MATHEMATICS H311 — FALL 2015

H. J. Sussmann

HOMEWORK ASSIGNMENT NO. 1, DUE ON WEDNESDAY, SEPTEMBER 9

1. Book, Exercise 1.5.4 on Page 30. (HINTS: For Part (a), use the function arc tan, suitably translated and rescaled. For Part (c), remember the "infinite hotel phenomenon": suppose there is an "infinite hotel", with rooms labelled by the natural numbers, so that there is Room 1, Room 2, Room 3, and so on, one room—called Room n—for every natural number n. Suppose that all the rooms are occupied, and then a new guest arrives and you want to find a room for this guest. If the hotel was finite, then you would have to say "sorry, we are full; there is no room for you." But in the case of an infinite hotel you can find a room for the new guest as follows: for each $n \in \mathbb{N}$, move the guest that occupies Room n to Room $n+1$. (So the guest in Room 1 is moved to Room 2, the guest in Room 2 is moved to Room 3, and so on.) Now every one of the guests that was there before still has a room, but Room 1 is free, so you can put the new guest in Room 1.)

2. Prove that if S is an arbitrary set then there does not exist a one-to-one function from $\mathcal{P}(S)$ to S. (Here $\mathcal{P}(S)$ is the power set of S, that is, the set of all subsets of S. In other words, $\mathcal{P}(S) = \{X : X \subseteq S\}$.) HINT: This problem is closely related to Cantor's Theorem, i.e., Theorem 1.6.2 on Page 34 of the book.

3. Exercise 1.4.8 on Page 24 of the book.

NOTE: I promised in class that there was going to be a homework question on proving that the axiom "$0 \neq 1$" cannot be proved from the other axioms of an ordered field. This question is going to appear in the set of homework problems for Week 2.