

MATHEMATICS 300 — SPRING 2015

Introduction to Mathematical Reasoning

H. J. Sussmann

REVIEW PROBLEMS FOR THE APRIL 24 MIDTERM EXAM

The following list of problems consists mostly of questions that are answered in the notes, or are discussed in the book, or were discussed in class. But there are a few problems for which you will have to do some thinking.

The problems in the April 24 midterm exam are going to be identical or very similar to problems in this list.

1. Define “rational number” and “irrational number”.
2. Define “divisible”.
3. Define “prime number”.
4. Define “subset”.
5. Define “empty set”.
6. Define “power set”.
7. Define “inductive set”.
8. Define “union” (of two sets).
9. Define “intersection” (of two sets).
10. Define “coprime”.
11. Define “greatest common divisor”.

12. For each of the following pairs A, B of sets, determine whether the sets A, B are equal, and in each case explain why they are, or why they are not, equal.

1. $A = \{2, 3, 4\}, B = \{4, 3, 2\}$.

2. $A = \{2, 3, 4\}, B = \{4, 3, 2, 2\}$.

3. $A = \{2, 3, 4\}, B = \{4, 3, 1\}$.

4. $A = \mathbb{N}, B = \{n \in \mathbb{Z} : n \geq 0\}$.

5. $A = \mathbb{N}, B = \{n \in \mathbb{Z} : n > 0\}$.

6. $A = \mathbb{R}, B = \mathbb{Q}$.

7. $A = \{x \in \mathbb{R} : x^2 = 2x + 1\}, B = \{x \in \mathbb{Z} : x^2 = 2x + 1\}$.

8. $A = \{x \in \mathbb{R} : x^2 = 2x + 8\}, B = \{x \in \mathbb{Z} : x^2 = 2x + 8\}$.

13. Let $A = \{x \in \mathbb{R} : x^2 = 2x + 1\}, B = \{x \in \mathbb{R} : x^2 = 2x + 8\}$. Determine $A \cup B$ and $A \cap B$.

14. Prove or disprove each of the following statements (remember that capital letters stand for sets, so for example the quantifier “ $(\forall X)$ ” says “for all sets X ”):

1. $(\forall X)\emptyset \subseteq X$.

2. $(\forall X)\emptyset \in X$.

3. $(\forall X)X \cup \emptyset = X$.

4. $(\forall X)X \cup \emptyset = \emptyset$.

5. $(\forall X)X \cap \emptyset = X$.

6. $(\forall X)X \cap \emptyset = \emptyset$.

7. $(\forall X)(\forall Y)(X \cup Y = \emptyset \implies X = Y)$.

8. $(\forall X)(\forall Y)(X \cap Y = \emptyset \implies X = Y)$.

15. Consider the statement

(*) If A, B, C are sets such that A is a subset of B and B is a subset of C , then A is a subset of C .

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.

- II. Prove statement (*).
16. The *distributive law of union with respect to intersection* says that:
- (*) If A, B, C are sets then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.
- II. Prove statement (*).
17. The *distributive law of intersection with respect to union* says:
- (*) If A, B, C are sets then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.
- II. Prove statement (*).
18. Prove that if A, B are sets then $(A \cup B) \cap A = A$.
19. Prove that if A, B are sets then $(A \cap B) \cup A = A$.
20. The *division theorem* says that
- (*) If a, b are integers, and $b \neq 0$, then there exist unique integers q, r such that $a = bq + r$ and $0 \leq r < |b|$.
- I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives, and without using the “!” symbol. (This means that, instead of saying “ $(\exists!x) \dots$ ”, you have to say what this means without using “!”.)
- II. Prove the existence part.
21. The *division theorem* says that
- (*) If a, b are integers, and $b \neq 0$, then there exist unique integers q, r such that $a = bq + r$ and $0 \leq r < |b|$.

- I. Write statement (*) in formal language, using the appropriate quantifiers, and logical connectives, and without using the “!” symbol. (This means that, instead of saying “ $(\exists!x) \dots$ ”, you have to say what this means without using “!”.)
- II. Prove the uniqueness part.
22. State the well-ordering principle, both in ordinary language and in formal language.
23. The Fibonacci numbers f_n are defined by

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1, \\ f_{n+2} &= f_n + f_{n+1} \text{ for } n \in \mathbb{N}. \end{aligned}$$

Prove that the parity of the Fibonacci numbers is “odd-odd-even-odd-odd-even-...”. Precisely, prove that

$$(\forall n \in \mathbb{N})(f_{3n-2} \text{ is odd} \wedge f_{3n-1} \text{ is odd} \wedge f_{3n} \text{ is even}).$$

24. The Fibonacci numbers f_n are defined by

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1, \\ f_{n+2} &= f_n + f_{n+1} \text{ for } n \in \mathbb{N}. \end{aligned}$$

Define real numbers φ, ψ by

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad \psi = \frac{1 - \sqrt{5}}{2}.$$

- I. Verify that $\varphi^2 = \varphi + 1$ and $\psi^2 = \psi + 1$.
- II. Prove *Binet’s formula*:

$$(\forall n \in \mathbb{N}) f_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n).$$

(HINT: For the proof, use the well-ordering principle.)

25. The *coprime representation theorem* says that

(*) If r is a rational number, then there exist integers m, n such that $n \neq 0$, m and n are coprime, and $r = \frac{m}{n}$.

I. Write statement (*) in formal language, using $COP(x, y)$ for “ x and y are coprime”.

II. Prove statement (*).

(HINT: For the proof, use the well-ordering principle.)

26. Prove that $\sqrt{2}$ is irrational.

27. Prove that $\sqrt{3}$ is irrational.

28. Prove that $\sqrt{5}$ is irrational.

29. Prove that $\sqrt{6}$ is irrational.

30. Prove that $\sqrt{12}$ is irrational.

31. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

32. Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

33. Prove that $\sqrt{2} + \sqrt[3]{2}$ is irrational.

34. For each of the following four statements:

I. Write the statement in formal language.

II. Prove it or disprove it.

1. The sum of two rational numbers is a rational number.

2. The sum of two irrational numbers is an irrational number.

3. The sum of a rational number and an irrational numbers is an irrational number.

4. The product of a rational number and an irrational numbers is an irrational number.

35. *Bézout's lemma* says that

(*) The greatest common divisor of two integers a, b that are not both equal to zero is the smallest integer linear combination of a and b .

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives, and using " $GCD(x, y)$ " for "the greatest common divisor of x and y ".

II. Prove statement (*).

36. Find the greatest common divisor of 33 and 47 and express it as an integer linear combination of 33 and 47.

37. Find the greatest common divisor of 24 and 46 and express it as an integer linear combination of 24 and 46.

38. Find the greatest common divisor of 24 and 49 and express it as an integer linear combination of 24 and 49.

39. *Euclid's lemma* says that

(*) If p is a prime number, a and b are integers, and p divides the product ab , then p divides a or p divides b .

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives, and using " $x|y$ " for " x divides y ", " $PR(x)$ " for " x is a prime number".

II. Prove statement (*). (HINT: use Bézout's lemma.)

40. Consider the statement

(*) If p, q are coprime integers, a is an integer, p divides a and q divides a , then pq divides a .

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives, and using " $x|y$ " for " x divides y ", " $CR(x)$ " for " x is a prime number", and " $COP(x, y)$ " for " x and y are coprime".

II. Prove statement (*). (HINT: use Bézout's lemma.)

41. Prove, *without using any general theorems such as Bézout's lemma or its corollaries, or the fundamental theorem of arithmetic*, that if an integer is divisible by 7 and by 11 then it is divisible by 77. (HINT: $22 - 21 = 1$.)
42. Prove, *without using any general theorems such as Bézout's lemma or its corollaries, or the fundamental theorem of arithmetic*, that if an integer is divisible by 5 and by 13 then it is divisible by 65. (HINT: $26 - 25 = 1$.)
43. Consider the statement
- (*) If a is an odd integer then there exists an integer k such that $a = 4k + 1$ or $a = 4k + 3$.
- I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.
- II. Prove statement (*). (HINT: use the division theorem.)
44. Consider the statement
- (*) If a is an odd integer then there exists an integer k such that $a^2 = 4k + 1$, and there does not exist an integer k such that $a^2 = 4k$ or $a^2 = 4k + 2$ or $a^2 = 4k + 3$.
- I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.
- II. Prove statement (*). (HINT: use the division theorem.) (Note: you will have to use the division theorem a few times, both for a and for a^2 . And the uniqueness part of the theorem is going to play a crucial role.)
45. Prove that if $\mathbf{x} = (x_j)_{j=1}^n$ is a finite list of nonzero real numbers, then the product $\prod_{j=1}^n x_j$ is nonzero. (That is, if you prefer: if x_1, x_2, \dots, x_n are real numbers that are all different from zero, then the product $x_1 x_2 \cdots x_n$ is also different from zero.) NOTE: The special case $n = 2$ is a theorem that we proved earlier in the course, but you are *not* allowed to use this theorem. You should give a completely self-contained proof, including the proof for $n = 2$.

46. Prove that if p is a prime number, and $\mathbf{a} = (a_j)_{j=1}^n$ is a finite list of integers such that p divides the product $\prod_{j=1}^n a_j$, then p must divide one of the factors. (That is, if you prefer: if p is a prime number, a_1, a_2, \dots, a_n are integers, and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_j$ for some $j \in \{k \in \mathbb{N} : k \leq n\}$. NOTE: The special case $n = 2$ is Euclid's Lemma, which was proved earlier in the course, but you are *not* allowed to use this result. You should give a completely self-contained proof, including the proof for $n = 2$. (In other words: you have to prove Euclid's lemma.)
47. Prove that if n is a natural number and $n \geq 2$ then there exists a finite list $\mathbf{p} = (p_j)_{j=1}^m$ of prime numbers such that $\prod_{j=1}^m p_j = n$.
48. Prove the uniqueness assertion of the fundamental theorem of arithmetic:
- (*) If $\mathbf{p} = (p_j)_{j=1}^m$ and $\mathbf{q} = (q_j)_{j=1}^r$ are ordered finite lists of prime numbers such that $\prod_{j=1}^m p_j = \prod_{j=1}^r q_j$, then $\mathbf{p} = \mathbf{q}$.
- (NOTE: a list $\mathbf{x} = (x_j)_{j=1}^m$ of real numbers is ordered if $x_j \leq x_{j+1}$ for $j \in \mathbb{N}$, $1 \leq j \leq m - 1$. And " $\mathbf{p} = \mathbf{q}$ " means " $r = m$ and $p_j = q_j$ for every j in the set $\{k \in \mathbb{N} : k \leq m\}$ ".)
49. Prove that if $n \in \mathbb{N}$ and n has a prime factorization $n = p_1 p_2 p_3$ and another prime factorization $n = \prod_{j=1}^m q_j$, where the p_j and the q_j satisfy $p_1 \leq p_2 \leq p_3$ and $q_1 \leq q_2 \leq \cdots \leq q_{m-1} \leq q_m$, then it follows that $m = 3$, $q_1 = p_1$, $q_2 = p_2$, $q_3 = p_3$. (This is a special case of the uniqueness assertion of the fundamental theorem of arithmetic. Do **not** use the fundamental theorem of arithmetic. Prove the result directly, by using the same method that we used in the proof of the general theorem. That is, start by proving that $p_3 = q_m$, and then $p_1 p_2 = \prod_{j=1}^{m-1} q_j$, and so on.)