Predictions and Primitive Ontology in Quantum Foundations: A Study of Examples

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Abstract

A major disagreement between different views about the foundations of quantum mechanics concerns whether for a theory to be intelligible as a fundamental physical theory it has to involve a "primitive ontology" (PO), i.e., variables describing the distribution of matter in 4-dimensional space-time. In this paper, we want to convey some of the reasons why we think a PO is indispensable. We focus on the role that the PO plays for extracting predictions from a given theory and discuss valid and invalid ways of extracting predictions. To this end, we investigate a number of examples based on toy models built from the elements of known theories about the foundations of quantum mechanics.

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Dedicated to Tim Maudlin on the occasion of his 50th birthday

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1 Introduction

We consider in this paper certain quantum theories without observers, particularly the Ghirardi–Rimini–Weber (GRW) theory [30, 14], the simplest proposal for implementing spontaneous collapses of the wave function; see [14, 31, 38, 28, 3] for introductory presentations. A number of empirical predictions of this theory have been derived [30, 37, 34, 1, 8, 9]; they deviate from those of quantum mechanics but only so slightly that no experimental test has been possible so far [7]. However, the logical clarity of these derivations leaves to be desired. We do not dispute that the claimed predictions are indeed predictions of the GRW theory; we see a gap in the derivation, though. The situation is similar to that of a calculation that yields the correct result but is not mathematically rigorous. Here, the problem is not one of mathematical rigor but of philosophical clarity. We argue that any valid derivation needs to be based on the behavior of the *primitive ontology* (PO) of the theory (i.e., the variables describing the distribution of matter in space and time). The derivations cited focus exclusively on the wave function and do not refer to the PO, and that means that they either use tacit assumptions about the PO or contain steps that are not justified.

Our goal in this paper is to convey how a valid derivation of empirical predictions works and where it uses the PO and the laws governing it. We do so mainly by means of examples. Some of the examples (Section 3) are valid derivations found in the literature concerning GRW or other serious theories, but most of them (Section 4) are novel and concern toy theories that we have made up for the purposes of this paper. Some of these toy theories make completely wrong predictions; but that does not keep them from exemplifying valid derivations of predictions.

We simply say "predictions" for "empirical predictions," i.e., those predictions that can be tested empirically, and "empirical contents" for the sum of all empirical predictions of a theory. We do not, in this paper, compute any specific predictions for specific experiments.

The versions of GRW theory that we refer to are GRWm and GRWf, which correspond to two different choices of PO [3]: the *matter density ontology* (GRWm) and the *flash ontology* (GRWf). We recall their definitions in Section 2. Our discussion focuses on the *non-relativistic* GRWm and GRWf theory. See [44] for a discussion of how the point of view of PO resolves the paradoxes that some researchers have seen in the GRW theories.

2 The GRWm and GRWf Theories

Detailed introductions to the GRWm and GRWf theories have been given recently in [3, 42, 33]. Here we give only a brief description.

In both GRWm and GRWf the evolution of the wave function follows, instead of the Schrödinger equation, a stochastic jump process in Hilbert space, called the GRW process. Consider a quantum system of (what would normally be called) N "particles," described by a wave function $\psi = \psi(x_1, \dots, x_N)$, $x_i \in \mathbb{R}^3$, $i = 1, \dots, N$. For any point x in $\mathbb{R}^3$, define on the Hilbert space of the system the *collapse rate operator*

$$\Lambda_i(x) = \frac{1}{(2\pi\sigma^2)^{3/2}} \, e^{-\frac{(\widehat{Q}_i - x)^2}{2\sigma^2}} \,, \tag{1}$$

where $\widehat{Q}_i$ is the position operator of "particle" i. Here σ is a new constant of nature of order of $10^{-7}\,\mathrm{m}$. Let λ be another new constant of nature of order of $10^{-15}\,\mathrm{s}^{-1}$, called the collapse rate per particle. At random times $T_1, T_2, \dots$ that occur with constant rate $N\lambda$, the wave function "collapses," i.e., it changes according to

$$\psi_{T_k} \mapsto \psi_{T_k+} = \frac{\Lambda_{I_k}(X_k)^{1/2}\psi_{T_k}}{\|\Lambda_{I_k}(X_k)^{1/2}\psi_{T_k}\|} \,, \tag{2}$$

where I_k is chosen at random (independently of the past) from $\{1, \dots, N\}$ with uniform distribution and X_k is chosen at random from $\mathbb{R}^3$ with distribution[1]

$$\mathbb{P}(X_k \in dx | T_1, I_1, X_1, \ldots, T_{k-1}, I_{k-1}, X_{k-1}, T_k, I_k) = \langle \psi_{T_k} | \Lambda_{I_k}(x) | \psi_{T_k} \rangle dx. \qquad (3)$$

Between the collapse times $T_1, T_2, \ldots$, the wave function evolves unitarily, according to Schrödinger's equation, so that

$$\psi_t = U_{t-T_k} \psi_{T_k+}, \qquad (4)$$

where T_k is the last collapse time before t and U_t is the unitary operator $U_t = e^{-\frac{i}{\hbar}Ht}$ corresponding to the standard Hamiltonian H governing the system, e.g., given, for N spinless particles, by

$$H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \nabla_k^2 + V, \qquad (5)$$

where m_k, $k = 1, \ldots, N$, are the masses of the particles, and V is the potential energy function of the system. Since the T_k, I_k, X_k are random, ψ_t is also random.

In other words, when the wave function is ψ a collapse with center x and label i occurs at rate

$$r(x, i|\psi) = \lambda \langle \psi | \Lambda_i(x) | \psi \rangle \qquad (6)$$

and when this happens, the wave function changes to $\Lambda_i(x)^{1/2}\psi$ (times a normalizing factor). (There exist generalizations of this mathematical scheme, according to which the collapse rate operators $\Lambda_i(x)$ do not have to be of the form (1) but could be other positive operators, could depend on time and on the previous flashes. See [40, 43] for details.)

We have described the law for the evolution of the wave function. We now turn to the primitive ontology (PO). In the subsections below we present two versions of the GRW theory, based on two different choices of the PO, namely the *matter density ontology* (in Section 2.1) and the *flash ontology* (in Section 2.2).

2.1 GRWm

GRWm postulates that, at every time t, matter is continuously distributed in space with density function $m(x, t)$ for every location $x \in \mathbb{R}^3$, given by

$$m(x, t) = \sum_{i=1}^{N} m_i \int_{\mathbb{R}^{3N}} dq_1 \cdots dq_N \, \delta(q_i - x) \left| \psi_t(q_1, \ldots, q_N) \right|^2. \qquad (7)$$

In words, one starts with the $|\psi|^2$–distribution in configuration space $\mathbb{R}^{3N}$, then obtains the marginal distribution of the i-th degree of freedom $q_i \in \mathbb{R}^3$ by integrating out all

[1]Hereafter, when no ambiguity could arise, we use the standard notations of probability theory, according to which a capital letter, such as X, is used to denote a random variable, while the the values taken by it are denoted by small letters; $X \in dx$ is a shorthand for $X \in [x, x + dx]$ in one dimension, etc.

other variables q_j, $j \neq i$, multiplies by the mass associated with q_i, and sums over i. Alternatively, (7) can be rewritten as

$$m(x,t) = \langle \psi_t | \tilde{\Lambda}(x) | \psi_t \rangle \tag{8}$$

with $\tilde{\Lambda}(x) = \sum_i m_i \, \delta(\widehat{Q}_i - x)$.

2.2 GRWf

According to GRWf, the PO is given by "events" in space-time called flashes, mathematically described by points in space-time. In GRWf matter is neither made of particles following world lines, nor of a continuous distribution of matter such as in GRWm, but rather of discrete points in space-time, in fact finitely many points in every bounded space-time region.

In the GRWf theory, the space-time locations of the flashes can be read off from the history of the wave function: every flash corresponds to one of the spontaneous collapses of the wave function, and its space-time location is just the space-time location of that collapse. The flashes form the set

$$F = \{(X_1, T_1), \ldots, (X_k, T_k), \ldots\} \tag{9}$$

(with $T_1 < T_2 < \ldots$). Alternatively, we may postulate that flashes can be of N different types ("colours"), corresponding to the mathematical description

$$F = \{(X_1, T_1, I_1), \ldots, (X_k, T_k, I_k), \ldots\}\,. \tag{10}$$

Note that if the number N of the degrees of freedom in the wave function is large, as in the case of a macroscopic object, the number of flashes is also large (if $\lambda = 10^{-15}$ s^{-1} and $N = 10^{23}$, we obtain 10^8 flashes per second). Therefore, for a reasonable choice of the parameters of the GRWf theory, a cubic centimeter of solid matter contains more than 10^8 flashes per second. That is to say that large numbers of flashes can form macroscopic shapes, such as tables and chairs. That is how we find an image of our world in GRWf.

3 Predictions and Primitive Ontology

When saying,

"the pointer of the apparatus is pointing to the value z," (11)

what does one mean? Two answers are:

(a) The wave function lies (approximately) in the subspace corresponding to the pointer pointing to z.

(b) The PO is such that the matter of the pointer is in the position corresponding to z.

These two options lead to two very different attitudes towards the derivation of the empirical predictions of GRW theories, in particular towards what needs to be done in a valid derivation. One could argue for (a) that if we asked physicists what *they* mean when making a statement like (11), the majority would opt for (a), as most physicists do not think in terms of a PO. However, this response largely reflects their philosophical attitude, which one may disagree with. One could argue for (b) that the normal understanding of the statement (11), the understanding prior to any education in quantum physics, is that the pointer consists of some amount of matter, and this matter resides in the location corresponding to z. Thus, the normal understanding is much closer to (b). Our attitude, and our considerations in this paper, are based on option (b).

3.1 Calibration Functions

As a consequence of option (b) above, the PO plays a key role in deriving the empirical predictions, since to say that the outcome Z of a particular experiment was z means that the PO looks in a certain way. In particular, the outcome Z is a function of the PO,

$$Z = \zeta(PO)\,. \tag{12}$$

That is, in GRWf $Z = \zeta_{\mathrm{GRWf}}(F)$, and in GRWm $Z = \zeta_{\mathrm{GRWm}}(m)$. Similarly, the PO plays a key role for the claim of empirical equivalence between two theories (for which there will be several examples in this paper).

Let us elaborate a bit on what the function ζ should look like. In GRWm, it is natural that Z should be a functional of $m(\cdot, t)$, the distribution of matter at the time t when the experiment is completed. What the ζ function does is essentially to read off the outcome from the display of the apparatus. For example, if the outcome is displayed by means of the position of a needle on a scale, ζ should read off from m the position of the needle, and may concretely be the following: Suppose the experiment is so arranged that the region $R \subseteq \mathbb{R}^3$, for simplicity a cuboid $R = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, contains no other matter than the needle, and suppose the scale is along the x_1-axis between a_1 and b_1. Then the mean x_1 coordinate of the matter distribution at time t inside R is given by

$$\langle x_1 \rangle = \frac{\int_R dx_1\, dx_2\, dx_3\, x_1\, m(x_1, x_2, x_3, t)}{\int_R dx_1\, dx_2\, dx_3\, m(x_1, x_2, x_3, t)}\,, \tag{13}$$

and a typical choice of calibration function would be

$$\zeta(m) = z_0 + \alpha \langle x_1 \rangle \tag{14}$$

with suitable proportionality constant α. Note that ζ is a functional of $m(\cdot, t)$. If we think that Z is a discrete variable, we may replace that with a suitable step function, such as

$$\zeta(m) = z_0 + [\alpha \langle x_1 \rangle]\,, \tag{15}$$

where $[z]$ denotes the nearest integer to the real number z.

In GRWf, Z must depend on the history in an entire time interval, say $[t, t+\Delta t]$ with t the time when the experiment is completed and Δt, say, a millisecond. In the example of the needle pointing to the outcome, ζ needs to read off the position of the needle from the flashes and rescale it accordingly. As a concrete example, suppose again the region R contains no other matter than the needle, and suppose the scale is along the x_1-axis between a_1 and b_1; then the mean x_1 coordinate of the flashes during $[t, t+\Delta t]$ in R is given by

$$\langle x_1 \rangle = \frac{\sum_k 1_{t \le T_k \le t+\Delta t}\, 1_{X_k \in R}\, (X_k)_1}{\sum_k 1_{t \le T_k \le t+\Delta t}\, 1_{X_k \in R}}\,, \tag{16}$$

and a typical choice of calibration function would be given in terms of this $\langle x_1 \rangle$ by (14) or (15).

3.2 Taking the PO Seriously

As a consequence of the view that the PO represents the matter, we are forced to take the PO seriously. Here is a simple example: In a relativistic context, the PO is not allowed to change in an arbitrary way when changing the Lorentz frame (or the spacelike surface of curved coordinates); if the PO tells us how much matter is in a certain cubic centimeter of space in a certain second of time, then this quantity must not depend on how the surfaces of coordinate time are chosen in another galaxy.

As another example for what it means to take the PO seriously, we look at a difficulty that arose in [6] when Bassi gave an over-simplified discussion of how to derive in GRWm that, in a suitable situation, a pointer is pointing to z. Bassi decomposed the coordinates $x_1, \ldots, x_N \in \mathbb{R}^3$ of the configuration space $\mathbb{R}^{3N}$ of a pointer consisting of N "particles" into the center of mass $x_{\mathrm{cm}} = \sum_i m_i x_i / \sum_j m_j \in \mathbb{R}^3$ and relative coordinates $r_1, \ldots, r_{3N-3}$, and derived that in this decomposition the wave function, in a suitable situation, is approximately a product $\psi_{\mathrm{cm}}(x_{\mathrm{cm}})\, \psi_{\mathrm{rel}}(r_1, \ldots, r_{3N-3})$, where the first factor, the center-of-mass wave function ψ_{cm}, is a very narrow wave packet. Bassi suggests that therefore the matter density associated with the center-of-mass, $m_{\mathrm{cm}}(x,t) = |\psi_{\mathrm{cm}}(x,t)|^2$, is very narrow, too (thus making the position of the pointer precisely defined). The difficulty with this argument is that m_{cm} is not the right quantity to look at: It is not the PO, not the matter, not real, just a mathematical quantity. What counts is whether m as defined in (7) is concentrated in the right location, and there is no simple relation between m and m_{cm}. For example, the width of m_{cm} is, in realistic examples, 10^{-13} m whereas that of m is 10^{-3} m (= the width of the pointer). So Bassi's argument fails to take the PO seriously.

The issue of taking the PO seriously also arises in the context of studying the limitations on knowledge in GRWm and GRWf, a topic we plan to discuss in detail in a future work (and that we have outlined in [33]): Inhabitants of a GRWf or GRWm world cannot measure the times and locations of the collapses, although these values are well defined according to these theories. That is, some things that are real cannot always be measured with arbitrary accuracy. But if something cannot be measured, one may be tempted to not take it seriously. So, if there are limitations to measuring the variables representing the PO, one may be tempted not to take the PO seriously. Needless to say, we recommend to resist the temptation. We have given some discussion in [42] why the conclusion "unobservable, therefore unreal" is not a good one.

David Albert [personal communication, 2008] has expressed the view that GRWm and GRWf are actually not different theories, and the flashes and the m function, as they both supervene on the wave function, are just different tools for reading off physical facts from the wave function, in much the same way as the temperature of a macroscopic body might be measured by means of either a gas thermometer or a mercury thermometer but there is no point in discussing which of the two results, if they differ, is more correct than the other. This is an example of a view that explicitly rejects taking the PO seriously; it is a view we do not share. We will come back to this issue in the course of this paper; we mention now that GRWf becomes relativistic when the equations are suitably modified whereas GRWm does not, a fact that would not be possible if they were the same theory.

Another remark concerns the fine difference between the names "matter density" and "mass density" for the m function. The fact that its definition (7) involves the quantity m_i usually called "the mass of particle i" suggests the name "mass density" for m. The name "matter density," on the other hand, reflects the fundamental meaning of the m function. To illustrate this difference, let us suppose we had postulated, instead of (7), the following formula for the m function:

$$m(x,t) = \sum_{i=1}^{N} e_i \int\limits_{\mathbb{R}^{3N}} dq_1 \cdots dq_N \, \delta(q_i - x) \, \big|\psi_t(q_1, \ldots, q_N)\big|^2 . \tag{17}$$

This is the same equation as (7), but with m_i, the "mass of particle i," replaced by e_i, the "charge of particle i." In this case, it would evidently not be appropriate any more to call the m function the mass density; rather "charge density" would seem more appropriate. At the same time, the physical and metaphysical role and meaning of the m function has not changed in any way—we have only changed the law governing it. That is why this role is better expressed by the name "matter density." In particular, the matter that we postulate in GRWm and whose density is given by the m function does not, by itself, have any such properties like mass or charge; it can only assume various levels of density. A certain disadvantage about the terms "mass density" and "charge density" is that they may easily suggest that both functions coexist: Is it not natural to expect that an extended object possesses both a mass density and a charge density (different from each other)? But in GRWm, depending on whether we postulate (7) or (17), only one of the two functions represents something real (more precisely, represents the PO), whereas the other is a pure mathematical fiction. In other words, in GRWm with (7) as the law of m, the formula on the right hand side of (17) lacks any physical significance, just as it does in GRWf.

3.3 Examples From the Literature

We now look at arguments that have previously been put forward in the literature and that are based on the connection we are discussing between empirical predictions and PO.

1. The first example is a proof of no-signaling (i.e., the impossibility of transmitting messages faster than light between two distant observers, each acting on and observing one of two entangled quantum objects) in GRWf due to Bell [14]. The proof shows that for two non-interacting systems (here, the system can be taken to be one object together with nearby apparatus), the marginal distribution of the flashes pertaining to system 1 does not depend on the entangled wave function except through its reduced density matrix (with system 2 traced out), nor on the Hamiltonian of system 2, and thus not of any message that observer 2 may wish to transmit. Now, *since outcomes of experiments are functions of the flashes*, the outcome that observer 1 sees cannot depend on the message observer 2 may have wished to transmit, qed. In Bell's words [14],

   > Events in one system, considered separately, allow no inference about events in the other, nor about external fields at work in the other, ... nor even about the very existence of the other system. There are no "messages" in one system from the other. The inexplicable correlations of quantum mechanics do not give rise to signalling between noninteracting systems.

   A similar proof was given in [39, 41] for a relativistic version of GRWf.

2. The second example is provided by derivations of empirical predictions from Bohmian mechanics, which can be found in many papers, e.g., [20, 12, 27, 2]. The PO of Bohmian mechanics consists of particles with trajectories, and the outcomes of experiments are (of course) read off from the particle trajectories, and not directly from the wave function.

3. The third example is taken from the study on superselection rules by Colin *et al.* [21]. For superselection rules it is crucial that some self-adjoint operators are *not* observables. In the terminology of Colin *et al.*, a "weak superselection rule" means that no experiment can distinguish between a *superposition* of vectors from different superselection sectors in Hilbert space and a suitable *mixture* thereof. They prove this under suitable conditions for Bohmian mechanics (and certain generalizations to quantum field theory) as well as for GRWm and GRWf, and the proof uses that the outcome of any experiment is a function of the PO. In more detail, they show that (under suitable conditions) the distribution of the flashes in GRWf does not distinguish between a superposition and a suitable mixture ("strong superselection"), and conclude from this that the distribution of outcomes of experiments does not distinguish either. By empirical equivalence between GRWf and GRWm, the same experiment does not distinguish in GRWm.

4. Colin and Struyve [22] discuss whether their Dirac sea model is empirically equivalent with orthodox quantum field theory. This question amounts to whether macroscopic facts such as outcomes of experiments can be read off from their PO.

5. In [33] it has been shown (and outlined before in [3]) that GRWm and GRWf are empirically equivalent.[2]

6. In [33] we have derived a *GRW formalism*, the analog of the quantum formalism for GRWm and GRWf and the general scheme of the predictions of GRWm and GRWf. We now discuss the GRW formalism in some detail in the next section.

3.4 The GRW Formalism

Just like the quantum formalism, the GRW formalism is an algorithm, formulated in terms of operators in Hilbert space, for computing the predictions. Its derivation in [33] from GRWf is based on the PO and does not have the gap we complained about in the opening paragraph of this paper; it provides an example of a valid derivation of predictions. We describe the GRW formalism in this section and a part of its derivation in the next.

Put succinctly, the difference between the quantum and the GRW formalism is: *different evolution, different operators.* "Different evolution" means that the unitary Schrödinger evolution is replaced by a master equation for the density matrix ρ_t (a Lindblad equation, or quantum dynamical semigroup):

$$\frac{d\rho_t}{dt} = -\tfrac{i}{\hbar}[H,\rho_t] + \lambda \sum_{k=1}^{N} \int d^3x\, \Lambda_k(x)^{1/2}\, \rho_t\, \Lambda_k(x)^{1/2} - N\lambda\rho_t\,. \tag{18}$$

Here, H is the Hamiltonian, $\lambda > 0$ is a constant, and the positive operators $\Lambda_k(x)$ are the collapse rate operators as defined in (1). "Different operators" means that a given experiment may be associated with a different operator as the observable in GRWm/GRWf than in quantum mechanics.

To formulate the GRW formalism (or, in fact, the quantum formalism) with appropriate generality, we need the mathematical concepts of *POVM* and *superoperator*, which we now explain briefly.

POVMs. While many quantum experiments are associated with self-adjoint operators, this is not the most general case, which corresponds to *positive-operator-valued measures* (POVMs, also known as "generalized observables"; for an introduction see

[2]Interestingly, although the proof involves some inexactness (as connected to the notion "macroscopic"), it shows that GRWf and GRWm make, not approximately, but *exactly* the same predictions. It would be interesting to formalize the argument in a mathematical way, as its conclusion is *sharp* ("exactly the same predictions") and *absolute* ("always ... the same predictions"), not infected with the inexactness inherent in the argument, and not either with the inexactness that seems inherent in every formulation of what the predictions actually *are*. This suggests that it might somehow be possible to separate the inexactness from the core of the argument.

Section 4 of [27] or [24]). We outline the definition. On a discrete set Ω, a POVM $E(\cdot)$ associates with every $\omega \in \Omega$ a positive operator $E(\omega)$ on the Hilbert space $\mathscr{H}$ such that

$$\sum_{\omega\in\Omega} E(\omega) = I\,, \tag{19}$$

the identity operator. On a continuous set Ω, a POVM $E(\cdot)$ associates with every subset $B \subseteq \Omega$ a positive operator $E(B)$ on the Hilbert space $\mathscr{H}$ satisfying *normalization* $E(\Omega) = I$ and *σ-additivity*, i.e., for pairwise disjoint $B_1, B_2, \ldots \subseteq \Omega$,

$$E\Big(\bigcup_{k=1}^{\infty} B_k\Big) = \sum_{k=1}^{\infty} E(B_k)\,. \tag{20}$$

A relevant statement for us is (what we call) the *main theorem about POVMs*, valid in both quantum mechanics and GRW theory: *With every experiment $\mathscr{E}$, there is associated a POVM $E(\cdot)$ on the set $\mathscr{Z}$ of possible outcomes of $\mathscr{E}$, so that the probability distribution of the outcome Z of $\mathscr{E}$, when performed on a system with wave function ψ, is*

$$\mathbb{P}(Z = z) = \langle\psi|E(z)|\psi\rangle\,. \tag{21}$$

The self-adjoint operators correspond to special POVMs, the projection-valued measures (PVMs) on the real axis. The self-adjoint operators are in one-to-one correspondence to the PVMs on the real axis by virtue of the spectral theorem.

Apart from their role as generalized observables, POVMs also appear in the mathematics of the GRW theory: The joint distribution of all flashes, as a function of the initial wave function ψ_0, is given by a POVM $G(\cdot)$,

$$\mathbb{P}(F \in B) = \langle\psi_0|G(B)|\psi_0\rangle \tag{22}$$

for any set B of flash histories; see, e.g., [33, 40, 43] for details.

Superoperators. In an "ideal quantum measurement," according to the standard rules, the density matrix ρ changes according to

$$\rho \to \frac{P_z \rho P_z}{\mathrm{tr}(P_z \rho P_z)}\,, \tag{23}$$

where z is the outcome of the measurement and P_z the projection to the corresponding eigenspace of the observable measured. In less idealized cases, ρ changes according to

$$\rho \to \frac{\mathscr{C}_z(\rho)}{\mathrm{tr}\,\mathscr{C}_z(\rho)}\,, \tag{24}$$

where $\mathscr{C}_z$ is a mapping generalizing $\rho \mapsto P_z\rho P_z$; it maps density matrices to positive operators with finite trace; it is called a *superoperator* to express that it acts on density matrices rather than wave functions. Furthermore, $\mathscr{C}_z$ is *completely positive*; that is a positivity property defined as follows: for every integer $k \geq 1$ and every positive operator $\rho \in \mathbb{C}^{k\times k} \otimes TRCL(\mathscr{H})$, $(I_k \otimes \mathscr{C})(\rho)$ is a positive operator, where I_k denotes the identity operator on $\mathbb{C}^{k\times k}$. (Here, $TRCL$ means the trace class, i.e., the set of operators with finite trace. Completely positive superoperators are also often called completely positive maps. If for every density matrix ρ, $\operatorname{tr}\mathscr{C}(\rho) \leq 1$ (as will be the case for all superoperators that we consider in this paper) then $\mathscr{C}$ is also called a *quantum operation*.)

We are now ready to formulate the GRW formalism. It is so similar to the quantum formalism that the two can easily be formulated together as follows.

The Quantum/GRW Formalism.

- A system isolated from its environment has at every time t a density matrix ρ_t which evolves in the quantum formalism according to the unitary Schrödinger evolution
$$\frac{d\rho_t}{dt} = -\tfrac{i}{\hbar}[H, \rho_t]\,, \tag{25}$$
and in the GRW formalism according to the master equation (18).

- With every experiment $\mathscr{E}$ on a system there is associated a discrete set $\mathscr{Z}$ and a POVM $E(\cdot)$ on $\mathscr{Z}$ acting on the Hilbert space $\mathscr{H}_{\mathrm{sys}}$ of the system. When the experiment $\mathscr{E}$ is performed on a system with density matrix ρ, the outcome Z is random with probability distribution
$$\mathbb{P}(Z = z) = \operatorname{tr}\big(\rho\, E(z)\big)\,. \tag{26}$$

- Suppose that $\mathscr{E}$ begins at time s and ends at time t. With $\mathscr{E}$ is further associated a family $(\mathscr{C}_z)_{z\in\mathscr{Z}}$ of completely positive superoperators acting on $TRCL(\mathscr{H}_{\mathrm{sys}})$. In case $Z = z$, the density matrix of the system at time t immediately after the experiment $\mathscr{E}$ is
$$\rho_t = \frac{\mathscr{C}_z(\rho_s)}{\operatorname{tr}\mathscr{C}_z(\rho_s)}\,. \tag{27}$$
$E(z)$ and $\mathscr{C}_z$ are related according to
$$\operatorname{tr}\big(\rho\, E(z)\big) = \operatorname{tr}\mathscr{C}_z(\rho) \tag{28}$$
for all $\rho \in TRCL(\mathscr{H}_{\mathrm{sys}})$.

However, a difference between the quantum formalism and the GRW formalism lies in the *law of operators*, i.e., the rule determining the POVM $E(\cdot)$ and the superoperators $\mathscr{C}_z$ associated with a given experiment $\mathscr{E}$.

The Quantum Law of Operators.

- Suppose we are given the density matrix ρ_{app} for the ready state of the apparatus, its Hamiltonian H_{app}, and the interaction Hamiltonian H_I. Let

$$U_t = e^{-\frac{i}{\hbar}(H_{\text{sys}}+H_{\text{app}}+H_I)t} \tag{29}$$

be the unitary Schrödinger evolution operator for the composite (system + apparatus). Let the experiment $\mathscr{E}$ start at time s and be finished at time t, so that the result can be read off at t from the apparatus.[3] Let P_z^{app} be the projection to the subspace of apparatus states in which the pointer is pointing to the value z. Then

$$E^{\text{Qu}}(z) = \operatorname{tr}_{\text{app}}\Big([I_{\text{sys}} \otimes \rho_{\text{app}}]U^*_{t-s}[I_{\text{sys}} \otimes P_z^{\text{app}}]U_{t-s}\Big) \tag{30}$$

and

$$\mathscr{C}_z^{\text{Qu}}(\rho) = \operatorname{tr}_{\text{app}}\Big(U_{t-s}[\rho \otimes \rho_{\text{app}}]U^*_{t-s}[I_{\text{sys}} \otimes P_z^{\text{app}}]\Big)\,, \tag{31}$$

where $\operatorname{tr}_{\text{app}}$ denotes the partial trace over the Hilbert space of the apparatus.

In other words, the superoperator $\mathscr{C}_z^{\text{Qu}}$ is obtained by solving the Schrödinger equation for the apparatus together with the system, then collapsing the joint density matrix as if applying the collapse rule to a "quantum measurement" of the pointer position, and then computing the reduced density matrix of the system.

The GRW Law of Operators.

- Given the density matrix ρ_{app} for the ready state of the apparatus, its Hamiltonian H_{app}, and the interaction Hamiltonian H_I, so that $H = H_{\text{sys}} + H_{\text{app}} + H_I$. Let the experiment $\mathscr{E}$ start at time s and be finished at time t, and let $\zeta : \Omega_{[s,t)} \to \mathscr{Z}$ be the function that reads off the outcome of $\mathscr{E}$ from the flashes between s and t. Then $E^{\text{GRW}}(\cdot)$ is given by

$$E^{\text{GRW}}(z) = \operatorname{tr}_{\text{app}}\Big([I_{\text{sys}} \otimes \rho_{\text{app}}]G \circ \zeta^{-1}(z)\Big)\,, \tag{32}$$

or, equivalently,

$$E^{\text{GRW}}(z) = \operatorname{tr}_{\text{app}} \int\limits_{\zeta^{-1}(z)} df\; [I_{\text{sys}} \otimes \rho_{\text{app}}]\, L^*_{[s,t)}(f)\, L_{[s,t)}(f)\,, \tag{33}$$

and

$$\mathscr{C}_z^{\text{GRW}}(\rho) = \operatorname{tr}_{\text{app}} \int\limits_{\zeta^{-1}(z)} df\; L_{[s,t)}(f)\, [\rho \otimes \rho_{\text{app}}]\, L^*_{[s,t)}(f)\,. \tag{34}$$

[3] This assumption is to be understood in an operational sense: It is assumed that we humans can read off the result when looking at the apparatus. This is different from assuming that the result can be read off from the *wave function* of (the system and) the apparatus, which is notoriously not the case, a fact known as the measurement problem of quantum theory.

3.5 Derivation of the GRW Formalism From the Primitive Ontology

For a detailed discussion and derivation of the GRW formalism see [33]. Here we reproduce only the particularly simple argument that establishes the main theorem about POVMs from GRWf.

Recall that the joint distribution of all flashes after time t is given by a POVM $G(\cdot)$ and the wave function of the universe Ψ_t at time t. Let t be the time at which the experiment begins. Consider splitting the universe into a system (the object of the experiment) and its environment (the rest of the world, including all relevant apparatus of the experiment), corresponding to a splitting of the Hilbert space into $\mathscr{H} = \mathscr{H}_{\mathrm{sys}} \otimes \mathscr{H}_{\mathrm{env}}$. We assume independence between the system and the environment immediately before t, so that

$$\Psi_t = \psi \otimes \phi \,. \tag{35}$$

Here ϕ is fixed, being part of the characterization of the experiment, while ψ, the initial state of the system upon which the experiment is performed, is allowed to vary in the system Hilbert space $\mathscr{H}_{\mathrm{sys}}$. Let F be the history of flashes after time t and Ω the space of all possible histories of flashes after time t. The outcome Z of the experiment is a function of F,

$$Z = \zeta(F) \tag{36}$$

with $\zeta : \Omega \to \mathscr{Z}$, where $\mathscr{Z}$ is the *value space* of the experiment. That is so because the flashes define where the pointers point, and what the shape of the ink on a sheet of paper is. (It would even be realistic to assume that Z depends only on the flashes of the apparatus, but this restriction is not needed here.) Therefore, the distribution of the random outcome Z is given by

$$\mathbb{P}(Z = z) = \mathbb{P}\big(F \in \zeta^{-1}(z)\big) = \langle \Psi_t | G \circ \zeta^{-1}(z) | \Psi_t \rangle = \langle \psi | E(z) | \psi \rangle \quad \forall z \in \mathscr{Z}\,, \tag{37}$$

where the first scalar product is taken in the Hilbert space of the universe and the second in the Hilbert space of the object of the experiment, and $E(\cdot)$ is the POVM given by

$$E(z) = \langle \phi | G \circ \zeta^{-1}(z) | \phi \rangle \quad \forall z \in \mathscr{Z}\,, \tag{38}$$

where the scalar product is a partial scalar product in the Hilbert space of the environment. Thus, for every experiment in GRWf the distribution of outcomes is given by a POVM $E(\cdot) = E^{\mathrm{GRW}}(\cdot)$ on $\mathscr{Z}$, which is what we wanted to show.

4 A Set of Examples

A new, and useful, perspective on GRW theories arises from contrasting them with other theories, even unreasonable ones. For this purpose we develop in this section a set of example theories which we obtain by combining elements of the known theories GRWm, GRWf, and Bohmian mechanics in new, sometimes playful, ways. Some of the theories obtained in this way make completely wrong predictions but are instructive nonetheless since they illustrate the way in which predictions follow from a theory. Others make predictions in agreement with known empirical facts, yet nobody would seriously propose them as fundamental physical theories; still, they allow for illuminating comparisons with GRWm and GRWf.

Here is a "theory construction kit." Choose one of the three primitive ontologies of GRWm, GRWf, and Bohmian mechanics: continuous matter density, flashes, or particles with trajectories. Then choose one of two different evolution laws for the wave function: the unitary Schrödinger evolution or the stochastic GRW evolution. On top of that, either use either wave functions or instead directly density matrices—which might evolve according to the master equation (18) of the GRW theories, or in a stochastic way. Then consider all simple laws for how the wave function may govern the PO. In this way we arrive at about ten new theories.

Let us briefly recall the laws of *Bohmian mechanics*, laws that we will use for recombining. Bohmian mechanics is a (non-relativistic) theory of particles in motion. The motion of a system of N particles is provided by their world lines $t \mapsto Q_i(t)$, $i = 1, \ldots, N$, where $Q_i(t)$ denotes the position in $\mathbb{R}^3$ of the i-th particle at time t. These world lines are determined by Bohm's law of motion [20, 11, 19],

$$\frac{dQ_i}{dt} = v_i^\psi(Q_1, \ldots, Q_N) = \frac{\hbar}{m_i} \mathrm{Im} \frac{\psi^* \nabla_i \psi}{\psi^* \psi}(Q_1 \ldots, Q_N), \tag{39}$$

where the wave function ψ evolves according to Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \,, \tag{40}$$

where H is the usual nonrelativistic Schrödinger Hamiltonian; for spinless particles it is of the form (5).

4.1 Bohmian Trajectories and GRW Collapses

We begin with the particle ontology, and the particle trajectories governed by Bohm's law of motion. We consider several ways of combining this with the GRW evolution of the wave function or a similar one.

4.1.1 Bohm's Law and GRW's Law

First, suppose simply that the particles move according to the usual Bohmian law of motion (39), but that $\psi = \psi_t$ is the GRW wave function, so that the GRW process replaces the unitary evolution. In this theory, which we denote GRWp$_1$, the $|\psi|^2$ distribution for the configuration is not equivariant.

A world governed by this theory GRWp$_1$ has little resemblance with our world: It behaves in a very unstable way. For example, a system with the wave function of Schrödinger's cat has, even before the collapse into either $|\text{dead}\rangle$ or $|\text{alive}\rangle$, a configuration of either a dead or a live cat—but the collapse need not agree with that configuration. The cat could be alive before the collapse (i.e., its particle configuration is that of a live cat), and still the collapse could reduce ψ to a state vector close to $|\text{dead}\rangle$.

At this point, the reader may feel unsure whether to conclude that the cat is really alive or that the cat is really dead. That is, partly, what makes this theory philosophically useful, despite its empirical inadequacy: It nicely illustrates the role of the PO. Taking the PO seriously, we must conclude that the cat is really alive; after all, the PO in this theory consists of particles, and the particle configuration is one of a live cat. This illustrates that the mere fact that the wave function is one of a dead cat does not, in and of itself, mean that there is a dead cat.

From that fateful collapse onwards, the configuration is guided, in the sense of Bohm's law of motion (39), by that tiny part of ψ that remains of $|\text{alive}\rangle$ after the collapse, and who knows what happens then. For sure, the further behavior of the configuration will be catastrophic. The further evolution of the configuration is not just that of a live cat, but will be disturbed by two factors: first, by the fact that the Gaussian collapse factor will change the shape of $|\text{alive}\rangle$, and second, by the fact that the tails of $|\text{dead}\rangle$ will reach the support of $|\text{alive}\rangle$ under the Schrödinger evolution, and will dominate over the contribution from $|\text{alive}\rangle$.

What this example illustrates: First and foremost, it illustrates how the same wave function—the GRW wave function—can be combined with a different PO than usual, and thus helps us getting used to the distinction between the wave function and the PO. Second, it illustrates what it means to derive predictions from the PO rather than from the wave function, as the wave function is well behaved but the particle configuration is not. Third, GRWp$_1$ forces us to face the question: Do the predictions follow from the wave function or from the PO? Finally, GRWp$_1$ shows that the GRW wave function can be part of a theory making completely different predictions than GRWm and GRWf.

Another observation: The $|\psi|^2$ distribution is not equivariant in GRWp$_1$, and presumably there is no equivariant density formula at all. This undercuts the reasons for assuming the initial configuration was $|\psi|^2$ distributed, so that we lose the basis for deriving predictions at all. But even if we postulated the $|\psi|^2$ distribution at some point in time (e.g., the big bang), so that the theory would make unambiguous predictions, the distribution of the outcome Z of an experiment will not be given by a POVM, presumably not even approximately, and we see no reason why the empirical contents could be summarized by a formalism at all. Thus, GRWp$_1$ with $|\psi|^2$ distribution at the big bang is presumably an example of a theory *without a formalism*.

4.1.2 Bohm's Law and a Modified GRW Law

To improve GRWp$_1$, one may think of modifying it a bit: Instead of choosing the collapse center X at random, as prescribed by the GRW process, one could take

$$X = Q_I(T)\,, \tag{41}$$

so that the collapse is centered at the actual position of the corresponding particle. In other words, for every collapse the time T and the label I are chosen at random as in the GRW process, but the position of the (center of the) collapse is not, but is taken from the particle configuration instead.

Let us call this model GRWp$_2$; it was called GRWp in [3]. The behavior of a GRWp$_2$ world is less catastrophic than that of a GRWp$_1$ world, but still the $|\psi|^2$ distribution is not equivariant, and so probabilities cannot be expected to agree with $|\psi|^2$. As with GRWp$_1$, one could consider GRWp$_2$ with $|\psi|^2$ distribution at the big bang, but as with GRWp$_1$, one presumably obtains no POVM, and no formalism.

What this example illustrates: Even though GRWp$_1$ and GRWp$_2$ both use the GRW wave function and the particle ontology, they are very different theories. This shows that the particle ontology leaves room for different possible laws, each of which is simple and respects the symmetries of the GRW process (invariance under rotations, translations, time translations, and Galilean boosts).

With a little modification in its defining equations, GRWp$_2$ becomes a better behaved theory GRWp$_3$ [10, 45]: Instead of (41), take the collapse center X to be

$$X = Q_I(T) + Z \,, \tag{42}$$

where Z is a random 3-vector that is chosen independently of the past with a Gaussian distribution with mean 0 and covariance matrix $\mathrm{diag}(\sigma^2, \sigma^2, \sigma^2)$. It then follows [45] that the conditional distribution of $Q(t)$, given the X, I, T for all collapses up to time t (or given ψ_t), equals $|\psi_t|^2$; that the joint distribution of the ψ_t for all $t \geq 0$ is the same as for the GRW process; and that this theory is empirically equivalent to GRWm and GRWf.

What this example illustrates: The empirical content of the GRWf and GRWm theories can as well be obtained with the particle ontology, and thus is not limited to the flash and matter density ontologies.

4.1.3 Trajectories From the GRW Wave Function

For the next theory, GRWp$_4$, let us return to the GRW law for ψ_t, and consider a particle ontology with positions given by the wave function by means of the law

$$Q_i(t) = \langle \psi_t | \hat{Q}_i | \psi_t \rangle \,, \tag{43}$$

where $\hat{Q}_i$ is the position operator of particle number i. That is, the actual position $Q_i(t)$ is what would in orthodox quantum mechanics be the *average* position of particle i (if measured). In contrast to GRWp$_1$ and GRWp$_2$ (and Bohmian mechanics), this theory does not require any initial data about the particle configuration, as the configuration is a function of ψ_t. (GRWp$_4$ is not empirically adequate either, but we choose not to go into detail.)

What this example illustrates: Apart from being an example of the multitude of possible laws for the PO, an interesting trait of this theory is that the configuration of the PO supervenes on the wave function by means of the law (43), as it does in GRWm and GRWf. This addresses a view expressed by David Albert (see the end of Section 3.2), according to which GRWf and GRWm are the same theory as both F and m supervene on the wave function. GRWp$_4$ shows that among the many ways in which a configuration of matter (be it a particle configuration, a continuous matter distribution, or flashes) can supervene on the wave function, different possibilities may strongly disagree about the empirical predictions.

4.1.4 Configuration Jumps and GRW Law

Another theory, GRWp$_5$, has the following laws: The wave function ψ_t follows the GRW process, and the configuration moves according to Bohm's law of motion (39) between the GRW collapses. However, at the time T when ψ collapses around $X \in \mathbb{R}^3$ with label I, also the configuration Q jumps; more precisely, only the I-th particle jumps, and it jumps to the random center X of the GRW collapse:

$$Q_I(T+) = X\,. \tag{44}$$

Again, $|\psi|^2$ is not equivariant, and indeed, the behavior of the particles is quite catastrophic: If $\psi(T-)$ is the wave function of Schrödinger's cat, the configuration $Q(T-)$ is that of a live cat, and the wave function collapses to near that of a dead cat, it may do so with just a few collapses connected to a few particle labels, corresponding to particles that would have to be in different positions depending on whether the cat is dead or alive. As a consequence, the configuration after these few collapses will be one of a live cat, with a few particles moved to where they would have to be if the cat were dead. So, this configuration is very different from what one would normally associate with $|\text{dead}\rangle$. Also, this configuration may be well outside the support of both $|\text{dead}\rangle$ and $|\text{alive}\rangle$; it may be a configuration for which $|\psi|^2$ is literally zero, or much smaller than even the remains of $|\text{alive}\rangle$ after the collapse. And the behavior of such configurations should be expected to be catastrophic.

What this example illustrates: It is another example of how to derive consequences from the law governing the PO while strictly avoiding the use of any normal idea about what the reality is like when $\psi = |\text{dead}\rangle$.

4.1.5 Another Way of Configuration Jumps and GRW Law

Alternatively, we may consider another way for the configuration to jump: not just the I-th particle (the one associated with the collapse) jumps, but all particles jump. Specifically, choose $Q(T+)$ at random with distribution $|\psi_{T+}|^2$. That is, the wave function ψ_t follows the GRW process, and the configuration obeys Bohm's law of motion between the GRW collapses. Let us call this theory GRWp$_6$.

In this theory, the $|\psi|^2$ distribution is indeed equivariant, in the sense that the configuration $Q(t)$ will always have distribution $|\psi_t|^2$. And indeed, this theory is presumably empirically equivalent to GRWm and GRWf (in the sense that there is no experiment that could distinguish GRWp$_6$ from GRWm and GRWf). However, the particles in GRWp$_6$ do not necessarily behave in a reasonable way: For example, when ψ is the wave function of Schrödinger's cat, then the configuration may well be one of a dead cat, until the first collapse occurs, which may well favor the live cat, in which case the configuration jumps to one of a live cat.

What this example illustrates: Even if a theory is empirically equivalent to a reasonable theory (such as GRWm and GRWf) it need not itself be a reasonable theory.[4] It also shows how the law governing the PO can force the PO to look like what would normally be expected from a wave function such as $|\text{dead}\rangle$ or $|\text{alive}\rangle$.

4.2 MBM: Bohm-Like Trajectories From the Master Equation

The most interesting example in our list of toy theories is perhaps MBM; the abbreviation stands for "master equation Bohmian mechanics." That is why we now spend more time on MBM than on the previous toy theories. MBM resembles Bohmian mechanics in that it is deterministic and that its PO consists of particles, but at the same time it is empirically equivalent to GRWf and GRWm, as we will show in Section 4.2.3.

The law of motion (39) is replaced by the following equation (considered already in [12, 25]) using, in the role of the wave function, a density matrix ρ:

$$\frac{dQ_k}{dt} = v_k^\rho(Q_1,\dots,Q_N) = \frac{\hbar}{m_k}\mathrm{Im}\frac{\nabla_{q_k}\langle q|\rho|q'\rangle}{\langle q|\rho|q'\rangle}\Big|_{q=q'=(Q_1,\dots,Q_N)}. \tag{45}$$

The density matrix ρ evolves according to (18), which we repeat here for convenience:

$$\frac{d\rho_t}{dt} = -\tfrac{i}{\hbar}[H,\rho_t] + \lambda\sum_{k=1}^N\int d^3x\,\Lambda_k(x)^{1/2}\,\rho_t\,\Lambda_k(x)^{1/2} - N\lambda\rho_t\,. \tag{18}$$

We make a few comments on how these equations are to be understood.

Eq. (45) is the natural generalization of Bohmian mechanics to density matrices, and reduces to (39) in case of a pure state $\rho = |\psi\rangle\langle\psi|$. However, it is important to notice that the density matrix considered here is not the one that is normally regarded as the density matrix of a system, which arises by averaging $|\psi\rangle\langle\psi|$ (in case that the wave function ψ is random) or by tracing out the environment of the system. The density matrix in (45), in contrast, does not arise from averaging or partial traces but is assumed to be one of the fundamental variables of the theory. The complete description of the state is, instead of the pair (Q,ψ) in Bohmian mechanics, the pair (Q,ρ).

[4] This point is also illustrated by the following theory [13] that is empirically equivalent to Bohmian mechanics: Using a Schrödinger (i.e., non-collapsing) wave function ψ_t and a particle ontology, let the configuration $Q(t)$ be random with distribution $|\psi|^2$, independently of the past.

The master equation (18) is, in the GRW theories, a consequence of the GRW evolution of the wave function. This is different in MBM. In MBM there is no random wave function. In MBM, (18) holds *by fiat*, not as a theorem. The defining equations of MBM—its postulates—are (18) and (45).

What this example illustrates: It shows that the empirical content of GRWm and GRWf can be realized as well with a particle ontology. The derivation of predictions from MBM is completely analogous to that from Bohmian mechanics; indeed, the GRW formalism was first discovered starting from MBM. If experiments will confirm the GRW deviations from quantum mechanics, then Bohmian mechanics can be modified so as to reproduce the GRW predictions. MBM also shows that the empirical content of GRWm and GRWf can be realized with a deterministic theory, and in particular without wave function collapse: after all, MBM involves the master equation (18) but not literal wave function collapse as in (2).

4.2.1 The Empirical Content of Bohmian Mechanics

The empirical contents of Bohmian mechanics agrees exactly with the quantum formalism "whenever the latter is unambiguous." Here is an outline of how the derivation proceeds.

An important probability distribution in Bohmian mechanics is the quantum equilibrium distribution

$$p^{\psi}(q) = |\psi(q)|^2 \,. \tag{46}$$

(While the distribution density is usually denoted ρ, we write p here in order to reserve the letter ρ for density matrices.) As a consequence of Schrödinger's equation and of Bohm's law of motion, $|\psi|^2$ is *equivariant*. This means that if the configuration $Q(t) = (Q_1(t), \dots, Q_N(t))$ of a system is random with distribution $|\psi_t|^2$ at some time t, then this will be true also for any other time t. Thus, the *quantum equilibrium hypothesis*, which asserts that whenever a system has wave function ψ_t, its configuration $Q(t)$ is random with distribution $|\psi_t|^2$, can consistently be assumed. This hypothesis is not as hypothetical as its name may suggest: the quantum equilibrium hypothesis follows, in fact, by the law of large numbers from the assumption that the (initial) configuration of the universe is typical (i.e., not-too-special) for the $|\Psi|^2$ distribution, with Ψ the (initial) wave function of the universe [26]. The situation resembles the way Maxwell's distribution for velocities in a classical gas follows from the assumption that the phase point of the gas is typical for the uniform distribution on the energy surface.

It will be useful in the following to express the $|\psi|^2$ distribution by means of a PVM $P(\cdot)$: with every Hilbert space of the form $\mathscr{H} = L^2(\mathcal{Q}, \mathbb{C}^k)$ with $\mathcal{Q}$ the configuration space (mathematically, a measure space, often $\mathbb{R}^{3N}$ with the Lebesgue measure), there is automatically associated a PVM $P(\cdot)$ on $\mathcal{Q}$ acting on $\mathscr{H}$. It can be written as

$$P(S) = \int_S dq \, |q\rangle\langle q| \,. \tag{47}$$

The $|\psi|^2$ distribution is then nothing but the measure $\langle\psi|P(\cdot)|\psi\rangle$.

As a consequence of the quantum equilibrium hypothesis, a Bohmian universe, even if deterministic, appears random to its inhabitants. In fact, the probability distributions observed by the inhabitants agree exactly with those of the quantum formalism. To begin to understand why, note that any measurement apparatus must also consist of Bohmian particles. Calling $Q_{\rm sys}$ the configuration of the particles of the system to be measured and $Q_{\rm app}$ the configuration of the particles of the apparatus, we can write for the configuration of the big Bohmian system relevant to the analysis of the measurement $Q = (Q_{\rm sys}, Q_{\rm app})$. Let us suppose that the initial wave function ψ of the big system is a product state $\Psi(q) = \Psi(q_{\rm sys}, q_{\rm app}) = \psi(q_{\rm sys})\,\phi(q_{\rm app})$.

During the measurement, this Ψ evolves according to the Schrödinger equation, and in the case of an ideal measurement it evolves to $\Psi_t = \sum_z \psi_z\,\phi_z$, where z runs through the eigenvalues of the observable measured, ϕ_z is a state of the apparatus in which the pointer points to the value z, and ψ_z is the projection of ψ to the appropriate eigenspace of the observable. By the quantum equilibrium hypothesis, the probability for the random apparatus configuration $Q_{\rm app}(t)$ to be such as to correspond to the pointer pointing to the value z is $\|\psi_z\|^2$.

4.2.2 The Empirical Content of MBM

The analogue of the quantum equilibrium distribution p^ψ in MBM is the distribution

$$p^\rho(q) = \langle q|\rho|q\rangle\,. \tag{48}$$

Note that $p^\rho(q) \geq 0$, and $\int_{\mathcal{Q}} p^\rho(q)\,dq = \mathrm{tr}\,\rho = 1$. This distribution is *equivariant* as a consequence of (45) and (18). The essential reason is that the non-unitary (or diffusive) terms in (18) (i.e., the second and third term on the right hand side), do not contribute to the continuity equation for p^ρ. In detail, it follows from (18) for the Hamiltonian (5) that

$$\frac{\partial}{\partial t} p^{\rho_t}(q) = -\tfrac{i}{\hbar}\langle q|[H,\rho_t]|q\rangle - N\lambda\langle q|\rho_t|q\rangle + \lambda\sum_{k=1}^{N}\int d^3x\,\langle q|\Lambda_k(x)^{1/2}\,\rho_t\,\Lambda_k(x)^{1/2}|q\rangle =$$

$$= \sum_{k=1}^{n}\tfrac{i\hbar}{2m_k}\langle q|[\nabla^2_{q_k},\rho_t]|q\rangle - N\lambda\langle q|\rho_t|q\rangle + \lambda\sum_{k=1}^{N}\int d^3x\,\langle q|\rho_t|q\rangle\frac{1}{(2\pi\sigma^2)^{3/2}}\,e^{-\frac{(q_k-x)^2}{2\sigma^2}} =$$

$$= \sum_{k=1}^{n}\nabla_{q_k}\cdot\Big(\tfrac{i\hbar}{2m_k}\nabla_{q_k}\langle q|\rho_t|q'\rangle - \tfrac{i\hbar}{2m_k}\nabla_{q'_k}\langle q|\rho_t|q'\rangle\Big)\Big|_{q'=q} =$$

$$= \sum_{k=1}^{n}\nabla_{q_k}\cdot\big(p^{\rho_t}\,v_k^{\rho_t}\big)\,.$$

Since any probability distribution p on configuration space will be transported, under the flow (45), according to the continuity equation

$$\frac{\partial p}{\partial t} = -\sum_{k=1}^{N}\nabla_{q_k}\cdot\big(p\,v_k^\rho\big)\,, \tag{49}$$

we have that $p_t = p^{\rho_t}$ satisfies (49), which means equivariance.

4.2.3 Empirical Equivalence of MBM with GRWm and GRWf

For the same reasons as in Bohmian mechanics [26], the quantum equilibrium hypothesis holds in a typical MBM universe, i.e., we can assume that the configuration Q_t of any system with density matrix ρ_t is random with distribution $\langle q|\rho_t|q\rangle$. And therefore, the probability at time t of a certain macroscopic configuration is

$$p(S) = \int_S dq \, \langle q|\rho_t|q\rangle = \mathrm{tr}\big(\rho_t P(S)\big) \tag{50}$$

where S is the set of all microscopic configurations consistent with that macroscopic configuration, and $P(S)$ the projection operator corresponding to the set S as in (47).

In GRW theories, this quantity can be written in terms of the GRW wave function ψ_t as

$$\mathrm{tr}\big(\rho_t \, P(S)\big) = \int_{\mathscr{H}} \mathbb{P}(\psi_t \in d\phi) \, \|P(S)\phi\|^2 \, . \tag{51}$$

Since ψ_t is typically concentrated on a single macro-configuration, the probability distribution $\mathbb{P}(\psi_t \in d\phi)$ is typically concentrated on those ϕ with either $\|P(S)\phi\| \approx 0$ or $\|P(S)\phi\| \approx 1$; thus, the probability that ψ_t is (nearly) concentrated on S equals $p(S)$. And in this case, either the flashes of GRWf or the matter density of GRWm gives rise to the same macroscopic appearance as configurations from S. In other words, at any fixed time the MBM, GRWf, and GRWm world have the same probability distribution over the possible macro-states.

Now empirical equivalence follows immediately: If there was an experiment which had (probably) one outcome $Z = z_1$ in MBM and another one, $Z = z_2$ in GRWm and GRWf, then at the time when the experiment is finished, the probability distribution over the macro-states would have to be different in MBM than in GRWm and GRWf, but it is not.

We list a few consequences of the empirical equivalence between MBM, GRWm, and GRWf:

- The empirical content of MBM is summarized by the GRW formalism.
- The empirical content of the GRW theories is not indicative of collapse of the wave function, but indeed consistent with a no-collapse theory.
- The empirical content of the GRW theories is not indicative of stochastic laws, but indeed consistent with a deterministic theory.

Our argument concerning empirical equivalence also exemplifies that empirical equivalence is a statement about the PO, as in Section 3.

4.2.4 Derivation of the GRW Formalism From MBM

Although it follows already from the empirical equivalence between GRWf and MBM that the GRW formalism holds in MBM, it will be helpful to look at a direct derivation of the GRW formalism from MBM. Before that, as a preparation (and for the purpose of comparison), we recall the simple argument how POVMs arise from Bohmian mechanics.

We make the following assumptions: At time t, when the experiment begins, the wave function of system and apparatus is $\Psi_t = \psi \otimes \phi$, where ψ is the (arbitrary) wave function of the system, and ϕ is the ready state of the apparatus. The outcome Z of the experiment is a function of the configuration at time t', when the experiment ends, $Z = \zeta(Q_{t'})$.

Let $U_t = e^{-iHt/\hbar}$ be the time evolution operator of system and apparatus. Then

$$\mathbb{P}(Z = z) = \int\limits_{\zeta^{-1}(z)} dq\, |\Psi_{t'}(q)|^2 = \langle\psi|E(z)|\psi\rangle\,, \tag{52}$$

where the scalar product is taken in $\mathscr{H}_{\mathrm{sys}}$, and the POVM $E(\cdot)$ is defined as follows:

$$E(z) = \langle\phi|U^*_{t'-t}\, P(\zeta^{-1}(z))\, U_{t'-t}|\phi\rangle\,. \tag{53}$$

Here, the scalar product is a partial scalar product over $\mathscr{H}_{\mathrm{app}}$, and $P(\cdot)$ is the PVM on the configuration space $\mathcal{Q}$ of system and apparatus as in (47).

This simple argument establishes the *main theorem about POVMs* for Bohmian mechanics: *For every experiment $\mathscr{E}$ there is a POVM $E(\cdot)$ such that the distribution of the outcome Z of $\mathscr{E}$, when performed on a system with wave function ψ, is $\langle\psi|E(\cdot)|\psi\rangle$.*

(The same argument may be taken to show the main theorem about POVMs for any other version of quantum mechanics, provided merely that configurations are assumed to be $|\psi|^2$ distributed. However, the argument will have a gap if it is not assumed that particles exist and that the configuration of their positions is random with distribution $|\psi|^2$.)

We now turn to the derivation of the GRW formalism from MBM. It is based on the following assumptions:

- The experiments begins at time t and ends at time t'.
- The density matrix of system + apparatus at time t is $\rho_t = \rho_{\mathrm{sys}} \otimes \rho_{\mathrm{app}}$.
- The outcome Z is a function of the configuration at time t', $Z = \zeta(Q_{t'})$.

We will first derive that probabilities are given by a POVM $E(\cdot)$, and collapses by a superoperator $\mathscr{C}_z$, and then turn to the question whether $E(\cdot) = E^{\mathrm{GRW}}(\cdot)$ and $\mathscr{C}_z = \mathscr{C}_z^{\mathrm{GRW}}$.

Let $\mathscr{C}_{t,t'}$ be the (completely positive) superoperator defined by solving the master equation (18) from time t to time t', i.e., defined by

$$\rho_{t'} = \mathscr{C}_{t,t'}(\rho_t)\,. \tag{54}$$

By the theorem of Choi and Kraus, there are operators R_i such that

$$\mathscr{C}_{t,t'}(\rho) = \sum_i R_i\, \rho\, R_i^*\,. \tag{55}$$

We obtain that, for any $z \in \mathscr{Z}$,

$$\mathbb{P}(Z=z) = \mathbb{P}\big(Q_{t'} \in \zeta^{-1}(z)\big) = \int\limits_{\zeta^{-1}(z)} dq\, \langle q|\rho_{t'}|q\rangle =$$

$$= \int\limits_{\zeta^{-1}(z)} dq\, \langle q|\mathscr{C}_{t,t'}(\rho_t)|q\rangle = \mathrm{tr}\Big(P\big(\zeta^{-1}(z)\big)\, \mathscr{C}_{t,t'}(\rho_t)\Big) =$$

$$= \mathrm{tr}\Big(\rho_t \sum_i R_i^*\, P\big(\zeta^{-1}(z)\big)\, R_i\Big)$$

with $P(\cdot)$ the PVM on the configuration space $\mathcal{Q}$ of system and apparatus as in (47). Thus,

$$\mathbb{P}(Z=z) = \mathrm{tr}\big(\rho_t\, E(z)\big) \tag{56}$$

with the POVM

$$E(z) = \sum_i R_i^*\, P\big(\zeta^{-1}(z)\big)\, R_i\,. \tag{57}$$

Concerning the effective collapse, the future trajectory of the configuration Q depends only on the part of $\rho_{t'}$ corresponding to the actual outcome Z of the experiment. In other words, for the same reason as in Bohmian mechanics one can replace the quantum state by a collapsed quantum state without changing the trajectories, which means here to replace $\rho_{t'}$ with

$$\rho' = \frac{P_z^{\mathrm{app}}\, \rho_{t'}\, P_z^{\mathrm{app}}}{\mathrm{tr}(P_z^{\mathrm{app}}\, \rho_{t'}\, P_z^{\mathrm{app}})}\,, \tag{58}$$

with P_z^{app} the projection to the subspace of apparatus states in which the pointer is pointing to the value z. Tracing out the apparatus, we obtain the collapse rule (27) of the formalism with superoperator

$$\mathscr{C}_z^{\mathrm{MBM}}(\rho) = \mathrm{tr}_{\mathrm{app}}\Big(P_z^{\mathrm{app}} \mathscr{C}_t[\rho \otimes \rho_{\mathrm{app}}] P_z^{\mathrm{app}}\Big)\,. \tag{59}$$

We have thus obtained the formalism, but with a different law of operators:

The MBM Law of Operators.

- In MBM, the POVM $E^{\mathrm{MBM}}(\cdot)$ associated with an experiment $\mathscr{E}$ is given by (57), and the superoperators are given by (59).

Note that this law of operators can be obtained from the quantum law of operators by replacing the unitary evolution by the GRW master equation.

4.3 Master Equation and Matter Density

If one can consider a version of Bohmian mechanics in which the density matrix plays exactly the role of the wave function, then why not do the same trick with the matter density and flash ontologies?

For the matter density ontology, this would mean to postulate, in analogy to and replacing (8),

$$m(x,t) = \mathrm{tr}\big(\rho_t\, \tilde{\Lambda}(x)\big) \tag{60}$$

with $\tilde{\Lambda}(x) = \sum_i m_i\, \delta(\widehat{Q}_i - x)$, as before. Here, ρ_t is taken to evolve according to the master equation (18). We thus obtain a theory that could be called Mm, in which the fundamental objects are a density matrix (which, as in MBM, does *not* represent an ensemble, or the observer's limited knowledge, but is, by postulate, a fundamental object) and the continuous matter with density $m(x,t)$. This theory, though, is very different from GRWm! It has a many-worlds character. For example, if at some initial time $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = (|\mathrm{dead}\rangle + |\mathrm{alive}\rangle)/\sqrt{2}$ being the wave function of Schrödinger's cat, then after a short while the GRW function ψ_t will be either $|\mathrm{dead}\rangle$ or $|\mathrm{alive}\rangle$, but ρ_t will be $|\mathrm{dead}\rangle\langle\mathrm{dead}| + |\mathrm{alive}\rangle\langle\mathrm{alive}|$, up to a factor $\frac{1}{2}$. As a consequence, the m field of GRWm will be either m_{dead} or m_{alive}, but the m field of Mm will be $m_{\mathrm{dead}} + m_{\mathrm{alive}}$, up to a factor $\frac{1}{2}$. Both cats are there at once, but with reduced mass (which the cats, however, do not appear to notice). A very similar toy theory, with the unitary Schrödinger evolution instead of the master equation (18), has been described in some detail in [3, 4].

What this example illustrates: Foremost, the big difference, for quantum theories without observers, between a density matrix and a random wave function. Since with every probability distribution over wave functions there is associated a density matrix, and since for the purpose of computing predictions only the density matrix is relevant, it is common practice in quantum mechanics to immediately replace every probability distribution over wave functions by the density matrix. Here, however, the distinction is crucial: The random m function obtained from a random wave function that is either $|\mathrm{dead}\rangle$ or $|\mathrm{alive}\rangle$ is a reasonable picture of reality, but the deterministic m function obtained from the density matrix is not (more precisely, is a many-worlds picture).

4.4 Master Equation and Flashes

What happens to GRWf when we replace the wave function by a density matrix? Two versions of what this might mean come to mind:

MGRWf: Postulate that the flash rate equals

$$r(x, i|\rho) = \lambda\, \mathrm{tr}\big(\rho\, \Lambda_i(x)\big) \tag{61}$$

instead of (6). Postulate further that when a flash occurs, the density matrix changes to

$$\frac{\Lambda_i(x)^{1/2}\rho\Lambda_i(x)^{1/2}}{\mathrm{tr}(\Lambda_i(x)^{1/2}\rho\Lambda_i(x)^{1/2})}\,, \tag{62}$$

and that ρ_t evolves unitarily in between. Equivalently, the joint distribution of all flashes is given by the probability measure

$$\mathrm{tr}\big(\rho_0\, G(\cdot)\big)\,. \tag{63}$$

However, this theory, MGRWf, is physically equivalent to GRWf! Consider GRWf for an initial wave function ψ_0 that has been chosen at random from the unit sphere of Hilbert space $\mathbb{S}(\mathscr{H})$ with some distribution μ. Then the joint distribution of all flashes, given ψ_0, is $\langle\psi_0|G(\cdot)|\psi_0\rangle$, and therefore the unconditional joint distribution of all flashes is

$$\int_{\mathbb{S}(\mathscr{H})} \mu(d\psi)\, \langle\psi_0|G(\cdot)|\psi_0\rangle = \mathrm{tr}\big(\rho_0\, G(\cdot)\big)\,, \tag{64}$$

if ρ_0 is the density matrix of the ensemble μ.

Note that ρ_t is random and, in particular, does not evolve according to the master equation (18). However, there is another density matrix, namely

$$\tilde{\rho}_t = \mathbb{E}\rho_t\,, \tag{65}$$

where $\mathbb{E}$ means the expectation over the random flashes; $\tilde{\rho}_t$ does evolve deterministically according to (18).

What this example illustrates: First, it illustrates (like MBM and Mm) how it can make sense to speak of the density matrix of the entire universe and, more specifically, how a density matrix can be a fundamental object and part of the ontology, rather than just encoding statistical information. Second, it illustrates an aspect of the concept of physical equivalence: There is no physical difference between GRWf with a random initial wave function of the universe and MGRWf, as the distribution of the PO is the same. Third, it illustrates the different roles that a density matrix can play: On the one hand, it can be part of the ontology as one of the fundamental objects, such as the random density matrix ρ_t. On the other hand, the other density matrix $\tilde{\rho}_t$ is a mathematical object encoding information about the probability distribution of ρ_t. Thus, finally, it *also* illustrates that it is not necessarily well-defined to speak of "the" density matrix.

Mf: Another way of replacing the wave function in GRWf by a density matrix is to postulate (61) for the flash rate while insisting that ρ_t evolves deterministically according to the master equation (18). In other words, Mf arises from MGRWf by replacing ρ_t in the flash rate (61) by $\tilde{\rho}_t$.

This theory, too, has a many-worlds character, as the set F of flashes will be the union $F = F_{\mathrm{dead}} \cup F_{\mathrm{alive}}$ of a set of flashes of a live cat and a set of flashes of a dead cat (similar to the model "Sf" considered in [4]). Mf is empirically equivalent (to the extent that theories with many-worlds character can be empirically equivalent to normal theories) not to the quantum formalism but to the GRW formalism.

What this example illustrates: By way of contrast with MGRWf, it illustrates the difference between the "collapsing" density matrix of MGRWf and the deterministic one arising from the master equation (18). It thus illustrates why it is important that the law governing the PO be precisely formulated (in particular, specifies precisely which density matrix to use). It thus provides a warning against relying too much on the common practice in quantum mechanics to leave the exact PO unspecified, and the exact laws of the PO unspecified, trusting that it will always be essentially clear what the PO should (macroscopically) be like in any particular situation (and that it will always be essentially clear what "the" density matrix is).

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