Problem 1. **Prove** that, if \((X, \mathcal{M}, \mu)\) is a measure space, \(1 \leq p \leq \infty\), and \(\frac{1}{p} + \frac{1}{q} = 1\), then: If \(f : \rightarrow \mathbb{C}\) is a measurable function such that \(fg \in L^1(X, \mathcal{M}, \mu)\) for every \(g \in L^q(X, \mathcal{M}, \mu)\), then \(f \in L^p(X, \mathcal{M}, \mu)\). (NOTES: (1) The case \(p = 1\) is completely trivial. (2) If you know the Uniform Boundedness Principle, then this result follows easily from it, but you don’t really need that. It’s possible to give a direct proof.)

Problem 2. Folland, Problem 41, page 208.

Problem 3. If \(V\) is a set, a semimetric on \(V\) is a function \(s : V \times V \rightarrow \mathbb{R}\) such that

\[
\begin{align*}
\text{(Sm.1)} & \quad s(x, y) \geq 0 \text{ for all } x, y \in V, \\
\text{(Sm.2)} & \quad s(x, x) = 0 \text{ for all } x \in V, \\
\text{(Sm.3)} & \quad s(x, y) = s(y, x) \text{ for all } x, y \in V, \\
\text{(Sm.4)} & \quad s(x, z) \leq s(x, y) + s(y, z) \text{ for all } x, y, z \in V.
\end{align*}
\]

(Note: Semimetrics are also sometimes called “pseudometrics”, or “semidistances”, or “pseudodistances”.)

A semimetric space is a set \(V\) equipped with a semimetric \(s\) on \(V\).

A countably semimetric space is a set \(V\) equipped with a sequence

\[ s = (s_n)_{n=1}^{\infty} \]  \hspace{1cm} (1)

of semimetrics on \(V\).
NOTE: A semimetric space is a special kind of countably semimetric space. So every definition given below for countably semimetric spaces applies in particular to a semimetric space.

If \((V,s)\) is a countably semimetric space, with \(s\) given by (1), then we can define several notions similar to the ones you know for metric spaces:

**Convergence of sequences:** We say that a sequence \(x = (x_k)_{k=1}^{\infty}\) of points of \(V\) converges in \((V,s)\) to a point \(x \in V\) if \(\lim_{k \to \infty} s_n(x_k, x) = 0\) for every \(n\). (That is, \(x\) converges to \(x\) in \((V,s)\) if and only if \(x\) converges to \(x\) in the semimetric space \((V,s_n)\) for each \(n\).)

**Cauchy sequences:** We say that a sequence \(x = (x_k)_{k=1}^{\infty}\) of points of \(V\) is Cauchy in \((V,s)\) if for every \(n \in \mathbb{N}\) and every positive \(\varepsilon\) there exists a \(K \in \mathbb{N}\) such that \(s_n(x_j, x_k) < \varepsilon\) whenever \(j \geq K\) and \(k \geq K\). (That is, \(x\) is Cauchy in \((V,s)\) if and only if \(x\) is Cauchy in the semimetric space \((V,s_n)\) for each \(n\).)

**Hausdorff space:** \((V,s)\) is Hausdorff if the limit of a sequence, when it exists, is unique. (Equivalently: whenever a sequence \(x\) converges in \((V,s)\) to two points \(x_1, x_2\) of \(V\), it follows that \(x_1 = x_2\).)

**Complete space:** \((V,s)\) is complete if every Cauchy sequence converges.

**Closed sets:** A subset \(S\) of \(V\) is closed in \((V,s)\) if whenever \(x\) is a sequence of points of \(S\) and \(x\) converges to a point \(x\) of \(V\), it follows that \(x \in S\).

**Open sets:** A subset \(S\) of \(V\) is open in \((V,s)\) if the set \(V \setminus S\) is closed.

**Neighborhoods:** A subset \(S\) of \(V\) is a neighborhood of a point \(x\) of \(V\) if there exists an open subset \(U\) of \(V\) such that \(x \in U \subseteq S\).

**Prove** that:

1. If \((V,s)\) is a countably semimetric space, with \(s\) given by (1), then the function \(d_s : V \times V \mapsto \mathbb{R}\) given by

   \[
   d_s(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{s_n(x, y)}{1 + s_n(x, y)} \quad \text{for} \quad x, y \in V
   \]  

   is a semimetric on \(V\).

\(^1\)Just think of a sequence of semimetrics that are all equal.
2. The semimetric space \((V,d_s)\) is “equivalent” to the countably semimetric space \((V,s)\) in a number of ways. Precisely:

(a) Convergence of sequences in \((V,d_s)\) is exactly the same as convergence of sequences in \((V,s)\). That is, a sequence \(x\) converges to a point \(x\) in \((V,d_s)\) if and only if it converges to \(x\) in \((V,s)\).

(b) The Cauchy sequences in the countably semimetric space \((V,d_s)\) are exactly the same as the Cauchy sequences in \((V,s)\).

(c) \((V,d_s)\) is Hausdorff if and only if \((V,s)\) is Hausdorff.

(d) The closed subsets of \((V,d_s)\) are exactly the same as the closed subsets of \((V,s)\). (That is, if \(S \subseteq V\), then \(S\) is closed in \((V,d_s)\) if and only if \(S\) is closed in \((V,s)\).

(e) The open subsets of \((V,d_s)\) are exactly the same as the open subsets of \((V,s)\). (That is, if \(S \subseteq V\), then \(S\) is open in \((V,d_s)\) if and only if \(S\) is open in \((V,s)\).

(f) Neighborhoods in \((V,d_s)\) are exactly the same as neighborhoods in \((V,s)\). (That is, if \(S \subseteq V\) and \(x \in V\), then \(S\) is a neighborhood of \(x\) in \((V,d_s)\) if and only if \(S\) is a neighborhood of \(x\) in \((V,s)\).

3. The following conditions are equivalent:

(a) \((V,s)\) is Hausdorff.

(b) \((V,d_s)\) is Hausdorff.

(c) \((V,d_s)\) is a metric space.

(d) Whenever \(x, y \in V\), and \(x \neq y\), there exists \(n \in \mathbb{N}\) such that \(s_n(x,y) \neq 0\).

(e) Every singleton set \(\{x\}\), for \(x \in V\), is closed.

(f) Whenever \(x, y\) are points of \(V\) such that \(x \neq y\), there exists a neighborhood \(U\) of \(X\) such that \(y \neq U\).

(g) Whenever \(x, y\) are points of \(V\) such that \(x \neq y\), there exist neighborhoods \(W, Z\) of \(x, y\) such that \(W \cap Z = \emptyset\).

**Problem 4.** A countably seminormed space is a vector space \(V\) over \(\mathbb{F}\) (where \(\mathbb{F} = \mathbb{R}\) or \(\mathbb{F} = \mathbb{C}\)), equipped with a sequence

\[
p = (p_n)_{n=1}^{\infty}
\]
of seminorms on \( V \).

A semimetric \( s : V \times V \mapsto \mathbb{R} \) is translation invariant if

\[
s(x + z, y + z) = s(x, y) \quad \text{for all} \quad x, y, z \in V .
\]

**Be sure you know how to prove**\(^2\) that:

(\#) Every seminorm on a vector space \( V \) gives rise in a natural way to a translation-invariant semimetric. (For the seminorm \( p : V \mapsto \mathbb{R} \), define a semimetric \( s_p \) by letting \( s_p(x, y) = p(x - y) \). You have to prove that \( s_p \) is indeed a translation-invariant semimetric.)

It follows from the previous trivial observation that every countably seminormed space is a countably semimetric space, so it a semimetric space.

**Be sure you know how to prove** that, if \((V, p)\) is a countably seminormed space, with \( p \) given by (3), and \( s_p \) is the sequence of semimetrics

\[
s_p = (s_{p_n})_{n=1}^{\infty} ,
\]

where the \( s_{p_n} \) are the semimetrics constructed from the \( p_n \) as described in (\#) above, then the semimetric \( d_{s_p} \) constructed in (2) is translation-invariant.

**Prove** that the translation-invariant semimetric \( d_{s_p} \) need not arise from a norm, even if the space \((V, p)\) is Hausdorff. In order to do this, you should analyze the following specific example, and **prove** the facts listed there:

Let \( V \) be the space of all continuous functions \( f : \mathbb{R} \mapsto \mathbb{R} \). (Or you could take the continuous functions \( f : \mathbb{R} \mapsto \mathbb{C} \) if you prefer.) For each positive integer \( n \), define

\[
p_n(f) = \sup\{ |f(x)| : x \in [-n, n] \} .
\]

**Prove** that

1. If \( p = (p_n)_{n=1}^{\infty} \), then the countably seminormed space \((V, p)\) is Hausdorff and complete.

2. A sequence \((f_k)_{k=1}^{\infty}\) of functions in \( V \) converges to a function \( f \in V \) if and only if the \( f_k \) converge to \( f \) uniformly on every compact subset of \( \mathbb{R} \).

\(^2\)This is too trivial for me to insult your intelligence by asking you to prove it.
3. There does not exist a norm $\| \cdot \|$ on $V$ such that a sequence $f = (f_k)_{k=1}^{\infty}$ converges to a function $V$ in $(V, \mathbf{p})$ if and only if $f_k \to f$ in $(V, \| \cdot \|)$.

**Important concluding remark:** A Fréchet space is a complete Hausdorff countably seminormed space. What we have done in the previous example is this: we have constructed an example if a Fréchet space that is not a Banach space.

Fréchet spaces are very important in Analysis, because many useful spaces are Fréchet but not Banach. For example, in your Complex Variables course you must have seen the space $\text{Hol}(U)$ of all holomorphic functions $f : U \mapsto \mathbb{C}$, where $U$ is an open subset of the complex plane. This space can be equipped with a sequence of seminorms $p_n$, by letting $K = (K_n)_{n=1}^{\infty}$ be an exhausting sequence of compact subsets of $U$, and defining $p_n(f) = \sup\{|f(z)| : z \in K_n\}$. It is an easy exercise (if you know something about complex variables!) to prove that this space is Fréchet but not Banach.

**A recommendation:** Read the entry on “Fréchet spaces” in *Wikipedia*.

**Problem 5 (optional).** Prove that the space $\text{Hol}(U)$ defined above is Fréchet but not Banach.

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3 Even the case when $U = \mathbb{C}$ is important. In that case, of course, $\text{Hol}(U)$ is the space of all entire functions.

4 That is, a sequence of compact subsets of $U$ such that every compact subset of $U$ is a subset of some $K_n$. 