and effectively starting from the conclusion** and getting from it to a true statement, which doesn't prove anything.

A CORRECT PROOF THAT

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})((a > 0 \wedge b > 0) \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6) \qquad :$$

1. Let a, b be arbitrary real numbers.

2. Assume that $a > 0 \wedge b > 0$.

3. We want to prove that

$$\frac{9a}{b} + \frac{b}{a} \geq 6\,. \tag{1}$$

4. If we could prove that

$$9a^2 + b^2 \geq 6ab\,. \tag{2}$$

then (1) would follow, because (2) implies (1), since (1) follows from (2) by multiplying both sides of (2) by $\frac{1}{ab}$, which is possible because we are assuming that $a > 0$ and $b > 0$, so $ab > 0$.

5. Furthermore, (2) would follow if we prove

$$9a^2 + b^2 - 6ab \geq 0\,, \tag{3}$$

since adding $6ab$ to both sides of (3) yields (2).

6. But $9a^2 + b^2 - 6ab = (3a - b)^2$.

7. And $(3a - b)^2 \geq 0$, because the square of any real number is non-negative.

8. It follows from Steps 6 and 7 that (3) holds. Hence, as explained before, (2) is true, and then (1) is true.

9. So

$$(a > 0 \wedge b > 0) \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6\,. \tag{4}$$

10. Since (4) was proved for arbitrary $a, b \in \mathbb{R}$, we have shown that

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})((a > 0 \wedge b > 0) \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6)\,,$$

which is the desired conclusion.

ANOTHER CORRECT PROOF THAT

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})((a > 0 \wedge b > 0) \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6) \qquad :$$

1. Let a, b be arbitrary real numbers.

2. Assume that $a > 0 \wedge b > 0$.

3. It is clear that $9a^2 + b^2 - 6ab = (3a - b)^2$.

4. Also, $(3a - b)^2 \geq 0$, because the square of any real number is nonnegative.

5. Therefore $9a^2 + b^2 - 6ab \geq 0$.

6. Hence $9a^2 + b^2 \geq 6ab$.

7. Since $a > 0$ and $b > 0$, the number $\frac{1}{ab}$ is positive, so we may multiply both sides of the inequality of Step 6 by $\frac{1}{ab}$, and obtain

$$\frac{9a}{b} + \frac{b}{a} \geq 6\,. \qquad (5)$$

8. So

$$(a > 0 \wedge b > 0) \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6\,. \qquad (6)$$

9. Since (6) was proved for arbitrary $a, b \in \mathbb{R}$, we have shown that

$$(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})((a > 0 \wedge b > 0) \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6)\,,$$

which is the desired conclusion.

REMARK. Notice that in both proofs the implications go in the same direction. In both proofs we show that $(3a - b)^2 \geq 0$, then that $(3a - b)^2 \geq 0 \Longrightarrow 9a^2 + b^2 - 6ab \geq 0$, then that $9a^2 + b^2 - 6ab \geq 0 \Longrightarrow 9a^2 + b^2 \geq 6ab$, and then that $9a^2 + b^2 \geq 6ab \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6$.

This is completely clear in the second proof. You may think it is not so clear in the first proof, because in that proof we wrote "$\frac{9a}{b} + \frac{b}{a} \geq 6$" first, and we wrote "$9a^2+b^2 \geq 6ab$" afterwards. However, we did not assert that "$\frac{9a}{b}+\frac{b}{a} \geq 6$" is true and "$9a^2 + b^2 \geq 6ab$" follows from it. What we did assert is that *if we could prove "$9a^2 + b^2 \geq 6ab$" then "$\frac{9a}{b} + \frac{b}{a} \geq 6$" would follow*, and we gave a reason (multiplication of both sides by $\frac{1}{ab}$). In other words, we are still proving the implication $9a^2 + b^2 \geq 6ab \Longrightarrow \frac{9a}{b} + \frac{b}{a} \geq 6$, exactly as in the other proof.