

# Dirac Notation

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The “bra”-and-“ket” notation (introduced by Dirac) uses the symbols  $|\psi\rangle$  and  $\langle\psi|$  for vectors in and linear forms on Hilbert space.

In this notation, if  $\psi$  is a vector in Hilbert space  $\mathcal{H}$  then  $|\psi\rangle$  is just another notation for  $\psi$ , and  $\langle\psi|$  means the mapping  $\phi \mapsto \langle\psi|\phi\rangle$ , a *linear form*  $\mathcal{H} \rightarrow \mathbb{C}$  defined using the scalar product  $\langle\cdot|\cdot\rangle$  of  $\mathcal{H}$ . Turning  $|\psi\rangle$  into  $\langle\psi|$  is a conjugate-linear operation:  $\langle\phi + \psi| = \langle\phi| + \langle\psi|$  and  $\langle z\psi| = z^*\langle\psi|$  for  $z \in \mathbb{C}$ .

Linear forms are also called *co-vectors*, and the set of all linear forms is called the *dual space*. Thus,  $\langle\psi|$  is the co-vector naturally associated with the vector  $\psi$ . The difference between vectors and co-vectors is basically the same as the difference between a column and a row in matrix theory (linear algebra), or between the contravariant components  $u^\mu$  and the covariant components  $u_\mu$  of a 4-vector in relativity theory. The *Riesz lemma* of functional analysis implies that every continuous linear form  $\mathcal{H} \rightarrow \mathbb{C}$  (only the continuous one are usually considered) is of the form  $\phi \mapsto \langle\psi|\phi\rangle$  for a suitable  $\psi \in \mathcal{H}$ ; as a consequence, there is a one-to-one correspondence between vectors and co-vectors, and  $\mathcal{H}$  is, up to complex conjugation, its own *continuous dual space*.

As the notation suggests, the scalar product  $\langle\phi|\psi\rangle$  is the same as the linear form  $\langle\phi|$  applied to the vector  $|\psi\rangle$ . That is why Dirac called  $\langle\phi|$  a “bra” vector and  $|\psi\rangle$  a “ket” vector: bra + ket = bracket; that is, when written one after the other, they form the scalar product. When written in the opposite order,  $|\psi\rangle\langle\phi|$ , they form not a number but an operator  $|\chi\rangle \mapsto |\psi\rangle\langle\phi|\chi\rangle$ . In particular, if  $\|\psi\| = 1$  then  $|\psi\rangle\langle\psi|$  is the projection to the 1-dimensional subspace spanned by  $\psi$ . Moreover, if  $T$  is an operator then  $\langle\phi|T|\psi\rangle$  means the same as  $\langle\phi|T\psi\rangle$  or  $\langle T^*\phi|\psi\rangle$ .

The Dirac notation has another advantage: If some vectors  $\psi_n$  are indexed by some index  $n$  then one can write  $|n\rangle$  instead of  $|\psi_n\rangle$ , provided there is no danger of misunderstanding. For example, an orthonormal basis can be denoted  $|1\rangle, |2\rangle, |3\rangle, \dots$ , so that the matrix elements of an operator  $T$  can be written as  $T_{nm} = \langle n|T|m\rangle$ , the identity operator as

$$I = \sum_n |n\rangle\langle n|, \tag{1}$$

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and the orthonormality relation as

$$\langle n|m\rangle = \delta_{nm}. \quad (2)$$

An extension of the Dirac equation concerns *generalized orthonormal bases* (such as the position basis in quantum mechanics), which consist of a unitary isomorphism  $\mathcal{H} \rightarrow L^2(\mathcal{Q})$  and thus permits to write every vector  $\psi \in \mathcal{H}$  as a square-integrable function  $\psi(q)$  on some set  $\mathcal{Q}$  (such as  $\mathcal{Q} = \mathbb{R}^{3N}$ ), whereas an orthonormal basis in the ordinary sense permits to write a vector  $\psi \in \mathcal{H}$  as a sequence  $\langle 1|\psi\rangle, \langle 2|\psi\rangle, \dots$  of numbers, the components of  $\psi$ . The extended Dirac notation introduces the symbol  $|q\rangle$  as if the generalized basis was an ordinary basis, and to treat this symbol as if it denoted a vector in  $\mathcal{H}$ . (In quantum mechanics, in fact,  $|q\rangle$  of the position basis represents the Dirac delta function  $\delta(\cdot - q)$ , which is not a square-integrable function and thus does not belong to  $\mathcal{H}$ ; similarly, the kets of the momentum basis  $|k\rangle$  represent the non-normalizable functions  $x \mapsto e^{ikx}$ .) One writes

$$\psi(q) = \langle q|\psi\rangle, \quad (3)$$

the orthonormality relation

$$\langle q|q'\rangle = \delta(q - q'), \quad (4)$$

and the identity operator as

$$I = \int |q\rangle\langle q| dq. \quad (5)$$