Pauli Spin Matrices

Roderich Tumulka*

August 31, 2007

The Pauli spin matrices are the following 3 complex 2×2 matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

These matrices represent the spin observables along the x- (respective y- and z-)axis of physical 3-space for a spin-$\frac{1}{2}$ particle, relative to an orthonormal basis of spin space consisting of eigenvectors of σ_z. (Spin observables are measured, e.g., in the Stern–Gerlach experiment.) The spin observable along any direction in physical 3-space defined by the unit vector $\vec{n} = (n_x, n_y, n_z)$ is given, relative to the same basis, by

$$\sigma_{\vec{n}} = n_x\,\sigma_x + n_y\,\sigma_y + n_z\,\sigma_z = \vec{n}\cdot\vec{\sigma} \tag{2}$$

with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The spin observable is related to the angular momentum observable $J_{\vec{n}}$ along $\vec{n}$ according to

$$J_{\vec{n}} = \tfrac{\hbar}{2}\sigma_{\vec{n}} + L_{\vec{n}}\,, \tag{3}$$

where $L_{\vec{n}} = \vec{n}\cdot\vec{L}$ is the $\vec{n}$-component of the orbit angular momentum operator $\vec{L} = \vec{q}\times\vec{p}$.

The Pauli spin matrices, named after Wolfgang Pauli (1900–1958), are self-adjoint (= Hermitian) and unitary. Each of them (as well as $\sigma_{\vec{n}}$ for every unit vector $\vec{n}$) has trace equal to zero, determinant equal to -1, and eigenvalues 1 and -1.

The Pauli matrices belong to the fundamental structure of spin space, as spin space is defined to be a 2-dimensional complex vector space $\mathscr{H}_{\mathrm{spin}} \cong \mathbb{C}^2$ coupled to physical 3-space by a law specifying how the elements of spin space transform under rotations. The law involves the Pauli matrices and asserts that the rotation through the angle $\varphi \in \mathbb{R}$ about the axis spanned by the unit vector $\vec{n} \in \mathbb{R}^3$ transforms the vector $\psi \in \mathscr{H}_{\mathrm{spin}}$ from spin space into

$$\psi' = \pm e^{-(i/2)\varphi\sigma_{\vec{n}}}\psi\,. \tag{4}$$

(Exponentiation of a matrix can be defined by means of the power series $e^x = \sum x^k/k!$.) As a consequence, for the rotation through an infinitesimal angle $\delta\varphi$ one can write, neglecting higher order terms in $\delta\varphi$,

$$\psi' = \psi - \tfrac{i}{2}\delta\varphi\,\sigma_{\vec{n}}\,\psi\,. \tag{5}$$

*Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA. E-mail: tumulka@math.rutgers.edu

From this equation one can read off that the matrix $-(i/2)\sigma_{\vec{n}}$ represents the rate of change of ψ per angle when rotating around $\vec{n}$.

Expressing these facts in a technical way, spin space is endowed with an irreducible projective representation of the rotation group $SO(3)$ (the set of all orthogonal real 3×3 matrices with determinant 1), called the "spin-$\frac{1}{2}$ representation." Using the fact that $SO(3)$ can be "unfolded" yielding the group $SU(2)$ (the set of all unitary complex 2×2 matrices with determinant 1), the irreducible projective representation of $SO(3)$ can be translated into an irreducible representation of $SU(2)$, in fact the natural representation on $\mathbb{C}^2$ defined by matrix multiplication. In this translation, the rotation by φ about $\vec{n}$ corresponds to the matrix $\pm e^{-(i/2)\varphi\sigma_{\vec{n}}} \in SU(2)$, where the sign ambiguity arises from the "unfolding." The Lie algebra $su(2)$ associated with the Lie group $SU(2)$ consists of the infinitesimal generators of $SU(2)$, and thus of all matrices of the form $-(i/2)\varphi\sigma_{\vec{n}}$, and that is the 3-dimensional real vector space of all traceless skew-adjoint 2×2 matrices, of which $i\sigma_x, i\sigma_y, i\sigma_z$ form a basis.

The Pauli matrices satisfy the commutation relations

$$[\sigma_i, \sigma_j] = 2i\sigma_k \tag{6}$$

if ijk is any cyclic permutation of xyz. Except for the factor 2, these relations are the same as those of any angular momentum operators; the reason is that these are the defining relations of the Lie algebra $su(2)$, which is also the Lie algebra of the rotation group $SO(3)$, and thus are relations characteristic of rotations in physical 3-space.

Higher spins: For spin-s particles, $s \in \frac{1}{2}\mathbb{Z}$, the matrices analogous to the Pauli spin matrices are 3 complex $(2s+1) \times (2s+1)$ matrices. Higher dimensions: If physical space had dimension d instead of 3, there would be $d(d-1)/2$ Pauli spin matrices, as that number is the dimension of the rotation group $SO(d)$.