Math 250–Section #4   Hourly #1

Name: ____________________

1.[10 pts] Evaluate the determinant using cofactor expansions

$$\begin{vmatrix} 2 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 4 & 1 \\ 2 & 3 & 0 & 0 \end{vmatrix}$$

Answer:
We expand along the last row because it contains two zeros, and compute the determinants of the 3×3 matrices

$$\begin{aligned} \det &= 2(-1)^{4+1}\begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & -1 \\ -1 & 4 & 1 \end{vmatrix} + 3(-1)^{4+2}\begin{vmatrix} 2 & -3 & 1 \\ 0 & 2 & -1 \\ 3 & 4 & 1 \end{vmatrix} \\ &= -2((4-3+4)-(-2-3-8)) + 3((4+9+0)-(6-8+0)) \\ &= -2(18) + 3(15) = 9 \end{aligned}$$

2. [12 pts] Find all the values for t for which the resulting system of equations (a) has no solution, (b) a unique solution, and (c) infinitely many solutions.

$$\begin{array}{ccccccc} x & + & y & - & z & = & 2 \\ x & + & 2y & + & z & = & 3 \\ x & + & y & + & (t^2-5)z & = & t \end{array}$$

Answer: Let us us Gauss–Jordan's on this system

$$\left.\begin{array}{ccr} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & t^2-5 \end{array}\right| \begin{array}{c} 2 \\ 3 \\ t \end{array}$$

Using the 1 in the position $(1,1)$ as a pivot, we get

$$\left.\begin{array}{ccr} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & t^2-4 \end{array}\right| \begin{array}{r} 2 \\ 1 \\ t-2 \end{array}$$

We are now ready: If $t \neq \pm 2$, we get a unique solution since we have 3 pivots. If $t = 2$, we have an infinite number of solutions. If $t = -2$, it is impossible.

3. [12 pts] Let V be the set of all 3×3 matrices with the format

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

Show that V is a *subspace* of the vector space of all 3×3 matrices.

Answer: Just observe that if you add two matrices of this kind

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & a' & b' \\ -a' & 0 & c' \\ -b' & -c' & 0 \end{bmatrix} = \begin{bmatrix} 0 & a+a' & b+b' \\ -a-a' & 0 & c+c' \\ -b-b' & -c-c' & 0 \end{bmatrix}$$

we get another matrix of the same kind.
Also, if we multiply it by a scalar,

$$r \cdot \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & ra & rb \\ -ra & 0 & rc \\ -rb & -rc & 0 \end{bmatrix}$$

we get another matrix of the same kind.

4. [10 pts] Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Answer: Using either of the methods we know [Gauss-Jordan algorithm or the formula involving the adjoint matrix] you should get:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

5.[12 pts] (a) Explain what is a *linear combination* of vectors.
(b) Find out whether the vector $(1, 0, 1, 2)$ is a linear combination of $(2, 1, 3, 0)$, $(7, 3, 1, -1)$ and $(0, 1, 4, 3)$.

Answer: (a) A linear combination of the *vectors* $v_1, \ldots, v_n$ is a vector

$$v = c_1 v_1 + \cdots + c_n v_n,$$

where $c_1, \ldots, c_n$ are scalars.

(b): To answer this item, you must check whether it is possible to solve for c_1, c_2 and c_3 the equation

$$(1, 0, 1, 2) = c_1(2, 1, 3, 0) + c_2(7, 3, 1, -1) + c_3(0, 1, 4, 3).$$

Using Gaussian elimination, it turns out to be impossible.

6. [12 pts] Let $u = (1, 2, -1)$ and $v = (3, -1, 1)$ be vectors of $\mathbb{R}^3$.
(a) Show that u is perpendicular [orthogonal] to v.
(b) Find a third nonzero vector $w = (a, b, c)$ perpendicular to both u and v.

Answer: (a) $u \cdot v = 1 \cdot 3 + 2 \cdot (-1) + (-1) \cdot 1 = 3 - 2 - 1 = 0$. Since we know the formula $u \cdot v = ||u|| \cdot ||v|| \cos\theta$, θ being the angle between the vectors: $\theta = 90^{\text{deg}}$

If (a, b, c) is $\perp$ to both u and v, this sets up the system of equations

$$\begin{array}{rcrcrcl} a & + & 2b & - & c & = & 0 \\ 3a & - & b & + & c & = & 0 \end{array}$$

for which we want a nonzero solution. Easy: adding we get $b = -4a$, so we set $a = 1$ and get $b = -4$ and $c = -7$ and a vector is $(1, -4, -7)$. (Any nonzero multiple also works, such as $(-1, 4, 7)$ as many people found.)

7. [10 pts] If A, B and C are $n \times n$ matrices of determinants 2, 1 and 3 respectively. Explain the statements:
(a) The product ABC is invertible.

(b) The inverse of ABC is

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

Answer: (a) $\det(ABC) = \det(A)\det(B)\det(C) = (2)(1)(3) = 6 \neq 0$. This means that ABC has an inverse.

$$(ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} = ABB^{-1}A^{-1} = AA^{-1} = I_n,$$

which checks the statement.

8. [10 pts] Prove that the set of all polynomials $f(t)$, of degree at most 2, and such that $f'(1) = 0$ [$f'(t)$ denotes the derivative] is a subspace.

Answer: Our polynomials are of the form $f(t) = at^2 + bt + c$ whose derivative for $t = 1$ is zero, $2a + b = 0$. There are at least two ways to write down the answer:
1.

$$(at^2 - 2at + c) + (dt^2 - 2dt + e) = (a+d)t^2 - 2(a+d)t + (c+e)$$

which has the same format. There is a similar argument for $r(at^2 - 2at + c)$.

2. It is better just to observe: If $f'(1) = g'(1) = 0$, then $(f(t) + g(t))'(1) = f'(1) + g'(1) = 0 + 0$ and $(rf(t))'(1) = rf'(1) = 0$.

9. [12 pts] Using Cramer's rule, solve the system of equations

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{bmatrix} \cdot \mathbf{x} = \mathbf{b},$$

for 3 choices of $\mathbf{b}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Further, if you use the solutions, respectively, as the columns of a 3×3 matrix B, how is it related to the matrix of the system?

Answer:
You should get for solutions the columns of the matrix

$$\begin{bmatrix} 6 & -7/2 & 1/2 \\ -8 & 6 & -1 \\ 3 & -5/2 & 1/2 \end{bmatrix}$$

Note that you really looking for $AB = I_3$.