MATHEMATICS H311 — FALL 2015

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HOMEWORK ASSIGNMENT NO. 9, DUE ON TUESDAY, NOVEMBER 10

The following is a list of 2 (multipart) problems for you to hand in.

Problem 1. *The purpose of this problem is to introduce the basic trigonometric functions and the number π.*

The "sine" and "cosine" functions are functions from $\mathbb{R}$ to $\mathbb{R}$ defined as follows: if $x \in \mathbb{R}$, then

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\,, \tag{0.1}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\,, \tag{0.2}$$

(0.3)

that is,

$$\sin x = \sum_{n\in\mathbb{N},\, n \text{ odd}} (-1)^{\frac{n-1}{2}} \frac{x^n}{n!}\,, \tag{0.4}$$

$$\cos x = \sum_{n\in\mathbb{N}\cup\{0\},\, n \text{ even}} (-1)^{\frac{n}{2}} \frac{x^n}{n!}\,, \tag{0.5}$$

or

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\,, \tag{0.6}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\,. \tag{0.7}$$

I. **Prove** that for every x the series for $\sin x$ and $\cos x$ converge absolutely, so sin and cos are well defined functions from $\mathbb{R}$ to $\mathbb{R}$. (Use comparison with a geometric series.)

II. **Prove**, using series multiplication, that the identities

$$\begin{aligned}\sin(x+y) &= \sin x \cos y + \cos x \sin y,\\ \cos(x+y) &= \cos x \cos y - \sin x \sin y\,,\end{aligned}$$

(called the *addition formulas for the trigonometric functions*) hold for every $x \in \mathbb{R}$ and every $y \in \mathbb{R}$. (NOTE: The only hard part of the proof is doing the algebra, grouping terms together to get to the point where you are able to apply the binomial theorem as we did in class to prove that $e^{x+y} = e^x e^y$. I recommend that you work with Formulas (0.4) and (0.5), because this will make the algebraic work easier.)

III. **Prove** that

$$\sin(-x) = -\sin x$$

and

$$\cos(-x) = \cos x$$

for every real x.

IV. **Prove** that

$$\sin^2 x + \cos^2 x = 1 \quad \text{for every} \quad x \in \mathbb{R}\,.$$

(HINT: Observe first that $\cos 0 = 1$. Then write $0 = x + (-x)$ and apply the addition formulas together with the results of Part III.

V. **Conclude** from Part IV that

$$-1 \le \sin x \le 1 \quad \text{and} \quad -1 \le \cos x \le 1 \quad \text{for every} \quad x \in \mathbb{R}\,.$$

VI. **Prove** that

$$\sin x \le x \quad \text{whenever} \quad x \in \mathbb{R} \text{ and } x \ge 0\,. \tag{0.8}$$

(HINT: Write the series for $\sin x$ as

$$\sin x = x \cdot f(x)\,,$$

where

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}\,.$$

Observe that for $0 \le x \le \sqrt{6}$ the series for $f(x)$ is an alternating series, and conclude from this that $0 \le f(x) \le 1$ whenever $0 \le x \le \sqrt{6}$, so $\sin x \le x$ whenever $0 \le x \le \sqrt{6}$. Since $\sqrt{6} > 1$, conclude that (0.8) holds.)

VII. **Prove** that

$$\sin x \geq \frac{2x}{3} \quad \text{whenever} \quad x \in \mathbb{R} \text{ and } 0 \leq x \leq \sqrt{2}\,. \tag{0.9}$$

(HINT: Use the same method as in Part VI. Since the series for $f(x)$ is an alternating series for $0 \leq x \leq \sqrt{6}$, conclude that $f(x) \geq 1 - \frac{x^2}{6}$ for $0 \leq x \leq \sqrt{6}$, and derive from this the conclusion that $f(x) \geq \frac{2}{3}$ for $0 \leq x \leq \sqrt{2}$.)

VIII. **Prove** that

$$\sin 0.7 < \cos 0.7\,.$$

and

$$\sin 1.1 > \cos 1.1\,.$$

(HINT: Conclude from Part VI that $\sin 0.7 < 0.7$, so $\sin^2 0.7 < \frac{1}{2}$. Using Part IV, prove that $\cos^2 0.7 > \frac{1}{2}$, so $\sin 0.7 < \cos 0.7$. Then use Part VII to show that $\sin 1.1 > \frac{22}{30}$. Then $\sin^2 1.1 > \frac{1}{2}$, and you can use Part IV to prove that $\cos^2 1.1 < \frac{1}{2}$.)

IX. *In this part, you are allowed to use the fact that the sine and cosine functions are continuous. (This will be proved later in the course.)*

Prove that there exists a real number α such that[1]

$$0.7 < \alpha < 1.1 \tag{0.10}$$

and

$$\sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}}\,. \tag{0.11}$$

(HINT: $\sin 0.7 < \cos 0.7$ and $\sin 1.1 > \cos 1.1$. Then use the Intermediate Value Theorem.)

X. **Prove** that there exists a smallest real number α such that (0.10) and (0.11) hold. (HINT: Let S be the set of all real numbers α for which (0.10) and (0.11) hold. Then $S \neq \emptyset$ by Part IX. And S is obviously bounded below. Let β be the infimum of S. Prove that $\beta \in S$.)

[1] You probably know from high-school trigonometry that the smallest positive angle θ for which $\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$ is "45 degrees", that is, $\frac{\pi}{4}$. And you also know that π is about 3.14. So $\frac{\pi}{4}$ is about 0.785. So the bound $0.7 < \alpha < 1.1$ puts α in the correct range.

XI. If α is the number of Part X, ***we define the number*** π ***to be*** 4α. Hence $2.8 < \pi < 4.4$ which, fortunately, is in the correct range. (Later we will discuss how to calculate π much more accurately than that.)

Prove that

$$\sin\frac{\pi}{2} = 1,$$

$$\cos\frac{\pi}{2} = 0,$$

$$\sin\pi = 0,$$

$$\cos\pi = -1,$$

$$\sin 2\pi = 0,$$

and

$$\cos 2\pi = 1.$$

(HINT: Use the addition formulas for the sine and cosine functions, starting with the fact that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.)

XII. **Prove** that the sine and cosine functions are periodic with period 2π, that is:

$$\sin(x + 2\pi) = \sin x$$

and

$$\cos(x + 2\pi) = \cos x$$

for every real number x.

Problem 2. *In this problem we discuss how to turn the spaces* $\mathbb{R}^d$ *(i.e., the* d*-dimensional Euclidean spaces) into metric spaces.*

If d is a natural number, the symbol $\mathbb{R}^d$ denotes the set of all d-tuples

$$\vec{v} = (v_j)_{j=1}^d$$

of real numbers. Precisely, we use $\mathbb{N}_d$ to denote the set defined by

$$\{n \in \mathbb{N} : n \leq d\},$$

so $\mathbb{N}_d$ is the set $\{1, 2, \ldots, d\}$. Then a finite sequence of length d, or d-tuple, or finite list of length d, is a function whose domain is $\mathbb{N}_d$. We will use the notation[2] $(v_j)_{j=1}^d$ for "the d-tuple whose j-th entry is v_j for $j = 1, \ldots, d$". And we also sometimes write[3] $(v_1, v_2, \ldots, v_d)$ instead $(v_j)_{j=1}^d$.

If $\vec{v} = (v_j)_{j=1}^d$ is a d-tuple then, for each $j \in \mathbb{N}_d$, v_j is the j-th component, or j-th coordinate, of $\vec{v}$. (So, for example, v_1 is the first coordinate of $\vec{v}$, v_2 is the second coordinate of $\vec{v}$, v_3 is the third coordinate of $\vec{v}$, and so on, until we get to v_d, the d-th coordinate of $\vec{v}$.

Naturally, a d-tuple $\vec{v} = (v_j)_{j=1}^d$ is a d-tuple of real numbers if all the components of $\vec{v}$ are real numbers, i.e., if $(\forall j \in \mathbb{N}_d) v_j \in \mathbb{R}$.

As explained before, the expression $\mathbb{R}^d$ stands for the set of all d-tuples of real numbers. The members (or "points") of $\mathbb{R}^d$ are called d-dimensional real vectors, and the set $\mathbb{R}^d$ is called d-dimensional space, or d-dimensional real space, or d-dimensional Euclidean space[4].

The space $\mathbb{R}^2$ is called the plane, or the Euclidean plane, or, sometimes, two-dimensional space.

The space $\mathbb{R}^3$ is called three-dimensional space, or three-dimensional Euclidean space.

The sum of two d-dimensional vectors $\vec{v} = (v_j)_{j=1}^d$, $\vec{w} = (w_j)_{j=1}^d$, is the vector $\vec{v} + \vec{w}$ given by

$$\vec{v} + \vec{w} \stackrel{\text{def}}{=} (v_j + w_j)_{j=1}^d \,,$$

so $\vec{v} + \vec{w}$ is the vector $(x_j)_{j=1}^d$ whose components are given by

$$x_j = v_j + w_j \quad \text{for} \quad j = 1, 2, \ldots, d \,.$$

The product of a scalar[5] r by a vector $\vec{v}$ is the vector $r\vec{v}$ given by

$$r\vec{v} \stackrel{\text{def}}{=} (rv_j)_{j=1}^d \quad \text{if} \quad \vec{v} = (v_j)_{j=1}^d \,.$$

(So, for example, if $\vec{v} = (v_1, v_2, v_3, v_4)$ and $r \in \mathbb{R}$, then $r\vec{v}$ is the vector (rv_1, rv_2, rv_3, rv_4).)

[2]Notice the obvious similarity of this with the notation $(v_j)_{j=1}^\infty$ for infinite sequences.

[3]But we do ***not*** write $\{v_1, v_2, \ldots, v_d\}$ for a d-tuple. The expression $\{v_1, v_2, \ldots, v_d\}$ stands for a ***set***, not a d-tuple. The d-tuple is written $(v_1, v_2, \ldots, v_d)$.

[4]In this course, we only work with d-tuples of real numbers. One could also work with the space $\mathbb{C}^d$ of all d-tuples of complex numbers, called d-dimensional complex space, whose members would be called d-dimensional complex vectors, or with the space $\mathbb{F}^d$ of all d-tuples of memberso of $\mathbb{F}$, where $\mathbb{F}$ is some field. But we will not do that here, so for us "d-dimensional space" means $\mathbb{R}^d$.

[5]When discussing vectors in $\mathbb{R}^d$, it is customary to refer to the real numbers as "scalars".

The dot product (or scalar product, or inner product) of two vectors $\vec{v} = (v_j)_{j=1}^d$, $\vec{w} = (w_j)_{j=1}^d$ is the scalar $\vec{v} \cdot \vec{w}$ given by

$$\vec{v} \cdot \vec{w} \stackrel{\text{def}}{=} \sum_{j=1}^{d} v_j w_j \quad \text{if} \quad \vec{v} = (v_j)_{j=1}^d, \vec{w} = (w_j)_{j=1}^d \,.$$

(So, for example, if $\vec{v} = (v_1, v_2, v_3, v_4)$ and $\vec{w} = (w_1, w_2, w_3, w_4)$, then $\vec{v} \cdot \vec{w}$ is the real number given by $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4$.)

The square-length of a vector $\vec{v} = (v_j)_{j=1}^d$ is the real number $SQL(\vec{v})$ given by

$$SQL(\vec{v}) \stackrel{\text{def}}{=} \vec{v} \cdot \vec{v} \,.$$

That is, $SQL(\vec{v})$ is the dot product of $\vec{v}$ with itself. Hence

$$SQL(\vec{v}) = \sum_{j=1}^{d} v_j^2 \quad \text{if} \quad \vec{v} = (v_j)_{j=1}^d \,.$$

The length of a vector $\vec{v}$ is the square root[6] of $SQL(\vec{v})$.

If $v \in \mathbb{R}^d$, we write $\|\vec{v}\|$ to denote the length of $\vec{v}$, that is,

$$\|\vec{v}\| \stackrel{\text{def}}{=} \sqrt{SQL(\vec{v})} \,.$$

It follows that

$$\|\vec{v}\| = \sqrt{\sum_{j=1}^{d} v_j^2} \quad \text{if} \quad \vec{v} = (v_j)_{j=1}^d \,.$$

Furthermore, it follows from the definition that the square of $\|\vec{v}\|$ is precisely the number $SQL(\vec{v})$, so from now on we will write $\|v\|^2$ rather than $SQL(\vec{v})$, which means that the expression $SQL(\vec{v})$ is never going to appear again.

The distance between two vectors $\vec{v}, \vec{w} \in \mathbb{R}^d$ is the number $\text{dist}(\vec{v}, \vec{w})$ given by

$$\text{dist}(\vec{v}, \vec{w}) \stackrel{\text{def}}{=} \|\vec{v} - \vec{w}\| \,.$$

[6] If follows from the definition of the square lenght that the square-length of a vector $\vec{v} \in \mathbb{R}^d$ is always a nonnegative real number. So the square root of this number exists. Furthemore, the notation $\sqrt{x}$, for a nonnegative real number x, *always* stands for the unique nonnegative square root of x. So in particular the number $\sqrt{SQL(\vec{v})}$ is a perfectly well-defined nonnegative real number.

I. **Prove** that dist is a metric on $\mathbb{R}^d$, so $\mathbb{R}^d$, equipped with this metric, is a metric space. NOTE: The hard part is proving the triangle inequality. This requires some work. You should first prove that

$$\|\vec{v}+\vec{w}\| \le \|\vec{v}\| + \|\vec{w}\| \quad \text{for all} \quad \vec{v},\vec{w} \in \mathbb{R}^d\,. \tag{0.12}$$

(Once you have this, the triangle inequality for the metric dist follows easily).

And to prove (0.12) you can do it by brute force, just writing everything out, which would involve writing lot sof summations, and would be complicated but doable. Or you could use the following ingenious trick: First **prove** that the dot product satisfies the following laws:

1. *Commutativity:*

$$\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v} \quad \text{whenever} \quad \vec{v},\vec{w} \in \mathbb{R}^d\,.$$

2. *Distributivity:*

$$(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w} \quad \text{whenever} \quad \vec{u},\vec{v},\vec{w} \in \mathbb{R}^d\,,$$

and

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} \quad \text{whenever} \quad \vec{u},\vec{v},\vec{w} \in \mathbb{R}^d\,.$$

3. *Scalar multiplication law:*

$$(r\vec{u})\cdot\vec{v} = \vec{u}\cdot(r\vec{v}) = r(\vec{u}\cdot\vec{v}) \quad \text{whenever} \quad \vec{u},\vec{v} \in \mathbb{R}^d, r \in \mathbb{R}\,.$$

Then, using the above laws for sums and products, take two vectors $\vec{v},\vec{w} \in \mathbb{R}^d$, and define a function $p : \mathbb{R} \mapsto \mathbb{R}$ by letting

$$p(r) = (\vec{v}+r\vec{w})\cdot(\vec{v}+r\vec{w}) \quad \text{for} \quad r \in \mathbb{R}\,.$$

Observe that $p(r) \ge 0$ for every real number r, write out $p(r)$ as a polynomial ar^2+br+c, and use the necessary and sufficient condition for such a polynomial never to have negative values. (The condition is: $a > 0$ and $4ac \le b^2$.) This will give you the **Cauchy-Schwarz inequality**[7]:

$$\vec{v}\cdot\vec{w} \le \|\vec{v}\|\cdot\|\vec{w}\|\,.$$

[7] Known in Russia as "the Bunyakovsky inequality", or "the Cauchy-Schwarz-Bunyakovsky inequality".

(This is analogous to the one-dimensional inequality $a \cdot b \leq |a| \cdot |b|$ that I am sure you know very well.)

And, finally, once you have proved the Cauchy-Schwarz inequality, the triangle inequality for the metric follows easily. (Just write

$$\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\dot{(}\vec{v}+\vec{w})\,,$$

conclude from that that

$$\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\vec{v}\cdot\vec{w}\,,$$

use the Cauchy-Schwarz inequality, and take square roots of both sides.)

II. If d is a natural number, then a subset S of $\mathbb{R}^d$ is bounded if there exists a real number B such that

$$\|\vec{v}\| \leq B \quad \text{for every} \quad \vec{v} \in S\,.$$

A sequence $(\vec{v}_n)_{n=1}^{\infty}$ of vectors in $\mathbb{R}^d$ is bounded if the set $\{\vec{v}_n : n \in \mathbb{N}\}$ is bounded. That is, the sequence $(\vec{v}_n)_{n=1}^{\infty}$ is bounded if and only if there exists a real number B such that

$$\|\vec{v}_n\| \leq B \quad \text{for every} \quad n \in \mathbb{N}\,.$$

If $(\vec{v}_n)_{n=1}^{\infty}$ is a sequence of vectors in $\mathbb{R}^d$, the general definition of convergence of a sequence of points of a metric space says that the sequence $(\vec{v}_n)_{n=1}^{\infty}$ *converges* to a vector $\vec{v} \in \mathbb{R}^d$ if

$$\lim_{n\to\infty} \text{dist}(\vec{v}_n, \vec{v}) = 0\,,$$

that is, equivalently, if

$$\lim_{n\to\infty} \|\vec{v}_n - \vec{v}\| = 0\,.$$

We say that a sequence $(\vec{v}_n)_{n=1}^{\infty}$ converges coordinatewise to the vector $\vec{v}$ if for each index $j \in \mathbb{N}_d$, the sequence of j-th coordinates of the $\vec{v}_n$ converges to the j-th coordinate of $\vec{v}$. That is, if we write the vectors $\vec{v}_n$ as[8] $\vec{v}_n = (v_{nj})_{j=1}^{d}$, $\vec{v} = (v_j)_{j=1}^{d}$, then $(\vec{v}_n)_{n=1}^{\infty}$ converges coordinatewise to vector $\vec{v}$ if and only if

$$\lim_{n\to\infty} v_{nj} = v_j \quad \text{for every} \quad j \in \mathbb{N}_d\,,$$

[8] At this point, we need *two* subscripts: the subscript n identifies which vector in the sequence $(\vec{v}_n)_{n=1}^{\infty}$ we are talking about, and the subscript j identifies the coordinate, so v_{nj} stands for the j-th coordinate of the n-th vector.

or, if you prefer,

$$\lim_{n\to\infty} v_{nj} = v_j \quad \text{for} \quad j = 1, 2, \ldots, d\,.$$

Prove that a sequence $(\vec{v}_n)_{n=1}^{\infty}$ of vectors in $\mathbb{R}^d$ converges to a vector $\vec{v} \in \mathbb{R}^d$ if and only if $(\vec{v}_n)_{n=1}^{\infty}$ converges coordinatewise to $\vec{v}$. (For example, if $d = 2$, you have to prove that, if $\vec{v}_n = (x_n, y_n)$, and $\vec{v} = (x, y)$, then $\lim_{n\to\infty} \vec{v}_n = \vec{v}$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Equivalently, you have to prove that

$$\lim_{n\to\infty} \sqrt{(x_n - x)^2 + (y_n - y)^2} = 0$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y\,,$$

which should be very easy.)

III. **Prove** the *Bolzano-Weierstrass theorem in several variables*: If $d \in \mathbb{N}$, then every sequence $(\vec{v}_n)_{n=1}^{\infty}$ of vectors in $\mathbb{R}^d$ which is bounded has a convergent subsequence. (HINT: Use the result of Part II.)

IV. A Cauchy sequence in a metric space S with metric d is a sequence $(s_n)_{n=1}^{\infty}$ of points of S such that

(CS) For every positive real number ε there exists a natural number N such that

$$d(s_n, s_m) < \varepsilon \quad \text{whenever} \quad n, m \in \mathbb{N},\, n \geq N,\, m \geq N\,.$$

A metric space S with metric d is complete if every Cauchy sequence of points of S converges.

Prove that the metric spaces $\mathbb{R}^d$ are complete[9]. (HINT: You may consider using the same technique that was used in one dimension: prove that a Cauchy sequence is necessarily bounded, and prove that if a Cauchy sequence **s** has a convergent subsequence then **s** converges to a point of S.)

[9] We have already proved in class (and it is proved in the book) that one-dimensional space $\mathbb{R}$ si complete. Now I am asking you prove that all the Euclidean spaces $\mathbb{R}^d$ are complete.