MATHEMATICS H311 — FALL 2015

H. J. Sussmann

HOMEWORK ASSIGNMENT NO. 7, DUE ON FRIDAY, OCTOBER 23

The following is a list of 4 problems. In each problem, it is indicated which parts you are asked to hand in.

Problem 1. **(TO HAND IN)** ***(NOTE: PART III IS THE HARDEST AND MOST CHALLENGING. Try to do it, but do not worry too much if you cannot figure it out.)***

A sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ of real numbers is

1. summable, if the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent,
2. square-summable, if the series $\sum_{n=1}^{\infty} x_n^2$ is convergent, (NOTE: Here I could equally well have said "absolutely convergent". Since $\sum_{n=1}^{\infty} x_n^2$ is a series of nonnegative terms, the series is convergent if and only if it it absolutely convergent,)
3. cube-summable, if the series $\sum_{n=1}^{\infty} x_n^3$ is absolutely convergent.

(For example: the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is square-summable but not summable; the sequence $\left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty}$ is cube-summable but not square-summable.)

The sum of two sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}$, $\mathbf{y} = (y_n)_{n=1}^{\infty}$, is the sequence $\mathbf{x} + \mathbf{y}$ given by $\mathbf{x} + \mathbf{y} = (x_n + y_n)_{n=1}^{\infty}$.

The product of two sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}$, $\mathbf{y} = (y_n)_{n=1}^{\infty}$, is the sequence $\mathbf{x} \cdot \mathbf{y}$ given by $\mathbf{x} \cdot \mathbf{y} = (x_n \cdot y_n)_{n=1}^{\infty}$.

I. **Prove** each of the following statements:

1. The sum of two square-summable sequences is square-summable.
2. The sum of two cube-summable sequences is cube-summable.
3. The product of two square-summable sequences is summable.
4. The product of a summable sequence and a bounded sequence is summable.

II. **Prove or disprove** each of the following statements:

1. The product of two cube-summable sequences is square-summable.

2. The product of two cube-summable sequences is square-summable.
3. The product of two cube-summable sequences is square-summable.
4. The product of two cube-summable sequences is square-summable.
5. The product two cube-summable sequences is cube-summable.
6. The product two cube-summable sequences is square-summable.
7. The product two cube-summable sequences is summable.

III. **Prove** that

1. If a sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ has a the property that for every summable sequence $\mathbf{y} = (y_n)_{n=1}^{\infty}$ the product sequence $\mathbf{x} \cdot \mathbf{y}$ is summable, then $\mathbf{x}$ is bounded. (HINT: Suppose $\mathbf{x}$ is unbounded. Produce a contradiction by finding a summable sequence $\mathbf{y} = (y_n)_{n=1}^{\infty}$ such that $\mathbf{x} \cdot \mathbf{y}$ is not summable. For this purpose, find a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $\mathbf{x}$ such that the numbers x_{n_k} are very large (say larger than k, or than k^2, something of that sort). Then construct the y_n by letting each y_{n_k} be equal to x_{n_k} divided by some power $|x_{n_k}|^p$. And let $y_n = 0$ if n is not one of the n_k. Figure out how to choose p so that the sequence $\mathbf{y}$ is summable but the product sequence $\mathbf{x} \cdot \mathbf{y}$ is not.)
2. If a sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ has the property that for every square-summable sequence $\mathbf{y} = (y_n)_{n=1}^{\infty}$ the product sequence $\mathbf{x} \cdot \mathbf{y}$ is summable, then $\mathbf{x}$ is square-summable. (This could be hard, and this times I am not giving you a hint. Do it if you can but, as I said before, don't worry if you cannot figure it out.)

Problem 2. This problem is about the "extended real line". We choose two objects that are not real numbers and are not equal to each other, and call them "plus infinity" and "minus infinity". We use the symbols $+\infty$ and $-\infty$ for these two new objects. We then form a new set, called the <u>extended real line</u>, by addding $+\infty$ and $-\infty$ to the real line $\mathbb{R}$. We use the symbol $\overline{\mathbb{R}}$ for the extended real line, so

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}\,.$$

Having introduced the new objects $+\infty$ and $-\infty$, we give some definitions that tell us how these new objects are used[1].

Definition. Let $(x_n)_{n=1}^{\infty}$ be a sequence of. real numbers.

1. We say that $(x_n)_{n=1}^{\infty}$ goes to plus infinity, and write
$$\lim_{n\to\infty} x_n = +\infty\,,$$
if for every real number M there exists a natural number N such that
$$(\forall n)\Big(n \geq N \Longrightarrow x_n \geq M)\,.$$

2. We say that $(x_n)_{n=1}^{\infty}$ goes to minus infinity, and write
$$\lim_{n\to\infty} x_n = -\infty\,,$$
if for every real number M there exists a natural number N such that
$$(\forall n)\Big(n \geq N \Longrightarrow x_n \leq M)\,.$$

3. We say that the sum of the series $\sum_{n=1}^{\infty} x_n$ is plus infinity, and write
$$\sum_{n=1}^{\infty} x_n = +\infty\,,$$
if the sequence $(S_n)_{n=1}^{\infty}$ of partial sums of the series (where the S_n are defined by letting $S_n = \sum_{k=1}^{n} x_k$) goes to $+\infty$, i.e., if for every real number M there exists a natural number N such that
$$(\forall n)\Big(n \geq N \Longrightarrow \sum_{k=1}^{n} x_k \geq M)\,.$$

[1] In case you ask "who are $+\infty$ and $-\infty$?", the answer is "it does not matter". Plus infinity could be the set $\mathbb{R}$ itself, or the ordered pair $(2,\pi)$, or George Washington, or the planet Mars. And $-\infty$ could be the triple $(6,5,4)$, or my uncle Jimmy. All that matters is that $+\infty$ and $-\infty$ should not be real numbers, and that they should not be equal to each other. They are ***labels*** that we will use for certain things. For example, later we will agree to say, about certain sequences such as $(n)_{n=1}^{\infty}$, that those sequences "go to plus infinity", or that "the limit of the sequence is plus infinity. But this does ***not*** mean that such sequences converge, or that they have a limit. The sequence $(n)_{n=1}^{\infty}$ is *divergent*, not convergent, and it does not have a limit. And when we write "$\lim_{n\to\infty} = +\infty$", this does *not* mean that the sequence converges, or that it has a limit. Saying of a sequence that it goes to infinity does ***not*** mean that the sequence has a limit and that the limit is $+\infty$, any more than saying "I believe in nothing" means that "I believe in something and the thing I believe in is called 'nothing'".

4. We say that <u>the sum of the series</u> $\sum_{n=1}^{\infty} x_n$ <u>is minus infinity</u>, and write

$$\sum_{n=1}^{\infty} x_n = -\infty \,,$$

if the sequence $(S_n)_{n=1}^{\infty}$ of partial sums of the series goes to $-\infty$, i.e., if for every real number M there exists a natural number N such that

$$(\forall n)\Big(n \geq N \Longrightarrow \sum_{k=1}^{n} x_k \leq M\Big)\,.$$

And we add the following remark, as a clarification: the definition of what it means for a sequence or a series to <u>converge</u> is unchanged. A sequence $(x_n)_{n=1}^{\infty}$ <u>converges</u> if there exists a real number[2] a such that $\lim_{n\to\infty} x_n = a$. The sequence $(x_n)_{n=1}^{\infty}$ <u>diverges</u> if it does not converge. It follows that if $\lim_{n\to\infty} x_n = +\infty$ or $\lim_{n\to\infty} x_n = -\infty$ then the sequence (x_n) *diverges.*

Now we are ready to start proving things:

I. **(TO HAND IN) PROVE** that $\lim_{n\to\infty} \sqrt{n} = +\infty$.

IIa. **(TO HAND IN)** For each of the following sequences **indicate** whether

(a) the sequence converges,

(b) the sequence goes to plus infinity,

(c) the sequence goes to minus infinity,

(d) none of (a), (b), (c) holds,

and explain why.

1. $\Big(\frac{\sqrt{n}+1}{\sqrt[3]{n}+5}\Big)\Big)_{n=1}^{\infty}$,
2. $(n-n^2)_{n=1}^{\infty}$,
3. $\Big(\frac{\sqrt[3]{n}+1}{\sqrt[3]{n}+5}\Big)\Big)_{n=1}^{\infty}$,
4. $\Big(\frac{\sqrt[3]{n}+1}{\sqrt{n}+5}\Big)\Big)_{n=1}^{\infty}$,
5. $\Big(\frac{n^n}{n!}\Big)\Big)_{n=1}^{\infty}$,
6. $\Big(\frac{n!}{n^n}\Big)\Big)_{n=1}^{\infty}$.

[2] And I *do* mean a real number; not $+\infty$ or $-\infty$.

IIb. **(TO HAND IN)** For each of the following series indicate whether

(a) the series converges,
(b) the sum of the series is plus infinity,
(c) the sum of the series is to minus infinity,
(d) none of (a), (b), (c) holds.

and explain why.

1. $\sum_{n=1}^{\infty} \frac{1}{n}$,
2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$,
3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$,
4. $\sum_{n=1}^{\infty} (\frac{1}{n} - n)$,
5. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$,
6. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$,
7. $\sum_{n=1}^{\infty} 2^{-n} n^3$,
8. $\sum_{n=1}^{\infty} 2^{-n} n^n$,
9. $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$.

III. **(TO HAND IN) PROVE** that

1. if (x_n) is an increasing sequence of real numbers, then either the sequence converges or $\lim_{n \to \infty} x_n = +\infty$,
2. if (x_n) is a decreasing sequence of real numbers, then either the sequence converges or $\lim_{n \to \infty} x_n = -\infty$.

IV. **CONCLUDE** from the results of Part III that

1. if $\sum_{n=1}^{\infty} x_n$ is an series of nonnegative real numbers, then either the series converges or $\sum_{n=1}^{\infty} x_n = +\infty$,
2. if $\sum_{n=1}^{\infty} x_n$ is an series of nonpositive real numbers, then either the series converges or $\sum_{n=1}^{\infty} x_n = -\infty$,
3. if the terms of a series $\sum_{n=1}^{\infty} x_n$ are eventually[3] nonnegative then either the series converges or $\sum_{n=1}^{\infty} x_n = +\infty$,
4. if the terms of a series $\sum_{n=1}^{\infty} x_n$ are eventually nonpositive then either the series converges or $\sum_{n=1}^{\infty} x_n = -\infty$.

[3] We say that something happens eventually if it happens for all n greater than some N. Precisely: a predicate $P(n)$ holds eventually if $(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n \geq N \Longrightarrow P(n))$.

V. **PROVE** that if the terms x_n of a sequence are eventually positive[4] then $\lim_{n\to\infty} x_n = +\infty$ if and only if $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

Problem 3. This problem continues the discussion of the extended real line that was started in Problem 2. Now that we have created the extended real line, we would like to extend to it the order relation $<$. So we would like to say what "$x < y$" means for two extended real numbers x, y. The answer is quite simple:

Definition. Let x, y be extended real numbers. We say that x is smaller than y, or less than y, and write $x < y$, if either

1. $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$,

or

2. $x \in \mathbb{R}$ and $y = +\infty$,

or

3. $x = -\infty$ and $y \in \mathbb{R}$,

or

4. $x = -\infty$ and $y = +\infty$.

We then define the relations $\leq$, $>$, $\geq$, in the following way: "$x \leq y$" means "$x < y$ or $x = y$", "$x > y$" means "$y < x$", and "$x \geq y$" means "$x > y$ or $x = y$".

Armed with these definitions, I want you to prove a couple of completely trivial things:

I. **Prove** that the new order relation satisfies the *transitive law*, the *trichotomy law*, and the *irreflexivity law*:

$$\begin{aligned}
&(\forall x, y, z \in \overline{\mathbb{R}})\Big((x < y \wedge y < z) \Longrightarrow x < z\Big)\\
&(\forall x, y \in \overline{\mathbb{R}})(x < y \vee x = y \vee y < x)\\
&(\forall x, y \in \overline{\mathbb{R}})(x < y \Longrightarrow x \neq y))\,.
\end{aligned}$$

II. **(TO HAND IN) Prove** that every subset A of $\overline{\mathbb{R}}$ has a least upper bound[5].

---

[4] "Positive" means "> 0".

[5] The precise definitions are as follows: An upper bound in $\overline{\mathbb{R}}$ of a subset A of $\overline{\mathbb{R}}$ is a member u of $\overline{\mathbb{R}}$ such that $(\forall a \in A)a \leq u$. A least upper bound in $\overline{\mathbb{R}}$ of a subset A of $\overline{\mathbb{R}}$ is a member s of $\overline{\mathbb{R}}$ such that $s \leq u$ for every upper bound u of A in $\overline{\mathbb{R}}$. The definitions of "lower bound" and "greatest lower bound" are similar.

III. **(TO HAND IN) Prove**, in particular, that the least upper bound of the empty set $\emptyset$ is $-\infty$ and the greatest lower bound of $\emptyset$ is $+\infty$.

IV. **(TO HAND IN) Prove** that a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if and only if $\sum_{n=1}^{\infty} |x_n| < +\infty$.

Problem 4. This problem continues the discussion of the extended real line that was started in Problem 2 and pursued in Problem 3. Our next step is to try to extend to $\overline{\mathbb{R}}$ the operations of addition, subtraction, multiplication, and division. So we would like to say who "$x+y$", "$x-y$", "$x \cdot y$" and "$x \div y$" are, for any two extended real numbers x, y. The answer is not as simple as for the order relation.

First, let us clarify what it is that we want to achieve. Let # be a binary operation. (For example, # could be $+$, or $-$, or $\cdot$, or $\div$.) If it was just a matter of assigning a value to $x\#y$, for the cases when we do not already know what those values are[6], then we could, for example, decree that $5+(+\infty)$ is 73, or some other crazy thing like that. What we really want is *to define # on the extended real line in such a way that the algebraic limit theorems are still valid.* For example, we want it to be true if a, b are extended real numbers, that

(A1) *If* $\mathbf{x} = (x_n)_{n=1}^{\infty}$, $\mathbf{y} = (y_n)_{n=1}^{\infty}$ *are sequences of real numbers such that* $\lim_{n\to\infty} x_n = a$ *and* $\lim_{n\to\infty} y_n = b$*, then* $\lim_{n\to\infty}(x_n + y_n) = a + b$.

Can this be done? The answer is that we can make (A1) true if a and b are ordinary real numbers (as we already know), and also if one of the two is an ordinary real number and the other one is $+\infty$ or $-\infty$, or if both are $+\infty$, or if both are $-\infty$. But it cannot be done if one of them is $+\infty$ and the other one is $-\infty$. More succintly:

(A1a) *The sum* $a+b$ *of two extended real numbers is well defined except when one of the numbers is* $+\infty$ *and the other one is* $-\infty$.

And now that I have told you what the answer is, I am asking you to prove it:

I. **Prove** (A1a). For this purpose, you have to prove that

1. If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers that converges to a real number a, and $(y_n)_{n=1}^{\infty}$ is a sequence of real numbers such that $\lim_{n\to\infty} y_n = +\infty$, then $\lim_{n\to\infty}(x_n + y_n) = +\infty$.

[6] We know that $x+y$, $x-y$ and $x \cdot y$ are well defined for every pair of real numbers x, y, and that $\frac{x}{y}$ is well defined if $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $y \neq 0$.

2. If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers that converges to a real number a, and $(y_n)_{n=1}^{\infty}$ is a sequence of real numbers such that $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty}(x_n + y_n) = -\infty$.
3. If $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences of real numbers such that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = +\infty$, then it follows that $\lim_{n\to\infty}(x_n + y_n) = +\infty$.
4. If $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences of real numbers such that $\lim_{n\to\infty} x_n = -\infty$ and $\lim_{n\to\infty} y_n = -\infty$, then it follows that $\lim_{n\to\infty}(x_n + y_n) = -\infty$.
5. There does not exist an extended real number a such that
   (*) If $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences of real numbers such that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = -\infty$, then it follows that $\lim_{n\to\infty}(x_n + y_n) = a$.

Once you have proved these five statements, it will follow that one can define $a + b$ as follows:

$a + b = +\infty$ if $a \in \mathbb{R} \wedge b = +\infty$ or if $a = +\infty \wedge b =\in \mathbb{R}$;

$a + b = -\infty$ if $a \in \mathbb{R} \wedge b = -\infty$ or if $a = -\infty \wedge b =\in \mathbb{R}$;

$(+\infty) + (+\infty) = +\infty$;

$(-\infty) + (-\infty) = -\infty$;

$(+\infty) + (-\infty)$ and $(-\infty) + (\infty)$ are not well defined.

II. **Do exactly the same** as we did in Part I for the subtraction operation. That is, determine for which pairs of extended real numbers a, b the difference $a - b$ is well defined. (NOTE: This is very easy, because it's almost the same as for the addition operation.)

III. **(TO HAND IN) Do exactly the same** as we did in Part I for the multiplication operation. That is, determine for which pairs of extended real numbers a, b the product $a \cdot b$ is well defined. (NOTE: This is trickier. For example, the products $0 \cdot (+\infty)$ and $0 \cdot (-\infty)$ are not well defined.)

IV. **(TO HAND IN) Do exactly the same** as we did in Part I for the division operation. That is, determine for which pairs of extended real numbers a, b the quotient $a \div b$ is well defined. (NOTE: This is also tricky. For example, the quotients $0 \div 0$ and $(+\infty) \div (+\infty)$ are not well defined.