MATHEMATICS H311 — FALL 2015

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HOMEWORK ASSIGNMENTS NOS. 10 AND 11, DUE ON TUESDAY, NOVEMBER 17 AND TUESDAY, NOVEMBER 24.

This is a long list of problems. You should practice and figure out how to do all these problems[1].

The problems marked "D10" are part of Assignment No. 10, to be handed in on November 17. The problems marked "D11" are part of Assignment No. 11, to be handed in on November 24. The other problems are also very important and you should try to do as many of them as you can.

Problem 1. Book, page 93, Exercise 3.2.1.

Problem 2. Book, page 93, Exercise 3.2.2.

Problem 3. **(D10)** Book, page 93, Exercise 3.2.3.

Problem 4. **(D10)** Book, page 93, Exercise 3.2.4.

Problem 5. Book, page 93, Exercise 3.2.5.

Problem 6. Book, page 93, Exercise 3.2.6.

Problem 7. **(D10)** Book, page 93, Exercise 3.2.7. *Make sure that you understand the difference between the set $\bar{A}$ (the closure of A) and L, the set of limit points of A. Both sets are closed, but they can be quite different. For example, A is necessarily a subset of $\bar{A}$, but it can happen that A is not a subset of L.*

Problem 8. **(D10)** Book, page 93, Exercise 3.2.8.

Problem 9. Book, pages 93-94, Exercise 3.2.9.

Problem 10. Book, page 94, Exercise 3.2.10.

Problem 11. **(D10)** Book, page 94, Exercise 3.2.11.

Problem 12. **(D10)** Book, page 94, Exercise 3.2.12. (NOTE: The statement of this problem contains a typo. You should change the word "divides" into "divide".)

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[1] It's always a good strategy to practice a lot by doing lots and lots of problems. In the particular case of this list, there is an extra reason: each one of these problems has a good chance of appearing in the next midterm exam and/or the final exam.

Problem 13. **(D10)** Book, page 94, Exercise 3.2.13.

Problem 14. **(D10)** Book, page 94, Exercise 3.2.14.

Problem 15. Book, page 94, Exercise 3.2.15.

Problem 16. **(D10)** Book, page 99, Exercise 3.3.2, Parts (a), (b), (d), (e).

Problem 17. Book, page 100, Exercise 3.3.3. *This was done in class. I am putting it in to remind you that you have to know how to do this problem.*

Problem 18. Book, page 100, Exercise 3.3.4.

Problem 19. Book, page 100, Exercise 3.3.5. *I don't like the wording of this problem. The words "The arbitrary" should, in my view, be replaced by "An arbitrary", both in Part (a) and Part (b). Also, Part (a) is unclear, because the intersection of the empty family of compact subsets of* $\mathbb{R}$ *is* $\mathbb{R}$*, which is not compact, but I don't know if this is what the author wanted you to say, or if by "arbitrary family" he meant "arbitrary nonempty family".*

Problem 20. **(D10)** Book, page 100, Exercise 3.3.6. *The word "verify" means "prove". The statement has two typos: the two sentences that begin with the word "which" should have a question mark at the end.*

Problem 21. Book, page 101, Exercise 3.3.8.

Problem 22. **(D11)** Book, page 99, Exercise 3.3.1.

Problem 23. **(D11)** Book, page 101, Exercise 3.3.11.

Problem 24. **(D11)** Book, page 101, Exercise 3.3.13.

Problem 25. **(D11)** *In this problem we discuss the meaning and basic properties of the notion of "continuous function" from a metric space X to a metric space Y.* ***The most important part of this problem is Part I.***

Definition 1. Let X, Y be metric spaces with distance functions d_X, d_Y, and let $f : X \mapsto Y$ be a function. We say that f is <u>continuous</u>[2] if the following is true:

(*) Whenever $(x_n)_{n=1}^{\infty}$ is a sequence of points of X that converges to a point x of X, it follows that $(f(x_n))_{n=1}^{\infty}$ (which is a sequence of points of Y) converges to $f(x)$. $\square$

[2]Can you guess why I have underlined the word "continuous"?

I. **Prove** that if X, Y, Z are metric spaces with distance functions d_X, d_Y, d_Z, and $f : X \mapsto Y$, $g : Y \mapsto Z$ are continous functions, then the composite function $g \circ f : X \mapsto Z$ is continuous.

II. **Prove** that if X is a metric space with distance function d_X, d is a natural number, and $f : X \mapsto \mathbb{R}^d$, $g : Y \mapsto \mathbb{R}^d$ are continuous functions[3], then the sum $f + g$ is a continuous function from X to $\mathbb{R}^d$. (The sum of f and g is the function $f + g$ given by $(f + g)(x) = f(x) + g(x)$ for every $x \in X$.)

III. **Prove** that if X is a metric space with distance function d_X, d is a natural number, and $f : X \mapsto \mathbb{R}^d$, $g : Y \mapsto \mathbb{R}^d$ are continuous functions, then the dot product $f \cdot g$ is a continuous function from X to $\mathbb{R}$. (The dot product of f and g is the function $f \cdot g$ given by $(f \cdot g)(x) = f(x) \cdot g(x)$ for every $x \in X$.)

Problem 26. (D11) *The purpose of this problem is to introduce the notion of "open subset" of a metric space, and to prove several important facts about open sets. The proofs are all very similar (almost identical) to the proofs of the same properties for the case of open subsets of* $\mathbb{R}$. ***All three parts of this problem are very important.***

Definition 2. If X is a metric space with distance d_X, ε is a positive real number, and x is a point of X, the ε-neigborhood of x is the set $V_\varepsilon(x)$ given by

$$V_\varepsilon(x) \stackrel{\text{def}}{=} \{u \in X : d_X(u, x) < \varepsilon\} . \qquad \square$$

Definition 3. If X is a metric space with distance d_X, then a subset U of X is open if for every $x \in U$ there exists a positive real number ε such that $V_\varepsilon(x) \subseteq U$.

The following definition was given earlier, and we repeat it now.

Definition 4. If X is a metric space with distance d_X, then a sequence $(x_n)_{n=1}^\infty$ of points of X converges to a point x of X if $\lim_{n\to\infty} d_X(x_n, x) = 0$. We write $\lim_{n\to\infty} x_n = x$ to indicate that $(x_n)_{n=1}^\infty$ converges to x. $\square$

Definition 5. If X is a set, a topology on X is a set $\mathcal{T}$ such that

[3] Remember that $\mathbb{R}^d$ is a metric space, with the metric that we defined in Homework Assignment No. 9.

1. $\mathcal{T} \subseteq \mathcal{P}(X)$; that is, $\mathcal{T}$ is a set all whose members are subsets of X. (I am using $\mathcal{P}(X)$ to denote the *power set* of X, that is, the set of all subsets of X.)
2. $\emptyset \in \mathcal{T}$.
3. $X \in \mathcal{T}$.
4. Whenever U, V belongs to $\mathcal{T}$, it follows that $U \cap V$ belongs to $\mathcal{T}$.
5. Whenever $(U_i)_{i \in I}$ is an indexed family of members of $\mathcal{T}$, it follows that the set[4] $\bigcup_{i \in I} U_i$ belongs to $\mathcal{T}$. □

Prove that, if X is a metric space with distance d_X, then

1. The set $\mathcal{TOP}(X)$ of all open subsets of X is a topology on X.
2. Whenever x, y are two points of X such that $x \neq y$, there exist open subsets U, V of X such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. (NOTE: This is called the *separation condition*, because it says that any two different points can be separataed by open sets. A topology that has this property is called a Hausdorff topology.)
3. For a sequence $(x_n)_{n=1}^{\infty}$ of points of X and a point x of X, the following three conditions are equivalent[5]:
   (a) $\lim_{n \to \infty} x_n = x$,
   (b) For every positive real number ε, the x_n are eventually[6] in $V_\varepsilon(x)$.
   (c) For every open set U such that $x \in U$, the x_n are eventually in U.

Problem 27. **(D11)** *In Problem 25 we gave a definition of "continuous function" from a metric space X to a metric space Y, based on convergence of sequences. The purpose of this problem is to present another point of*

[4]Recall that $\bigcup_{i \in I} U_i$ is the set $\{x : (\exists i \in I) x \in U_i\}$.

[5]To prove that three conditions (a), (b), (c) are equivalent, you do not need to prove all three equivalences (a)$\Longleftrightarrow$(b), (a)$\Longleftrightarrow$(c), (b)$\Longleftrightarrow$(c), which would amount to proving six implications: (a)$\Longrightarrow$(b), (b)$\Longrightarrow$(a), (b)$\Longrightarrow$(c),(c)$\Longrightarrow$(b), (a)$\Longrightarrow$(c), and (c)$\Longrightarrow$(b). It suffices to prove that (a)$\Longrightarrow$(b), (b)$\Longrightarrow$(c), and (c)$\Longrightarrow$(a), because the other three implications follow from these three.

[6]Remember that the statement "P is eventually true for the x_n" means "there exists $N \in \mathbb{N}$ such that P holds for x_n for all $n \in \mathbb{N}$ such that $n \geq N$". So, in particular, the statement "the x_n are eventually in $V_\varepsilon(x)$" means "there exists $N \in \mathbb{N}$ such that $(\forall n \in \mathbb{N})(n \geq N \Longrightarrow x_n \in V_\varepsilon(x))$".

view, in which continuous functions are characterized as those for which "the preimage of an open set is an open set." ***The most important part of this problem is Part II.***

Definition 6. If A, B are sets, and $f : A \mapsto B$ is a function, then for every subset S of B the preimage of S under f to be the subset $f^{-1}(S)$ given by

$$f^{-1}(S) \stackrel{\text{def}}{=} \{a \in A : f(a) \in S\} .$$

That is, $f^{-1}(S)$ is the set of all points of A that are mapped by f to a point of S. □

I. For each of the following choices of A, B, S, f, **determine** the preimage $f^{-1}(S)$.

1. $A = B = \mathbb{R}$, $S = [0, 1]$, and $f : A \to B$ is the function given by $f(x) = \sin x$.
2. $A = B = \mathbb{R}$, $S = (0, 1)$, and $f : A \to B$ is the function given by $f(x) = \sin x$.
3. $A = \mathbb{R}^2$, $B = \mathbb{R}$, $S = (-\infty, 1)$, and $f : A \to B$ is the function given by $f(x, y) = x^2 + y^2$.
4. $A = \mathbb{R}^2$, $B = \mathbb{R}$, $S = \{0\}$, and $f : A \to B$ is the function given by $f(x, y) = 3x + 2y - 5$.

II. **Prove** that If X, Y are metric spaces with distance functions d_X, d_Y, then a function $f : X \mapsto Y$ is continuous if and only if

(CT) For every open subset U of Y, the preimage $f^{-1}(U)$ is an open subset of X.

Problem 28. (D11) *The purpose of this problem is to present the usual notion of closed set in metric space, as defined in the book*[7]*, and show that it has an equivalent formulation that is "purely topological", that is, makes use of the topology of X and nothing else.* ***The most important part of this problem is Part II.***

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[7]Sure, the book just does it for $\mathbb{R}$, but by now you must have realized that a lot of what the book does for $\mathbb{R}$ can be done in exactly the same way for any metric spaces. ***Warning: it's not all the same. Some things are different. But we haven't encountered any of them yet.***

Definition 7. If X is a metric space with distance function d, the closure of S is the set $\bar{S}$ defined by

$$\bar{S} = \{x \in X : x = \lim_{n\to\infty} x_n \quad \text{for some sequence } (x_n)_{n=1}^{\infty} \text{ of points of } S\}\,.$$

(That is, the closure of S is the set of all points of X that are the limit of a sequence of points of S.) □

Definition 8. If X is a metric space with distance function d, a subset S of X is closed if $\bar{S} = S$. □

Definition 9. If A is a set and B is a subset of A, the complement of B in A is the set $A\backslash B$ defined by

$$A\backslash B = \{x : x \in A \wedge x \notin B\}\,.$$

(That is, $A\backslash B$ is the set of those points of A that are not members of B.)

Usually, when the set A is fixed and we are working with subsets of A, we write B^c (rather than $A\backslash B$) to denote the complement of B in A. □

Prove that, if X is a metric space with distance function d, then

I. If $S \subseteq X$, then $S \subseteq \bar{S}$.

II. If $S \subseteq X$, then S is closed if and only if the complement S^c of S in X is open.

Problem 29. **(D11)**

Definition 10. Let X be a metric space. A subset K of X is compact if every sequence $(x_n)_{m=1}^{\infty}$ of members of K has a subsequence that converges to a member x of K. □

I. **Prove** that if X, Y are metric spaces with distance functions d_X, d_Y, and $f : K \mapsto Y$ is a continuous[8] function, then the set $f(K)$ is compact. (NOTE: $f(K)$ is the set of all points of the form $f(x)$, for $x \in K$. That is, $f(K) = \{y : (\exists x \in K) f(x) = y\}$.)

[8] In case you wonder what a "continuous function from K to X" is, remember that a subset S of a metric space X is itself a metric space, with distance d_S defined by letting $d_S(x, y) = d_X(x, y)$ for $x, y \in X$. So we can regard K as a metric space, and then we know what a "continuous function from K to Y" is.

II. **Prove** in two difrerent ways the *Extreme Value Theorem*: If K is a compact subset of a metric space X and $f : K \mapsto \mathbb{R}$ is continuous, then f has a maximum and a minimum (that is, there exist points M, m in K such that $f(m) \leq f(x) \leq f(M)$ for every $x \in K$). The ***first proof*** should be using sequences, the way we did it in class for the case when K is a bounded closed interval $[a, b]$. The ***second proof*** should be using the result of Part I together with the result of Problem 22 of this problem list.