MATHEMATICS 300 — SPRING 2015

Introduction to Mathematical Reasoning

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REVIEW PROBLEMS FOR THE MARCH 6 MIDTERM EXAM

The following list of problems consists mostly of questions that are answered in the notes, or are discussed in the book, or were discussed in class. But there are several problems that you will have to think about.

The set of problems in the March 6 midterm exam is going to be a subset of the set of problems in this list.

1. Define "finite set" and "infinite set".
2. Define "divisible".
3. Define "prime number".
4. Define "subset".
5. Define "empty set".
6. Define "power set".
7. Define "inductive set".
8. Prove that $\mathbb{N}$, the set of all natural numbers, is an infinite set.
9. Prove that the set of prime numbers is infinite (Euclid's Theorem).
10. Prove that if n is a natural number and $n \geq 2$ then n has a prime factor.
11. For each of the following, indicate whether it is a one-argument function, a two-argument function, a one-argument predicate, or a two-argument predicate. (Note: "function" means exactly the same as "operation", and "predicate" means exactly the same as "relation".):

(a) equal,
(b) absolute value,
(c) divisible,
(d) divides,
(e) prime number,
(f) even number,
(g) odd number,
(h) addition of real numbers,
(i) multiplication of real numbers,
(j) subtraction of real numbers,
(k) minus (that is, the negative of a real number),
(l) square (that is, the square of a real number),
(m) subset,
(n) power set.

12. Using the definitions of 2, 3, 4, 5 and 6 (that is: $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$, $5 = 4 + 1$, $6 = 5 + 1$) prove that $3 \times 2 = 6$.

13. Define "absolute value".

14. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If a, b, c are arbitrary real numbers such that $a + b = a + c$, then $b = c$.

15. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If a, b, c are arbitrary real numbers such that $a.b = a.c$, and $a \neq 0$, then $b = c$.

16. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

(*) If x is an arbitrary real number, then $x.0 = 0$.

17. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If x, y are arbitrary real numbers such that $x.y = 0$, then $x = 0$ or $y = 0$.

18. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If a, b, c are arbitrary real numbers such that $a \geq b$ and $b \geq c$, then $a \geq c$.

    (Note: the definition of "$\geq$" is as follows: if x, y are real numbers, we say that x is greater than or equal to y, and write "$x \geq y$", if $y < x \vee y = x$.)

19. Give an example of three sentences P, Q, R such that one of the sentences "$P \Longrightarrow (Q \Longrightarrow R)$", "$(P \Longrightarrow Q) \Longrightarrow R$" is true but the other one is false.

20. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If p, q, r are arbitrary real numbers such that p is less than or equal to q and q is less than r, then p is less thatn r.

    (Note: the definition of "less than or equal to" is as follows: if x, y are real numbers, we say that x is less than or equal to y, and write "$x \leq y$", if $x < y \vee x = y$.)

21. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If q is an arbitrary real number such that $q \neq 0$, then $q^2 > 0$.

22. Explain why the following proof is wrong.

Claim. $0 \neq 2$.

Proof. $0 \neq 1$ by Axiom FA11.

Adding the inequality "$0 \neq 1$" to itself, we get $0 + 0 \neq 1 + 1$, that is, $0 \neq 2$. **Q.E.D.**

23. Prove that $1 > 0$.

24. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If q is an arbitrary real number such that $q > 0$, then $q + \frac{1}{q} \geq 2$.

25. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If q is an arbitrary real number such that $q > 0$, then $6q + \frac{1}{q} \geq 2\sqrt{6}$.

26. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If p, q are arbitrary real numbers, then $pq \leq 8p^2 + \frac{q^2}{32}$.

27. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If p, q are arbitrary real numbers, then the absolute value of pq is the product of the absolute values of p and q.

28. State and prove the triangle inequality for real numbers.

29. Prove that $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R}) \, \big| |x| - |y| \big| \leq |x - y|$.

30. In this problem

    - $\mathbb{R}^2$ is the set of all pairs (a, b) of real numbers.
    - The members of $\mathbb{R}^2$ are called <u>two-dimensional vectors</u>.

- The length of a two-dimensional vector $\vec{v} = (a, b)$ is the number $\|\vec{v}\|$ given by $\|\vec{v}\| = \sqrt{a^2 + b^2}$.
- The dot product of two two-dimensional vectors $\vec{v} = (a, b)$, $\vec{w} = (c, d)$, is the number $\vec{v} \cdot \vec{w}$ given by $\vec{v} \cdot \vec{w} = ac + bd$.

Prove the *Cauchy-Schwarz inequality*:

$$(\forall \vec{v} \in \mathbb{R}^2)(\forall \vec{w} \in \mathbb{R}^2)\, \vec{v} \cdot \vec{w} \leq \|\vec{v}\| \, \|\vec{w}\| \,.$$

31. In this problem

- $\mathbb{R}^2$ ia the set of all pairs (a, b) of real numbers.
- The members of $\mathbb{R}^2$ are called two-dimensional vectors.
- The length of a two-dimensional vector $\vec{v} = (a, b)$ is the number $\|\vec{v}\|$ given by $\|\vec{v}\| = \sqrt{a^2 + b^2}$.
- The dot product of two two-dimensional vectors $\vec{v} = (a, b)$, $\vec{w} = (c, d)$, is the number $\vec{v} \cdot \vec{w}$ given by $\vec{v} \cdot \vec{w} = ac + bd$.

Prove the *triangle inequality*:

$$(\forall \vec{v} \in \mathbb{R}^2)(\forall \vec{w} \in \mathbb{R}^2)\, \|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| \,.$$

You are allowed to use the *Cauchy-Schwarz inequality*:

$$(\forall \vec{v} \in \mathbb{R}^2)(\forall \vec{w} \in \mathbb{R}^2)\, \vec{v} \cdot \vec{w} \leq \|\vec{v}\| \, \|\vec{w}\| \,.$$

32. For each of the following sentences, indicate whether the sentence is true or false, and explain why.

(a) $6 < 5 \vee 4 > 3$,

(b) $6 < 5 \wedge 4 > 3$,

(c) $6 < 5 \Longrightarrow 4 > 3$,

(d) $6 < 5 \Longrightarrow 4 < 3$,

(e) $(\forall x \in \mathbb{R})(x^2 \leq 0 \wedge x \neq 0)$,

(f) $(\forall x \in \mathbb{R})(x^2 \leq 0 \vee x \neq 0)$,

(g) $(\forall x \in \mathbb{R})(x^2 \leq 0 \Longrightarrow x \neq 0)$,

(h) $(\exists x \in \mathbb{R})(x^2 < 0 \Longrightarrow x \neq 0)$,

(i) $(\exists x \in \mathbb{R})(x^2 < 0 \Longrightarrow 4 < 3)$,

(j) $(\exists x \in \mathbb{R})x^2 < 0 \Longrightarrow 4 < 3$,

(k) $(\forall x \in \mathbb{R})(x^2 < 0 \Longrightarrow 4 < 3)$,

(l) $(\forall x \in \mathbb{R})(x^2 \leq 0 \Longrightarrow 4 < 3)$,

(m) $(\forall x \in \mathbb{R})x^2 \leq 0 \Longrightarrow 4 < 3$.

33. For each of the following sentences,

    i. Translate the sentence into English.

    ii. Indicate whether the sentence is true or false, and explain why.

(a) $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})m \leq n$.

(b) $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})m < n$.

(c) $(\exists m \in \mathbb{Z})(\forall n \in \mathbb{Z})m \leq n$.

(d) $(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})m \leq n$.

(e) $(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z})m < n$.

34. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

Claim. If a, b, c are integers such that a divides c and b divides c and $a + b$ is odd then $a + b$ divides c.

Proof.

Let a, b, c be arbitrary integers.

Assume that a divides c and b divides c and $a + b$ is odd.

We want to prove that $a + b$ divides c.

Since a divides c, we can write $c = ak$, $k \in \mathbb{Z}$.

Since b divides c, we can write $c = bk$, $k \in \mathbb{Z}$.

Then, adding the two equations, we get $2c = ak + bk$, so $2c = (a+b)k$.

It follows that $(a+b)k$ is even.

But $a+b$ is odd, so k must be even. (Reason: if k was odd then, since $a+b$ is odd, the product $(a+b)k$ would be odd. But we have just shown that $(a+b)k$ is even.)

Since k is even, we can write $k = 2m$, $m \in \mathbb{Z}$.

Then $2c = (a+b) \times 2m$, so $2c = 2(a+b)m$, and then $c = (a+b)m$.

Therefore $a+b$ divides c. **Q.E.D.**

35. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

    Claim. If a, b, c are integers such that a is divisible by c and b is divisible by c, then $a+b$ is divisible by c.

    Proof.

    Let a, b, c be arbitrary integers.

    Assume that a is divisible by c and b is divisible by c.

    We want to prove that $a+b$ is divisible by c.

    Since a is divisible by c, we can write $a = ck$, $k \in \mathbb{Z}$.

    Since b is divisible by c, we can write $b = ck$, $k \in \mathbb{Z}$.

    Then, adding the two equations, we get $a + b = ck + ck = c \times 2k$.

    Therefore $a+b$ is divisible by c. **Q.E.D.**

36. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

Claim. If a, b, c are integers such that a is divisible by c and b is divisible by c, then $a + b$ is divisible by c.

Proof.

Let a, b, c be arbitrary integers.

Assume that a is divisible by c and b is divisible by c.

We want to prove that $a + b$ is divisible by c.

Since a is divisible by c, we can write $a = ck$, $k \in \mathbb{Z}$.

Since b is divisible by c, we can write $b = cj$, $j \in \mathbb{Z}$.

Then, adding the two equations, we get $a+b = ck+cj = c(k+j)$.

Therefore $a + b$ is divisible by c. **Q.E.D.**

37. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

Claim. 1 is the largest natural number.

Proof.

Let n be the largest natural number. [Introducing an object and giving it a name, using Rule $\exists_{use}$]

Then $n^2 \in \mathbb{N}$ [Because the square of a natural number is a natural number]

And $n^2 \leq n$ [Because $n$is the largest natural number, so n^2 cannot be larger than n, so $n^2 \leq n$]

But $n^2 \geq n$ [Because $n \geq 1$, since $n \in \mathbb{N}$, so $n^2 \geq n$]

So $n^2 = n$ [Because $n^2 \geq n$ and $n^2 \leq n$]

So $n^2 - n = 0$ [adding $-n$ to both sides]

But $n^2 - n = n(n-1)$ [Trivial]

So $n(n-1) = 0$ [Rule SEE]

So $n = 0 \vee n - 1 = 0$ [Theorem 4 of the Lecture 2-3-4 notes]

So $n = 0$ or $n = 1$. [Trivial]
But $n \neq 0$ [Because $n \in \mathbb{N}$ and $0 \notin \mathbb{N}$]
So $n = 1$ [Because $n = 0 \vee n = 1$ and $n \neq 0$]
So 1 is the largest natural number. **Q.E.D.**

38. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

    Claim. If a, b are real numbers, then $2ab \leq a^2 + b^2$.

    Proof.

    Since $2ab \leq a^2 + b^2$, we may subtract $2ab$ from both sides and conclude that

    $$0 \leq a^2 + b^2 - 2ab\,. \tag{0.1}$$

    But $a^2 + b^2 - 2ab = (a - b)^2$.

    And the square of every real number is nonnegative, so $(a - b)^2 \geq 0$.

    So $0 \leq a^2 + b^2 - 2ab$, which agrees with (0.1). **Q.E.D.**

39. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

    Claim. If a, b are real numbers, then $2ab \leq a^2 + b^2$.

    Proof.

    We prove the claim by contradiction.

    Assume that the inequality "$2ab \leq a^2 + b^2$" is not true.
    Then $2ab \geq a^2 + b^2$.
    Subtracting $2ab$ from both sides we get $0 \geq a^2 + b^2 - 2ab$.
    But $a^2 + b^2 - 2ab = (a - b)^2$.

So $0 \geq (a-b)^2$.

But the square of every real number is nonnegative, so $(a-b)^2 \geq 0$.

So we have established two contradictory facts, namely, that $0 \geq (a-b)^2$ and that $(a-b)^2 \geq 0$.

Since assuming that our desired claim was false has led us to a contradiction, we can conclude that the claim is true, i.e., that $2ab \leq a^2 + b^2$. **Q.E.D.**

So $0 \leq a^2 + b^2 - 2ab$, which agrees with (0.1). **Q.E.D.**

40. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

Claim. If a, b are real numbers, then $2ab \leq a^2 + b^2$.

Proof.

We prove the claim by contradiction.

Assume that the inequality "$2ab \leq a^2 + b^2$" is not true.

Then $2ab > a^2 + b^2$.

Subtracting $2ab$ from both sides we get $0 > a^2 + b^2 - 2ab$.

But $a^2 + b^2 - 2ab = (a-b)^2$.

So $0 > (a-b)^2$.

But the square of every real number is nonnegative, so $(a-b)^2 \geq 0$.

So we have shown two contradictory facts, namely, that $0 > (a-b)^2$ and that $(a-b)^2 \geq 0$.

Since assuming that our desired claim was false has led us to a contradiction, we can conclude that the claim is true, i.e., that $2ab \leq a^2 + b^2$. **Q.E.D.**

41. Translate the following statement into formal language (using quantifiers), determine if it is true or false, and prove it, if it is true, or prove

that it is false, if it is false. (The translation is worth 30%, and the proof is worth 70% each.)

(&) If X, Y, Z are arbitrary sets then if X is a subset of Y and Y is a subset of Z, it follows that X is a subset of Z. false.

42. Translate each of the following four statements into formal language (using quantifiers), determine if the statement is true or false, and then prove it, if is true, or prove that it is false, if it is false. (The translations are worth 5% each, the true-false questions are worth 5% each, and the proofs are worth 15% each.)

    (a) The empty set belongs to every set.

    (b) The empty set is a subset of every set.

    (c) There exists a set that belongs to every set.

    (d) There exists a set that is a subset of every set.

43. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If x is an arbitrary real number, then x^2 is equal to 9 if and only if either $x = 3$ or $x = -3$.

44. i. Define "even integer" and "odd integer".

    ii. Using your definitions of "even" and "odd", prove that the sum of an even integer and an odd integer is an odd integer.

45. Translate the following statement into formal language (using quantifiers) and prove it. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If a, b, c are real numbers, and c is positive, then $|a - b| < c$ if and only if $b - c < a < b + c$.

46. Translate the following statement into formal language (using quantifiers) and prove it by induction. (The translation is worth 30%, and the proof is worth 70%.):

(*) Every natural number is greater than or equal to 1.

47. Translate the following statement into formal language (using quantifiers) and prove it by induction. (The translation is worth 30%, and the proof is worth 70%.):

    (*) Every natural number is equal to 1 or greater than or equal to 2.

48. Translate the following statement into formal language (using quantifiers) and prove it by induction. (The translation is worth 30%, and the proof is worth 70%.):

    (*) Every natural number is equal to 1, or equal to 2, or greater than or equal to 3.

49. Translate the following statement into formal language (using quantifiers) and prove it by induction. (The translation is worth 30%, and the proof is worth 70%.):

    (*) If n is an arbitrary natural number and $n \neq 1$ then $n - 1$ is a natural number.

50. Translate the following statement into formal language (using quantifiers) and prove it by induction. (The translation is worth 30%, and the proof is worth 70%.):

    (*) The sum of two natural numbers is a natural number.

    (HINT: This problem involves a statement with two universally quantified variables. To prove it, you should fix one of the variables and do induction on the other variable.)

51. Translate the following statement into formal language (using quantifiers) and prove it by induction. (The translation is worth 30%, and the proof is worth 70%.):

    (*) The product of two natural numbers is a natural number.

    (HINT: This problem involves a statement with two universally quantified variables. To prove it, you should fix one of the variables and do induction on the other variable.)

52. i. Give an inductive definition of " a^n ", for a real number a and a natural number n.
    ii. Prove by induction that if n is an arbitrary natural number then $n < 2^n$.

53. i. Give an inductive definition of the "factorial" of a natural number.
    ii. Compute 6! using the inductive definition.
    iii. Prove by induction that if n is an arbitrary natural number then $n! \leq n^n$.

54. i. Give an inductive definition of the expression $\sum_{k=1}^{n} a_k$ (if n is a natural number and $a_1, \ldots, a_n$ are real numbers.
    ii. Compute $\sum_{k=1}^{5} k^3$ using the inductive definition.
    iii. Prove by induction that if n is an arbitrary natural number then $n! \leq n^n$.

55. Consider the statement

$$(*) \qquad (\forall n \in \mathbb{N})\Big(\sum_{k=1}^{n} k^2 \;=\; \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\Big).$$

    i. Verify (*) for $n = 1, 2$ and 3.
    ii. Prove (*) by induction.

56. Consider the statement

$$(*) \qquad (\forall n \in \mathbb{N})\Big(\sum_{k=1}^{n} k^3 \;=\; \frac{n^2(n+1)^2}{4}\Big).$$

    i. Verify (*) for $n = 1, 2$ and 3.
    ii. Prove (*) by induction.

57. Prove by induction that

$$(*) \qquad (\forall n \in \mathbb{N})(\forall x \in \mathbb{R})(x > 0 \Longrightarrow (1+x)^n \geq 1 + nx).$$

58. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

Claim. If n is a natural number and $(a_1, a_2, \ldots, a_n)$ is a list of n real numbers, then $a_1 = a_2 = \cdots = a_n$.

Proof. We prove our conclusion by induction.

Let $P(n)$ be the statement "if $(a_1, a_2, \ldots, a_n)$ is a list of n real numbers, then $a_1 = a_2 = \cdots = a_n$".

The base case. If $n = 1$, then $P(n)$ says that if a_1 is a real number then a_1 is equal to itself, which is clearly true. So $P(1)$ is true.

The inductive step. We want to prove that

$$(\#) \qquad (\forall n \in \mathbb{N})(P(n) \Longrightarrow P(n+1))\,.$$

Let n be an arbitrary natural number.

We want to prove that $P(n) \Longrightarrow P(n+1)$.

Assume that $P(n)$ is true.

We want to prove that $P(n+1)$ is true.

But $P(n+1)$ says that "if $(a_1, a_2, \ldots, a_n, a_{n+1})$ is a list of $n+1$ real numbers, then $a_1 = a_2 = \cdots = a_n = a_{n+1}$".

So, in order to prove $P(n+1)$, we let $(a_1, a_2, \ldots, a_n, a_{n+1})$ be an arbitrary list of $n+1$ real numbers, and prove that the $n+1$ numbers in this list are all equal.

The list $(a_1, a_2, \ldots, a_n)$ is a list of n real numbers,

And our inductive assumption (that $P(n)$ is true) says that if you have a list of n natural numbers then the numbers in the list are all equal.

So $a_1 = a_2 = \cdots = a_n$.

Also, the list $(a_2, a_3, \ldots, a_n, a_{n+1})$ is a list of n real numbers,

And our inductive assumption (that $P(n)$ is true) says that if you have a list of n natural numbers then the numbers in the list are all equal.

Therefore $a_2 = a_3 = \cdots = a_n = a_{n+1}$.

Since $a_1 = a_2 = \cdots = a_n$ and $a_2 = a_3 = \cdots = a_n = a_{n+1}$, all the a_j, for $j = 1, 2, \ldots, n+1$, are equal.

That is,

$$(\%) \qquad a_1 = a_2 = \cdots = a_n = a_{n+1}\,.$$

Since we have proved assertion (%) for a completely arbitrary list $(a_1, a_2, \ldots, a_n, a_{n+1})$ of $n+1$ real numbers, it follows that $P(n+1)$ is true.

Since we have proved $P(n+1)$ assuming $P(n)$, we have proved that $P(n) \Longrightarrow P(n+1)$.

Since we have proved that $P(n) \Longrightarrow P(n+1)$ for an arbitrary natural number n, we have shown that (#) is true.

Since $P(1)$ is true, and (#) is true, it follows from the principle of mathematical induction that $(\forall n \in \mathbb{N})P(n)$.

In other words, we have proved that, if n is an arbitrary natural number, and $(a_1, a_2, \ldots, a_n)$ is a list of n real numbers, then the numbers $a_1, a_2, \ldots, a_n$ are all equal. **Q.E.D.**