Complex-Conjugate Number

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The complex-conjugate number, or conjugate number, of a complex number $z = x + iy$ with real part x and imaginary part y is the number $x - iy$, usually denoted $\overline{z}$ or z^*. (The notation z^* is more frequent in quantum physics.)

The definition implies the following properties. Every complex number is the conjugate of its conjugate:

$$\overline{\overline{z}} = z\,, \quad \text{or} \quad (z^*)^* = z\,. \tag{1}$$

That is, conjugate numbers come in pairs, except for the cases in which a number is conjugate to itself; the latter case occurs if and only if the number $z = x + iy$ has vanishing imaginary part y, that is if and only if z is real:

$$z^* = z \Leftrightarrow z \in \mathbb{R}\,. \tag{2}$$

Conjugation, i.e., the operation of taking the conjugate, defines a mapping $^* : \mathbb{C} \to \mathbb{C}$. This mapping is real-linear, i.e.,

$$(z + w)^* = z^* + w^* \quad \text{and} \quad (\lambda z)^* = \lambda(z^*) \tag{3}$$

for all $z, w \in \mathbb{C}$ and $\lambda \in \mathbb{R}$. It is not complex-linear, as there exist $z, w \in \mathbb{C}$ for which $(zw)^* \neq z(w^*)$, but instead conjugation is multiplicative, i.e.,

$$(zw)^* = z^* w^*\,. \tag{4}$$

If the set of complex numbers is represented as a plane then conjugation corresponds to reflection across the real axis (see Fig. 1). Complex-conjugate numbers have equal modulus (absolute value), $r = |z| = |z^*|$, and opposite phase angles (arguments) $\varphi(z) = -\varphi(z^*)$. As a related fact, for all $\varphi \in \mathbb{R}$ and $z \in \mathbb{C}$,

$$(e^{i\varphi})^* = e^{-i\varphi} \quad \text{and} \quad (e^z)^* = e^{z^*}\,. \tag{5}$$

Moreover,

$$z^* z = |z|^2\,. \tag{6}$$

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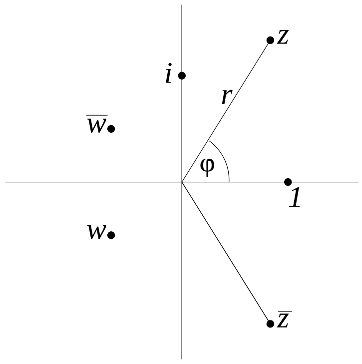

Figure 1: The complex plane, with example numbers z and w and their complex conjugate numbers $\overline{z}$ and $\overline{w}$.

The real and imaginary part of a complex number z can be expressed using z and z^*:

$$\operatorname{Re} z = \tfrac{1}{2}(z + z^*)\,, \quad \operatorname{Im} z = \tfrac{1}{2i}(z - z^*)\,. \tag{7}$$

For a function $f(z)$ of a complex variable z one defines the *Wirtinger derivatives*

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big) \tag{8}$$

$$\frac{\partial f}{\partial z^*} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial v}\Big)\,, \tag{9}$$

where $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $u = \operatorname{Re} f$, and $v = \operatorname{Im} f$.