MATHEMATICS 300 — SPRING 2015

Introduction to Mathematical Reasoning

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HOMEWORK ASSIGNMENT NO. 8, DUE ON FRIDAY, APRIL 10

Problem 1. In this problem, I am asking you to prove or disprove several statements. To *prove* one of these statements (which you should be able to do, if the statement is true[1]) you can use induction or well-ordering. (And you probably will need to prove first, as a lemma, the result for the special case $n = 2$.) To *disprove* it (which you should be able to do, if the statement is false), you should give a counterexample.

I will give you the solutions for the first two, and you are asked to do the other ones. ***You should read these solutions very carefully, and use the same pattern for all the other questions.***

Statement I. Prove or disprove the following statement: If n is a natural number and $x_1, \ldots, x_n$ are odd integers, then the product $\prod_{k=1}^{n} x_k$ is odd. (In other words: the product of several odd integers is odd.)

Solution. This statement is true. In order to prove it, we will first prove a lemma:

Lemma. If x and y are odd integers, then the product xy is odd.

Proof of the lemma. Let x, y be arbitrary odd integers. Since x and y are odd, we may pick integers j, k such that $x = 2j + 1$ and $y = 2k + 1$. Then

$$xy = (2j + 1)(2k + 1) = 4jk + 2j + 2k + 1 = 2(2jk + j + k) + 1\,.$$

Since $2jk + j + k$ is an integer, it follows that xy is odd. **Q.E.D.**

Proof of Statement I. We use induction on n. Let $P(n)$ be the statement "If $x_1, \ldots, x_n$ are odd integers, then the product $\prod_{k=1}^{n} x_k$ is odd". We want to prove that $(\forall n \in \mathbb{N})P(n)$.

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[1] Let me be very precise here. I am ***not*** saying that every true mathematical statement can be proved. In fact, there is a very deep result, known as the "Gödel Incompleteness Theorem", that says that there are mathematical statements that are true but cannot be proved. All I am saying is that *all the statements in this set of problems that are true can be proved, and you should be able to prove them.*

The base case. $P(1)$ says that "if x_1 is an odd integer, then the product $\prod_{k=1}^{1} x_k$ is odd". But, according to the inductive definition of "$\prod$", we have

$$\prod_{k=1}^{1} x_k = x_1\,,$$

so $P(1)$ says that "if x_1 is odd then x_1 is odd", and this is obviously true.

The inductive step. We have to prove that

$$(\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))\,. \tag{1}$$

For this purpose, we will take n to be an arbitrary natural number, assume $P(n)$, and prove $P(n+1)$.

Let $n \in \mathbb{N}$ be arbitrary.

Assume $P(n)$.

We want to prove $P(n+1)$.

That is, we want to prove that

$$(\forall x_1, \ldots, x_n, x_{n+1} \in \mathbb{Z})\Big(x_1, \ldots, x_n, x_{n+1} \text{ are odd} \Longrightarrow \prod_{k=1}^{n+1} x_k \text{ is odd}\Big). \tag{2}$$

To prove (2), we will take $x_1, \ldots, x_n, x_{n+1}$ to be arbitrary integers, assume that these integers are odd, and prove that $\prod_{k=1}^{n+1} x_k$ is odd.

Let $x_1, \ldots, x_n, x_{n+1}$ be arbitrary integers,

Assume that $x_1, \ldots, x_n, x_{n+1}$ are odd.

We want to prove that $\prod_{k=1}^{n+1} x_k$ is odd.

We know, from the inductive definition of "$\prod$", that[2]

$$\prod_{k=1}^{n+1} x_k = \Big(\prod_{k=1}^{n} x_k\Big).x_{n+1}\,. \tag{3}$$

[2] ***Pay attention to the use of parentheses!*** I you had written "$\prod_{k=1}^{n} x_k.x_{n+1}$", without parentheses, rather than "$\Big(\prod_{k=1}^{n} x_k\Big).x_{n+1}$", this would have meant a totally different thing, namely, the product of $x_1.x_{n+1}$ times $x_2.x_{n+1}$ times $\ldots$ times $x_n.x_{n+1}$, which is actually equal to $\Big(\prod_{k=1}^{n} x_k\Big)x_{n+1}^n$.

The inductive assumption $P(n)$ says that "if $x_1, \ldots, x_n$ are odd integers, then the product $\prod_{k=1}^{n} x_k$ is odd". In our case, we are assuming that $x_1, \ldots, x_n, x_{n+1}$ are odd integers, so in particular $x_1, \ldots, x_n$ are odd integers. Then $\prod_{k=1}^{n} x_k$ is odd.

And we are also assuming that x_{n+1} is odd.

So (3) tells us that $\prod_{k=1}^{n+1} x_k$ is the product of two odd integers.

And the lemma tells us that the product of two odd integers is odd.

Therefore $\prod_{k=1}^{n+1} x_k$ is odd.

We have proved that $\prod_{k=1}^{n+1} x_k$ is odd under the assumption that the integers $x_1, \ldots, x_n, x_{n+1}$ are odd.

And this was established for arbitrary integers $x_1, \ldots, x_n, x_{n+1}$.

So we have proved (2).

That is, we have proved $P(n+1)$.

Since we have proved $P(n+1)$ assuming $P(n)$, it follows that $P(n) \Longrightarrow P(n+1)$.

Since we have proved $P(n) \Longrightarrow P(n+1)$ for arbitrary $n \in \mathbb{N}$, we have shown that (1) holds.

This completes the inductive step.

It then follows from the Principle of Mathematical Induction that

$$(\forall n \in \mathbb{N}) P(n)\,,$$

which is our desired conclusion. **Q.E.D.**

Statement II. Prove or disprove the following statement: If n is a natural number and $x_1, x_2, \ldots, x_n$ are odd integers, then the sum $\sum_{k=1}^{n} x_k$ is odd.

Solution. Statement II is false. To see this, here is a counterexample: take $n = 2$, $x_1 = 5$, $x_2 = 3$. Then x_1 and x_2 are odd, but the sum $\sum_{k=1}^{n} x_k$ is equal to 8, which is not odd.

Statement III. Prove or disprove the following statement: If n is a natural number and $x_1, x_2, \ldots, x_n$ are real numbers that are all different from zero, then the sum $\sum_{k=1}^{n} x_k$ is different from zero.

Statement IV. Prove or disprove the following statement: If n is a natural number and $x_1, x_2, \ldots, x_n$ are real numbers that are all different from zero, then the product $\prod_{k=1}^n x_k$ is different from zero.

Statement V. Prove or disprove the following statement: If p is an integer, n is a natural number, and $x_1, x_2, \ldots, x_n$ are integers such that p divides x_k for every $k \in \mathbb{N}$ such that $k \leq n$, then the sum $\sum_{k=1}^n x_k$ is divisible by p. (That is[3],

$$(\forall p \in \mathbb{Z})(\forall n \in \mathbb{N})(\forall x_1, \ldots, x_n \in \mathbb{Z}) \\ \Big((\forall k \in \mathbb{N})(k \leq n \Longrightarrow p|x_k) \Longrightarrow p| \sum_{x_k} \Big).$$

In other words, "if an integer p divides several integers, then it divides the sum of those integers". Recall that "|" stands for "divides"[4].)

Statement VI. Prove or disprove the following statement: If p is an integer, n is a natural number, and $x_1, x_2, \ldots, x_n$ are integers such that the product $\prod_{k=1}^n x_k$ is divisible by p, then x_j is divisible by p for some $j \in \mathbb{N}$ such that $j \leq n$. (That is[5],

$$(\forall p \in \mathbb{Z})(\forall n \in \mathbb{N})(\forall x_1, \ldots, x_n \in \mathbb{Z}) \\ \Big(p| \prod_{k=1}^n x_k \Longrightarrow (\exists j \in \mathbb{N})(1 \leq j \leq n \wedge p|x_j)\Big).$$

In other words, "if an integer p divides a product of several integers, then it must divide one of those integers".)

Statement VII. Prove or disprove the following statement: If p is a prime number, n is a natural number, and $x_1, \ldots, x_n$ are integers such that the product $\prod_{k=1}^n x_k$ is divisible by p, then x_j is divisible by p for some $j \in \mathbb{N}$

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[3] ***Pay attention to the parentheses!***

[4] Many students seem to think that the vertical bar "|" denotes a fraction. ***It doesn't!***. The expression "3|6" means "3 divides 6", or, if you prefer, "6 is divisible by 3". It is ***not*** the name of the fraction $\frac{3}{6}$. In general, if x, y are integers, then "$x|y$" is a ***statement***, whereas "$\frac{x}{y}$" is the name of a number.

[5] ***Pay attention to the parentheses!***

such that $j \leq n$. (That is[6],

$$(\forall p \in \mathbb{Z})\Big(p \text{ is prime} \Longrightarrow$$
$$(\forall n \in \mathbb{N})(\forall x_1, \ldots, x_n \in \mathbb{Z})\Big(p | \prod_{k=1}^{n} x_k \Longrightarrow (\exists j \in \mathbb{N})(1 \leq n \wedge p | x_j)\Big)\Big)\,.$$

In other words: "if p is prime and divides a product of several integers, then it must divide one of these integers".)

Statement VIII. Prove or disprove the following statement: If n is a natural number, and $x_1, x_2, \ldots, x_n$ are real numbers, then there exist a $j \in \mathbb{N}$ such that $j \leq n$ and $(\forall k \in \mathbb{N})(k \leq n \Longrightarrow x_j \geq x_k)$. (NOTE: This just says that every finite set of real numbers has a maximum, i.e., a member of the set which is greater than or equal to every member of the set.)

Problem 2. In this problem, you are asked to prove or disprove several statements about linear independence of real numbers over the rationals. I am giving you the solutions of the first two, and you are asked to do the other ones.

But first, naturally, I have to give you the definition of "linear independence over $\mathbb{Q}$", and of the related concept of "linear dependence over $\mathbb{Q}$".

Definition of "linear independence over $\mathbb{Q}$***".*** If n is a natural number and $x_1, x_2, \ldots, x_n$ are real numbers, we say that the numbers $x_1, x_2, \ldots, x_n$ are linearly independent over $\mathbb{Q}$ if, whenever $c_1, \ldots, c_n$ are rational numbers such that $c_1x_1 + \cdots + c_nx_n = 0$, it follows that $c_1 = c_2 = \cdots = c_n = 0$. In formal language, $x_1, x_2, \ldots, x_n$ are linearly independent over $\mathbb{Q}$ if and only if

$$(\forall c_1, c_2, \ldots, c_n \in \mathbb{Q})\Big(\sum_{k=1}^{n} c_k x_k = 0 \Longrightarrow c_1 = c_2 = \cdots = c_n = 0\Big)\,.$$

Definition of "linear dependence over $\mathbb{Q}$***".*** If n is a natural number and $x_1, x_2, \ldots, x_n$ are real numbers, we say that the numbers $x_1, x_2, \ldots, x_n$ are linearly dependent over $\mathbb{Q}$ if they are not linearly independent over $\mathbb{Q}$.

Statement I. Prove or disprove the following statement: 1, $\sqrt{2}$, and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$.

Solution. This statement is true. Here is a proof.

[6] ***Pay attention to the parentheses!***

We want to prove that if c_1, c_2, c_3 are rational numbers such that

$$c_1 \times 1 + c_2 \times \sqrt{2} + c_3 \times \sqrt{3} = 0\,, \tag{4}$$

it follows that $c_1 = c_2 = c_3 = 0$.

Let c_1, c_2, c_3 be arbitrary rational numbers.

Assume that (4) holds, that is, that

$$c_1 + c_2\sqrt{2} + c_3\sqrt{3} = 0\,. \tag{5}$$

We want to prove that $c_1 = c_2 = c_3 = 0$.

Equation (5) implies that

$$c_2\sqrt{2} + c_3\sqrt{3} = -c_1\,. \tag{6}$$

Squaring both sides, we get

$$2c_2^2 + 3c_3^2 + 2c_2c_3\sqrt{6} = c_1^2\,. \tag{7}$$

Therefore

$$2c_2c_3\sqrt{6} = c_1^2 - 2c_2^2 - 3c_3^2\,. \tag{8}$$

Now, there are two possibilities: $c_2c_3 \neq 0$, and $c_2c_3 = 0$.

We are going to show, as a first step towards proving our desired conclusion, that the first possibility cannot arise. That is, we are going to show that c_2c_3 must be equal to 0.

Assume that $c_2c_3 \neq 0$.

In this case, we can divide both sides of (8) by $2c_2c_3$, and conclude that

$$\sqrt{6} = \frac{c_1^2 - 2c_2^2 - 3c_3^2}{2c_2c_3}\,. \tag{9}$$

Since c_1, c_2, c_3, 2, and 3 are rational, it follows that c_1^2, $2c_2^2$, $3c_3^2$, and $2c_2c_3$ are rational. Since the sum and difference of two rational numbers is rational, the number $c_1^2 - 2c_2^2 - 3c_3^2$ is rational. And, since the quotient $\frac{x}{y}$ of two rational numbers x, y is rational (as long as $y \neq 0$), the right-hand side of (9) is rational. Hence $\boxed{\sqrt{6} \text{ is rational}}$.

But $\boxed{\sqrt{6} \text{ is irrational}}$. (Proof: Assume that $\sqrt{6} \in \mathbb{Q}$. Then, by the Coprime Representation Theorem[7], we may pick coprime integers m, n such that $n > 0$ and $\sqrt{6} = \frac{m}{n}$. Then $n\sqrt{6} = m$, so $6n^2 = m^2$. Hence m^2 is even. Therefore m is even. So we may write $m = 2k$, for some integer k. Then $m^2 = 4k^2$. So $6n^2 = 4k^2$. Therefore $3n^2 = 2m^2$. Hence $3n^2$ is even, so n is even. But, since m and n are both even, they have the common factor 2, contradicting the fact that they are coprime.)

So we have proved the contradictory facts that $\sqrt{6}$ is rational and $\sqrt{6}$ is irrational. This shows that it is impossible that $c_2c_3 \neq 0$.

So $c_2c_3 = 0$.

It follows from this that $c_2 = 0 \vee c_3 = 0$. (This is a consequence of Theorem[8] 4 on Page 30 of the lecture notes for Lectures 2,3,4, which says that, if the product xy of two real numbers x, y is equal to zero, then one of the numbers x, y must be equal to zero.)

Since $c_2 = 0 \vee c_3 = 0$, we consider separately the cases when $c_2 = 0$ and when $c_3 = 0$.

Suppose that $c_2 = 0$.

Then

(10) $$c_3\sqrt{3} = -c_1 \,.$$

If c_3 was $\neq 0$, we would be able to divide both sides of (10) by c_3, and conclude that

(11) $$\sqrt{3} = -\frac{c_1}{c_3} \,.$$

Since c_1 and c_2 are rational, it follows that $\sqrt{3}$ is rational. But $\sqrt{3}$ is irational. So we have derived a contradiction.

[7]The Coprime Representation Theorem says that if r is a rational number then there exist integers m, n such that (1) $n > 0$, (2) $r = \frac{m}{n}$, and (3) m and n are coprime.

[8]This theorem is extremely important, and we have used it millions of times in proofs in this course. By looking at the first midterm, I can tell that many students do not understand this result and do not know how to prove it. ***You should study this theorem—and its proof—carefully***.

This shows that c_3 cannot be $\neq 0$.

So $c_3 = 0$.

Hence (since we are assuming that $c_2 = 0$), we have shown that $c_2 = c_3 = 0$.

Then (10) implies that $c_1 = 0$ as well.

So we have proved that $c_1 = c_2 = c_3 = 0$, in the case when $c_2 = 0$.

An identical argument works if $c_3 = 0$. So in this case it also follows that $c_1 = c_2 = c_3 = 0$.

So we have proved that $c_1 = c_2 = c_3 = 0$ both when $c_2 = 0$ and when $c_3 = 0$. And, since we know that one of these two possibilities occurs, it follows that $c_1 = c_2 = c_3 = 0$.

We have proved that $c_1 = c_2 = c_3 = 0$ under the assumption that $c_1 \times 1 + c_2 \times \sqrt{2} + c_3\sqrt{3} = 0$ and c_1, c_2, c_3 are arbitrary rational numbers.

Therefore

$$(\forall c_1, c_2, c_3 \in \mathbb{Q})\Big(c_1 \times 1 + c_2 \times \sqrt{2} + c_3\sqrt{3} = 0 \Longrightarrow c_1 = c_2 = c_3 = 0\Big)\,.$$

That is, we have shown that $1, \sqrt{2}, \sqrt{3}$ are linearly independent over $\mathbb{Q}$, which is our desired conclusion. **Q.E.D.**

Statement II. Prove or disprove the following statement: 1, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{2} + \sqrt{3}$ are linearly independent.

Solution. This statement is false. To prove this, we give a counterexample. That is, we exhibit rational numbers c_1, c_2, c_3, c_4 such that

$$c_1 \times 1 + c_2\sqrt{2} + c_3\sqrt{3} + c_4(\sqrt{2} + \sqrt{3}) = 0\,, \tag{12}$$

but

(13) it is not true that $c_1 = c_2 = c_3 = c_4 = 0$.

We take $c_1 = 0$, $c_2 = 1$, $c_3 = 1$, and $c_4 = -1$. Then it is clear that (12) and (13) hold. **Q.E.D.**

Statement III. Prove or disprove the following statement: The real numbers 1, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ are linearly independent over $\mathbb{Q}$.

Statement IV. Prove or disprove the following statement: The real numbers 1, $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt[4]{2}$ are linearly independent over $\mathbb{Q}$.

Statement V. Prove or disprove the following statement: If x, y are real numbers, then x, y are linearly independent over $\mathbb{Q}$ if and only if $x + y$ and $x - y$ are linearly independent over $\mathbb{Q}$.

Statement VI. Prove or disprove the following statement: If x, y, z are real numbers, then x, y, z are linearly independent over $\mathbb{Q}$ if and only if $x + y$, $y + z$ and $x + z$ are linearly independent over $\mathbb{Q}$.

Statement VII. Prove or disprove the following statement: If x, y, z are real numbers, then x, y, z are linearly independent over $\mathbb{Q}$ if and only if $x - y$, $y - z$ and $x - z$ are linearly independent over $\mathbb{Q}$.