MATHEMATICS H311 — FALL 2015

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HOMEWORK ASSIGNMENT NO. 6, DUE ON TUESDAY, OCTOBER 13

The following is a list of 5 problems that I am asking you to do.

1. **(TO HAND IN.)** This problem is about the definition of "interval". You have seen what a bounded closed interval is, what a bounded open interval is, what a bounded half-open half-closed interval is, what an open half-line is, and what a closed half-line is. But you probably have not seen a general definition of "interval" that covers all these possibilities. In this problem I give you the general definition, and ask you to prove that this agrees with, and brings together into one unified framework, the definitions of the various kinds of intervals you know.

   The definition of "interval" is as follows:

   DEFINITION. Let I be a subset of the real line $\mathbb{R}$. We say that I is an <u>interval</u> if the following is true:

   (INT) If a, b, c are arbitrary real numbers such that $a < b < c$, and a and c belong to I, it follows that $b \in I$. In other words: if $I \subseteq \mathbb{R}$, then I is an intevral if and only if
   $$(\forall a, b, c \in \mathbb{R})\Big((a < b \wedge b < c \wedge a \in I \wedge c \in I) \Longrightarrow b \in I\Big).$$

   Prove, *using the above definition of "interval"*, that

   (i) The empty set is an interval.

   (ii) If $p \in \mathbb{R}$ then the set $\{p\}$ is an interval.

   (iii) If $p \in \mathbb{R}$, $q \in \mathbb{R}$, and $p < q$, then the sets
   $$\begin{aligned}
   [p, q] &\stackrel{\text{def}}{=} \{x \in \mathbb{R} : p \le x \le q\}, \\
   (p, q) &\stackrel{\text{def}}{=} \{x \in \mathbb{R} : p < x < q\}, \\
   [p, q) &\stackrel{\text{def}}{=} \{x \in \mathbb{R} : p \le x < q\}, \\
   (p, q] &\stackrel{\text{def}}{=} \{x \in \mathbb{R} : p < x \le q\}
   \end{aligned}$$
   are intervals.

(iv) If $p \in \mathbb{R}$ then the sets

$$\begin{aligned}
[p, +\infty) &\stackrel{\text{def}}{=} \{x \in \mathbb{R} : p \le x\}, \\
(p, +\infty) &\stackrel{\text{def}}{=} \{x \in \mathbb{R} : p < x\}, \\
(-\infty, p] &\stackrel{\text{def}}{=} \{x \in \mathbb{R} : x \le p\}, \\
(-\infty, p) &\stackrel{\text{def}}{=} \{x \in \mathbb{R} : x < p\}
\end{aligned}$$

are intervals.

(v) The set $\mathbb{R}$ is an interval.

(vi) If I is an interval then I is necessarily one of the sets listed in Parts (i), (ii), (iii). (iv), (v) above. In other words: if I is an interval then either

$$I = \emptyset\,,$$

or

$$(\exists p \in \mathbb{R})\, I = \{p\}\,,$$

or

$$(\exists p, q \in \mathbb{R})\Big(p < q \wedge \Big(I = [p, q] \vee I = (p, q) \vee I = [p, q) \vee I = (p, q]\Big)\Big)\,,$$

or

$$(\exists p \in \mathbb{R})\Big(I = [p, +\infty) \vee I = (p, +\infty) \vee I = (-\infty, p] \vee I = (-\infty, p)\Big)\,,$$

or

$$I = \mathbb{R}\,.$$

The most important part of this problem is Question (vi).

2. **(TO HAND IN.) Prove** (using the general definition of "interval" given in Problem 1, together with the intermediate value theorem and the extreme value theorem) that

(i) If $I \subseteq \mathbb{R}$, I is an interval, and $f : I \mapsto \mathbb{R}$ is a continuous function, then $f(I)$ is an interval. (That is: "a continuous function maps intervals to intervals".)

(ii) If $I \subseteq \mathbb{R}$, I is a closed bounded interval, and $f : I \mapsto \mathbb{R}$ is a continuous function, then $f(I)$ is a closed bounded interval. (That is: "a continuous function maps closed bounded intervals to closed bounded intervals".)

NOTE:

a. If f is a function and S is a set, then $f(S)$ is "the image of S under f", that is, the set of all values $f(x)$ of f for all $x \in S$. (Formally: $f(S) = \{y : (\exists x \in \text{domain}(f) \cap S) f(x) = y.$)

b. A <u>closed bounded interval</u> is a subset I of $\mathbb{R}$ having the property that $(\exists p, q \in \mathbb{R})\, I = \{x \in \mathbb{R} : p \leq x \leq q\}$.

3. **(TO HAND IN.) Prove or disprove** that

(i) If I is a bounded open interval and $f : I \to \mathbb{R}$ is a continuous function, then $f(I)$ is a bounded set.

(ii) If I is a bounded open interval and $f : I \to \mathbb{R}$ is a continuous function, then $f(I)$ is an open interval.

(iii) If I is a bounded open interval and $f : I \to \mathbb{R}$ is a continuous function, then $f(I)$ cannot be a bounded closed interval.

NOTE:

1. A <u>bounded subset</u> of $\mathbb{R}$ is a subset S of $\mathbb{R}$ such that there exists a real number b such that

$$|x| \leq b \qquad \text{for every} \quad x \in S\,.$$

A <u>bounded open interval</u> is a subset I of $\mathbb{R}$ such that

$$(\exists p, q \in \mathbb{R}) I = \{x \in \mathbb{R} : p < x < q\}\,.$$

An <u>open half-line</u> is a subset I of $\mathbb{R}$ such that

$$(\exists p \in \mathbb{R})\Big(I = \{x \in \mathbb{R} : p < x\} \vee I = \{x \in \mathbb{R} : x < p\}\Big)\,.$$

The <u>full real line</u> is the set $\mathbb{R}$. An <u>open interval</u> is a set which is either an open bounded interval, or an open half-line, or the full real line.

4. **(TO HAND IN.)** In this problem you are allowed to use the following facts (which will be proved in the course very soon):

   (#) If $y \in \mathbb{R}$ and k is an odd natural number, then there exists a unique real number x such that $x^k = y$.

   (##) If $y \in \mathbb{R}$, $y \geq 0$, and k is an even natural number, then there exists a unique real number x such that $x \geq 0$ and $x^k = y$.

   The number x of Statements (#), (##) (which exists and is unique if either k is odd or $y \geq 0$) is called <u>the k-th root of y</u>, and we use $\sqrt[k]{y}$, or $y^{1/k}$, to denote it.

   (i) **Prove** that if k is an odd natural number then the function

   $$\mathbb{R} \ni y \mapsto \sqrt[k]{y}$$

   is continous. (The notation "$\mathbb{R} \ni y \mapsto \sqrt[k]{y}$" has the following meaning: "the function $\mathbb{R} \ni y \mapsto \sqrt[k]{y}$ " means "the function f such that (i) the domain of f is $\mathbb{R}$, and (ii) for any given $y \in \mathbb{R}$, $f(y) = \sqrt[k]{y}$ ". We read "the function $\mathbb{R} \ni y \mapsto \sqrt[k]{y}$ " as follows: "the function that takes any y in $\mathbb{R}$ to the k-th root of y.")

   (ii) **Prove** that if k is an even natural number then the function

   $$[0, +\infty) \ni y \mapsto \sqrt[k]{y}$$

   is continous. (The expression "the function $[0, +\infty) \ni y \mapsto \sqrt[k]{y}$ " is read as follows:

   > the function that takes any y in $[0, \infty)$ to the k-th root of y,

   or

   > the function that takes any nonnegative real number y to the k-th root of y,

   or

   > the function that maps a nonnegative real number y to the k-th root of y.)

   (iii) **Sketch the graphs** of the functions $\mathbb{R} \ni x \mapsto x^k$ for $k = 2$, 3, 4, and 5.

(iv) **Sketch the graphs** of the functions $\mathbb{R} \ni x \mapsto \sqrt[k]{x}$, or $[0, +\infty) \ni x \mapsto \sqrt[k]{x}$, for $k = 2$, 3, 4, and 5.

(v) **Explain** why, in the statements of (i), (ii) above, we cannot allow y to be negative and k even. (So either y has to be nonnegative or k has to be odd.)

5. **(TO HAND IN.)** In this problem you are given three statements of theorems, each one with an alleged proof. The theorems are false, and the proof must therefore be wrong. You are asked to

   I. **Prove** that the theorem is false.

   II. **Show** that the proof is invalid, by finding the mistake or mistakes in it. ***Please do not say "the proof is wrong because the theorem is false". Of course, if the theorem is false, then the proof has to be wrong, but I want to know exactly where the mistake in the proof is.***

   (NOTE: It cannot happen that the theorem is false but the proof is O.K. ***If a proof is correct then the conclusion is true***. Equivalently, ***If the conclusion of a proof is false then the proof has to be incorrect***.)

   Please do not make vague general comments such as "this proof is badly written", or "step 5 is not explained clearly". Give sharp, precise explanations of exactly where the author of the proof made a mistake, (For example, you should say things such as "step 453 is not valid because it goes from '$x < y$' to '$x^2 < y^2$', and this is not permissible in general; for example, if $x = -2$ and $y = -1$, then $x < y$ but it is not true that $x^2 < y^2$)".

   THEOREM-PROOF 1:

   THEOREM. *The number* 1 *is the largest natural number.*

   PROOF. Let n be the largest natural number.

   Then n^2 is a natural number, because the product of two natural numbers is a natural number.

   So $n^2 \leq n$, because n is the largest natural number, so any natural number must be $\leq n$, and in particular n^2 must be $\leq n$.

On the other hand, $n \geq 1$, because n is a natural number. So multipying both sides by n we get $n^2 \geq n$.

Since $n^2 \geq n$ and $n^2 \leq n$, it follows that $n^2 = n$.

Therefore $n^2 - n = 0$.

So $n(n-1) = 0$.

Since the product of two real numbers is zero only when one the numbers is equal to zero, we conclude that $n = 0$ or $n - 1 = 0$.

But $n = 0$ is impossible, because n is a natural number and 0 is not.

So $n - 1 = 0$.

Hence $n = 1$

Therefore $\boxed{\text{1 is the largest natural number}}$. Q.E.D.

THEOREM-PROOF 2:

THEOREM. $0 = 1$.

PROOF. Let S be the series $\sum_{n=1}^{\infty}(-1)^{n+1}$, so

$$S = 1 + (-1) + 1 + (-1) + (-1) + \cdots .$$

We rewrite S as

$$S = \Big(1 + (-1)\Big) + \Big(1 + (-1)\Big) + \Big(1 + (-1)\Big) + \cdots ,$$

and find that

$$S = 0 + 0 + 0 + \cdots ,$$

so $S = 0$.

Next we rewrite S as

$$S = 1 + \Big((-1) + 1\Big) + \Big((-1) + 1\Big) + \Big((-1) + 1\Big) + \cdots ,$$

and find that

$$S = 1 + 0 + 0 + 0 + \cdots ,$$

so $S = 1$.

Hence $S = 0$ and $S = 1$. So $\boxed{0 = 1}$. Q.E.D.

THEOREM-PROOF 3:

THEOREM. If x is a real number and k is a natural number then there exists a real number u such that $u^k = x$. In other words:

$$(\forall x \in \mathbb{R})(\forall k \in \mathbb{N})(\exists u \in \mathbb{R})u^k = x\,.$$

NOTE: ***The proof that follows is absolutely perfect. It is so well written, so clear and rigorous and detailed, so impeccably correct, that you should use it as a model of how to write proofs. It truly deserves almost full credit, because it's so beautiful. Unfortunately, the proof has one invalid step (just one, only one) and that suffices to kill the whole proof, and one ends up "proving" a false result. So, for that reason, if a student wrote a proof like this, I would give it a zero[1]. I would do so with tears in my eyes, but I would do it, because for proofs there has to be no partial credit: one wrong step suffices to prove any false statement you want, so proofs with one mistake are totally useless. Believe me, I would regret doing this as much as Donald Trump regrets firing people, but I would do it, because I have to.***

PROOF. Let x be an arbitrary real number.

Let k be an arbitrary natural number.

We want to prove that $(\exists u \in \mathbb{R})u^k = x$.

For this purpose, we first define a subset A of $\mathbb{R}$ as follows:

$$A = \{t \in \mathbb{R} : t^k \leq x\}\,.$$

(That is, A is the set of all real numbers t such that $t^k \leq x$.)

We will prove that A is bounded above.

[1] A proof is like a chain. One weak link sufices to make the chain break and become useless. If a chain is holding a piano right above your head, and one link breaks and the piano falls on your head and kills you, then you don't give the chain-maker partial credit, on the grounds that "after all, only one link broke; all the other links were fine, so the chain was almost perfect".

Let b be the largest of 1 and x (or, if you prefer, we could take b to be $1 + |x|$).

We prove that b is an upper bound for A.

We have to prove that $(\forall t \in A)\, t \leq b$.

Let t be an arbitrary member of A.

Then either $t \leq 1$ or $t > 1$.

If $t \leq 1$ then $t \leq b$, because $1 \leq b$.

If $t > 1$ then $t \leq t^k$, and $t^k \leq x$, because $t \in A$, so $t \leq x$, and then $t \leq b$, because $x \leq b$.

So $t \leq b$.

We have proved that $t \leq b$ if t is an arbitrary member of A.

So b is an upper bound for A.

So the set A is bounded above.

Since A is bounded above, it follows from the completeness axiom that A has a least upper bound.

Let u be the least upper bound of A.

We now prove that $u^k = x$.

For ths purpose, we will prove that $u^k \geq x$ and $u^k \leq x$.

Proof that $u^k \geq x$:

We do it by contradiction.

Suppose that $u^k < x$.

We know that $\lim_{n\to\infty}(u + \frac{1}{n}) = u$.

Then by Theorem 2.3.3 in the book, we can conclude that

$$\lim_{n\to\infty} \left(u + \frac{1}{n}\right)^k = u^k\,. \tag{0.1}$$

If we pick $\varepsilon = x - u^k$, then $\varepsilon > 0$ (because we are assuming that $u^k < x$).

It then follows from (0.1) that we may pick a natural number N such that $|(u + \frac{1}{n})^k - u^k| < \varepsilon$ for every $n \in \mathbb{N}$ such that $n \geq N$.

In particular, we may apply this with $n = N$, and conclude that

$$\left|\left(u + \frac{1}{N}\right)^k - u^k\right| < \varepsilon\,.$$

Hence

$$\left(u + \frac{1}{N}\right)^k - u^k < \varepsilon\,,$$

because $\alpha \leq |\alpha|$ for every $\alpha \in \mathbb{R}$.

Therefore

$$\begin{aligned}\left(u + \frac{1}{N}\right)^k &= \left((u + \frac{1}{N})^k - u^k\right) + u^k \\ &\leq \varepsilon + u^k \\ &= x - u^k + u^k \\ &= x\,.\end{aligned}$$

So

$$(u + \frac{1}{N})^k \leq x\,.$$

Therefore $u + \frac{1}{N}$ belongs to A.

But u is the least upper bound of A, so in particular u is an upper bound for A, that is, $(\forall v \in A) v \leq u$.

Since $u + \frac{1}{N} \in A$, we can conclude that

$$u + \frac{1}{N} \leq u\,.$$

On the other hand, $u + \frac{1}{N} > u$, because $\frac{1}{N} > 0$.

So we have proved that

$$u + \frac{1}{N} > u \wedge u + \frac{1}{N} \leq u\,. \tag{0.2}$$

Clearly, (0.2) is a contradiction.

So we have derived a contradiction, assuming that $u^k < x$. It follows that $\boxed{u^k \geq x}$.

Proof that $u^k \leq x$:.

Fix a natural number n.

We know that u is the least upper bound of A.

Then the number $u - \frac{1}{n}$ cannot be an uppen bound of A, because the smallest upper bound of A is u.

So there exists a member v_n of A such that $v_n > u - \frac{1}{n}$.

On the other hand, $v_n \leq u$, becaunse u is an upper bound for A and $v_n \in A$.

And $v_n^k \leq x$, because $v_n \in A$.

So we have found, for each $n \in \mathbb{N}$, a real number v_n such that

$$u - \frac{1}{n} < v_n \leq u\,, \quad \text{and} \quad v_n^k \leq x\,.$$

Since $\lim_{n\to\infty} u - \frac{1}{n} = u$, $\lim_{n\to\infty} u = u$, and $u - \frac{1}{n} < v_n \leq u$, it follows from the squeeze theorem that $\lim_{n\to\infty} v_n = u$.

Therefore $\lim_{n\to\infty} v_n^k = u^k$.

But $v_n^k \leq x$ for every n.

Therefore $\lim_{n\to\infty} v_n^k \leq x$.

But $\lim_{n\to\infty} v_n^k = u^k$.

So $\boxed{u^k \leq x}$.

So we have proved that $u^k \geq x$ and $u^k \leq x$. Therefore $u^k = x$.

This proves that $\boxed{\text{there exists a } u \in \mathbb{R} \text{ such that } u^k = x}$, which is our desired conclusion. Q.E.D.