MATHEMATICS H311 — FALL 2015

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HOMEWORK ASSIGNMENT NO. 8, DUE ON FRIDAY, OCTOBER 30

The following is a list of 5 problems for you to hand in.

Problem 1. A period of a function $f : \mathbb{R} \mapsto \mathbb{R}$ is a real number p such that

$$f(x+p) = f(x) \qquad \text{for every} \quad x \in \mathbb{R}\,.$$

A function $f : \mathbb{R} \mapsto \mathbb{R}$ is periodic if there exists a nonzero real number p which is a period of f. (NOTE: 0 is, trivially, a period of any function $f : \mathbb{R} \mapsto \mathbb{R}$. So if we didn't require p to be nonzero it would follow that every function is periodic.)

Prove the following statements:

I. A continuous periodic function $f : \mathbb{R} \mapsto \mathbb{R}$ has a maximum and a minimum.

II. If $f : \mathbb{R} \mapsto \mathbb{R}$ is a nonconstant continuous periodic function then

1. f has a smallest positive period (that is, there exists a real number p such that (i) $p > 0$, (ii) p is a period of f, and (iii) if q is any period of f such that $q > 0$, then $q \geq p$). NOTE: If P is the set of all positive periods of f, then you must prove that $P \neq \emptyset$ (which is very easy), that the set P is bounded below (also very easy), that the infimum p of P is itself a period (which is also easy) and, finally, that $p > 0$. (This is the hard part. You will need some work to do it.)
2. If p is the smallest positive period of f, then every period of f is an integer multiple of p. (That is, if q is a period then there exists an integer n such that $q = np$.)

III. There exists a periodic function $f : \mathbb{R} \mapsto \mathbb{R}$ whose set of positive periods is nonempty but does not have a smallest member. (Naturally, the function f cannot be continuous.)

Problem 2. If S is a set, a metric (or distance) on S is a function $d : S \times S \mapsto \mathbb{R}$ such that

1. *(nonnegativity)* $d(x, y) \geq 0$ for every ordered pair $(x, y) \in S \times S$;

2. *(symmetry)* $d(x,y) = d(y,x)$ for every ordered pair $(x,y) \in S \times S$;
3. *(the triangle inequality)* $d(x,z) \leq d(x,y) + d(y,z)$ for every triple (x,y,z) of members of S.
4. *(separation)* If x, y are any two members of S such that $x \neq y$, then $d(x,y) > 0$.

DEFINITION. A metric space is a set S equipped with a metric $d : S \times S \mapsto \mathbb{R}$.

Prove the following statements

I. If we take S to be $\mathbb{R}$, the set of real numbers, and define

$$d(x,y) = |x - y| \qquad \text{for} \quad x, y \in \mathbb{R}\,,$$

then d is a metric on S, so S, equipped with d, is a metric space.

II. If S is any metric space, with metric d, and S' is any subset of S, then S' is a metric space, with metric $d' : S' \times S' \mapsto \mathbb{R}$ defined by[1]

$$d'(x,y) = d(x,y) \qquad \text{for} \quad x, y \in S'\,.$$

Problem 3. In this problem we discuss how to turn the two-dimensional plane $\mathbb{R}^2$ into a metric space.

Recall that the symbol $\mathbb{R}^2$ denotes the set of all ordered pairs $\vec{v} = (x,y)$ of real numbers. The set $\mathbb{R}^2$ is called the plane, or the Euclidean plane, or, sometimes, two-dimensional space.

The members (points) or $\mathbb{R}^2$ are two-dimensional vectors.

The sum of two vectors $\vec{v}_1 = (x_1, y_1)$, $\vec{v}_2 = (x_2, y_2)$, is the vector $\vec{v}_1 + \vec{v}_2$ given by

$$\vec{v}_1 + \vec{v}_2 \stackrel{\text{def}}{=} (x_1 + x_2, y_1 + y_2)\,.$$

The dot product (or scalar product, or inner product) of two vectors $\vec{v}_1 = (x_1, y_1)$, $\vec{v}_2 = (x_2, y_2)$, is the real number $\vec{v}_1 \cdot \vec{v}_2$ given by

$$\vec{v}_1 \cdot \vec{v}_2 \stackrel{\text{def}}{=} x_1 y_1 + x_2 y_2\,.$$

The square length of a vector $\vec{v} = (x,y)$ is the real number $\|\vec{v}\|^2$ given by

$$|\vec{v}\|^2 \stackrel{\text{def}}{=} \vec{v} \cdot \vec{v}\,.$$

[1] The metric d' is called the restriction of d to S'

That is, $\|\vec{v}\|^2$ is the dot product of $\vec{v}$ with itself. Hence

$$|\vec{v}\|^2 = x^2 + y^2 \quad \text{if} \quad \vec{v} = (x, y)\,.$$

The length of a vector $\vec{v}$ is the square root of the square length of $\vec{v}$. We write $\|\vec{v}\|$ to denote the length of $\vec{v}$. It follows that

$$\|\vec{v}\| = \sqrt{x^2 + y^2} \quad \text{if} \quad \vec{v} = (x, y)\,.$$

The distance between two vectors $\vec{v}, \vec{w} \in \mathbb{R}^2$ is the number $d(\vec{v}, \vec{w})$ given by

$$d(\vec{v}, \vec{w}) \stackrel{\text{def}}{=} \|\vec{v} - \vec{w}\|\,.$$

Prove that d is a metric on $\mathbb{R}^2$, so $\mathbb{R}^2$, equipped with this metric, is a metric space. (NOTE: The hard part is proving the triangle inequality. This requires some work. You should first prove that

$$\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| \quad \text{for all} \quad \vec{v}, \vec{w} \in \mathbb{R}^2\,. \tag{0.1}$$

(Once you have this, the triangle inequality for the metric d follows easily).

And to prove (0.1) you can do it by brute force, just writing everything out, or by the following ingenious trick: let $p(r) = (\vec{v} + r\vec{w}) \cdot (\vec{v} + r\vec{w})$ for $r \in \mathbb{R}$; observe that $p(r) \ge 0$ for every real number r, write out $p(r)$ as a polynomial $ar^2 + br + c$, and use the necessary and sufficient condition for such a polynomial never to have negative values. (The condition is: $a > 0$ and $4ac \le b^2$.) This will give you the **Cauchy-Schwarz inequality**[2]:

$$\vec{v} \cdot \vec{w} \le \|\vec{v}\| \cdot \|\vec{w}\|\,.$$

(This is analogous to the one-dimensional inequality $a \cdot b \le |a| \cdot |b|$ that I am sure you know very well.)

And, finally, once you have proved the Cauchy-Schwarz inequality, the triangle inequality for the metric follows easily.

Problem 4. By now, you already know two examples of metric spaces: the real line $\mathbb{R}$ (with metric d defined by $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$), and two-dimensional space $\mathbb{R}^2$. This means that in fact you already know infinitely many metric spaces, because every subset of a metric space is a metric space, so every subset of $\mathbb{R}$ and every subset of $\mathbb{R}^2$ is a metric space.

In this problem you are asked to prove some theorems about metric spaces. But, as usual, we first need some definitions.

[2]Known in Russia as "the Bunyakovsky inequality", or "the Cauchy-Schwarz-Bunyakovsky inequality".

DEFINITION. Let S be a metric space with metric d. Then

1. A sequence $(s_n)_{n=1}^{\infty}$ of points of S converges to a point $s \in S$ if $\lim_{n\to\infty} d(s_n, s) = 0$. We write

$$\lim_{n\to\infty} s_n = s$$

   to indicate that $(s_n)_{n=1}^{\infty}$ converges to s.

2. A sequence $(s_n)_{n=1}^{\infty}$ of points of S is a Cauchy sequence if

$$(\forall \varepsilon \in \mathbb{R})\Big(\varepsilon > 0 \quad \Longrightarrow$$
$$(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(\forall m \in \mathbb{N})((n \geq N \wedge m \geq N) \Longrightarrow d(s_n, s_m) < \varepsilon)\Big) .$$

Prove the following statements:

1. Every convergent sequence is Cauchy.
2. Every subsequence of a convergent sequence converges to the same limit as the sequence.
3. If $\mathbf{s}$ is a sequence of members of S, and $s \in S$ is such that every subsequence of $\mathbf{s}$ has a subsequence that converges to s, then $\mathbf{s}$ converges to s.

Problem 5. **Prove or disprove** each of the following statements:

1. If S is a metric space with metric d, then every Cauchy sequence of members of S converges.
2. If S is a metric space with metric d, then if $\mathbf{s}$ is a Cauchy sequence that has a convergent subsequence it follows that $\mathbf{s}$ itself is convergent.
3. If S is a metric space with metric d, then if $\mathbf{s}$ is a sequence of members of S such that every subsequence of $\mathbf{s}$ has a convergent subsequence it follows that $\mathbf{s}$ is convergesnt.