p.711 #16 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$u(0,y)=0,\ u(2,y)=(2-y)y.$

$u(x,0)=0,\ u(x,2)=\begin{cases} x & 0\le x\le 1 \\ 2-x & 1\le x<2\end{cases}$

Sol^n: We add solutions to the following 2 problems:

① $v_{xx}+v_{yy}=0$

$v(0,y)=0,\ v(2,y)=0$

$v(x,0)=0,\ v(x,2)=\begin{cases} x & 0<x\le 1 \\ 2-x & 1\le x<2\end{cases}$

② $w_{xx}+w_{yy}=0$

$w(0,y)=0,\ w(2,y)=y(2-y)$

$w(x,0)=0,\ w(x,2)=0.$

Problem ①:

$$v(x,y)=\sum_{n=1}^{\infty}\left(A_n\cosh\frac{n\pi y}{2}+B_n\sinh\frac{n\pi y}{2}\right)\sin\frac{n\pi x}{2}.$$

$$v(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{2}=0\Rightarrow A_n=0 \text{ for all } n.$$

So $v(x,2)=\sum_{n=1}^{\infty}B_n\sinh n\pi\sin\frac{n\pi x}{2}=\begin{cases} x & 0<x\le 1 \\ 2-x & 1\le x<2\end{cases}$

So $B_n=\frac{1}{\sinh n\pi}\left[\int_0^1 x\sin\frac{n\pi x}{2}\,dx+\int_1^2(2-x)\sin\frac{n\pi x}{2}\,dx\right]$

$u=x \quad dv=\sin\frac{n\pi}{2}x$

$du=dx \quad v=-\frac{2}{n\pi}\cos\frac{n\pi x}{2}$

$w=2-x$

$dw=-dx$

$$B_n=\frac{1}{\sinh n\pi}\left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^1+\frac{2}{n\pi}\int_0^1\cos\frac{n\pi x}{2}dx-\frac{2(2-x)\cos\frac{n\pi x}{2}}{n\pi}\Big|_1^2-\frac{2}{n\pi}\int_1^2\cos\frac{n\pi x}{2}\right]$$

$$=\frac{1}{\sinh n\pi}\left[-\frac{2}{n\pi}\cos\frac{n\pi}{2}+\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{2}{n\pi}(-1)^n+\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{1}{\sinh n\pi}\left[\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{2}{n\pi}\left[(-1)^n+\cos\frac{n\pi}{2}\right]\right].$$