Homework Assignment no. 4, due on Thursday, Feb. 21

1. Let S be the statement "If x is a positive real number, then $x + \frac{4}{x} \geq 4$".

i. Write S in symbolic language. (Do not forget the quantifier, and make sure that your statement correctly indicates the scope of the quantifier!)

ii. Prove S.

2. Let S be the statement "If a, b are real numbers, then $ab \leq \frac{a^2+b^2}{2}$ ".

i. Write S in symbolic language. (Do not forget the quantifiers!)

ii. Prove S.

3. Let S be the statement "If a_1, a_2, b_1, b_2 are real numbers, then $a_1b_1+a_2b_2 \leq (a_1^2 + a_2^2)^{1/2}(b_1^2 + b_2^2)^{1/2}$ ".

i. Write S in symbolic language. (Do not forget the quantifiers!)

ii. Prove S.

4. A function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to be **convex** if, whenever x_1, x_2, t are real numbers such that $0 \leq t \leq 1$, it follows that $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$. Let S be the statement "the function f given by

$$f(x) = x^2 \quad \text{for all} \quad x \in \mathbb{R}$$

is convex".

i. Write S in symbolic language. (Do not forget the quantifiers! And do not include the word "convex" in the statement.)

ii. Prove S.

5. Book, exercises 1.4, problems 6 (all six parts of it), 7(a),(j),(m). (Note: it may happen that for at least one of the questions in these problems the answer is "this cannot be proved because it isn't true", in which case you would have to prove that the statement isn't true.)

6. For each of the following statements, explain why the statement is false. Give a *clear* explanation, that will convince someone who does not know that the statement is false and doesn't believe it is false until you persuade them.

i. An even integer cannot be prime.

ii. The number 1,001 is prime.

iii. The number 1,003 is prime.

iv. The number 1,007 is prime.

v. The sum of two prime numbers is prime.

vi. The sum of two prime numbers can never be prime.

vii. If a natural number p is such that whenever a, b are integers such that p divides ab it follows that p divides a or p divides b, then p is prime. (Write the statement in symbolic language first.)

viii. The proposition "$2 + 2 = 5$" is a contradiction.

ix. The proposition "$2 + 2 = 4$" is a tautology.

x. The Eiffel tower is in Paris if and only of the chemical symbol for Helium is H.

xi. $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})n < m$.

xii. $(\exists n \in \mathbb{Z})(\forall m \in \mathbb{Z})n < m$.

xiii. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(xz = yz \Longrightarrow x = y)$.

xiv. If a, b, c are integers and c is the greatest common divisor of a and b then $(\forall d \in \mathbb{Z})(d|a \wedge d|b)$.

xv. If a, b, c are integers and c is the greatest common divisor of a and b then $(\forall d \in \mathbb{Z})((d|a \wedge d|b) \wedge d \leq c)$.

xvi. If $P(x)$ and $Q(x)$ are predicates with the open variable x (such as, for example, "x is a cow", or "x likes to drink coffee", or "x is divisible by 43"), then $(\forall x)(P(x) \vee Q(x)) \Longrightarrow ((\forall x)P(x) \vee (\forall x)P(x))$. (You get to choose the range of the variable x, i.e., the set that the book calls "the universe".)

xvii. If $P(x, y)$ is a predicate with two open variables (such as, for example, "x is taller than y", or "x is y's mother", or "x is divisible by y"), then $(\forall x)(\exists y)P(x, y) \Longrightarrow (\exists y)(\forall x)P(x, y)$. (You get to choose the range of the variables x, y, i.e., the set that the book calls "the universe".)

xviii. To prove that a statement cannot be proved, it suffices to write down the beginning of a proof and show that at one point one cannot go on. (For example, in last week's problem about giraffes, cows and sheep, some students observed, correctly, that at one point in the proof they needed to use the fact that $(\exists x)C(x)$—that is, "cows exist"—and that was not in the axioms; the students then concluded that the conclusion cannot be proved.)