POVM

Roderich Tumulka*

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POVM: *positive-operator-valued measure*, also called *generalized observable.* A mathematical object, consisting of a family of operators on Hilbert space, that occurs in quantum theoretical formulas for the probability distribution of the random outcome of a quantum mechanical experiment. The concept of POVM contains, as a special case, that of observables represented by self-adjoint operators.

1 Overview

Outline of Definition. The word "measure" in "positive-operator-valued measure" is understood in the sense of mathematical probability and measure theory [5], where it means "additive set function". A *set function* $E(\cdot)$ is a function whose argument is a set (rather than a number, or a point in space). Possible arguments are subsets Δ of a basic set Ω. Typical relevant examples of Ω include the real line $\mathbb{R}$, n-space $\mathbb{R}^n$, or finite sets. A set function is called *additive* if for any two *disjoint* sets Δ_1, Δ_2 it is true that

$$E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2)\,. \tag{1}$$

(The full mathematical definition, see below, requires slightly more.)

Examples of measures include probability measures, for which $E(\Delta)$ is a number between zero and one, meaning the probability that a given random variable assumes a value in the set Δ. For a POVM, $E(\Delta)$ is a (bounded) positive operator on a Hilbert space $\mathscr{H}$. An operator T is called *positive* if $\langle\phi|T\phi\rangle \geq 0$ for all $\phi \in \mathscr{H}$; this is also sometimes called *positive semi-definite* in the literature; every (bounded) positive operator is self-adjoint. Finally, it is part of the definition of a POVM that it is normalized in the sense $E(\Omega) = I$, where I is the identity operator on $\mathscr{H}$, $I\psi = \psi$. In case Ω is a finite (or countable) set, $E(\Delta)$ can be expressed by singletons:

$$E(\Delta) = \sum_{\omega\in\Delta} E(\{\omega\})\,. \tag{2}$$

(Below we write $E\{\omega\}$ instead of $E(\{\omega\})$.)

*Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA. E-mail: tumulka@math.rutgers.edu

Probabilities from POVMs. From a POVM $E(\cdot)$ on a set Ω one can create probability measures on Ω in the following way: Given any vector $\psi \in \mathscr{H}$ with $\|\psi\| = 1$, then

$$\mathbb{P}_\psi(\Delta) = \langle\psi|E(\Delta)|\psi\rangle \tag{3}$$

defines a probability measure $\mathbb{P}_\psi(\cdot)$ on Ω. To see this, note that $\langle\psi|E(\Delta)|\psi\rangle$ is a non-negative real number since $E(\Delta)$ is a positive operator, and

$$\mathbb{P}_\psi(\Omega) = \langle\psi|E(\Omega)|\psi\rangle = \langle\psi|I|\psi\rangle = \|\psi\|^2 = 1. \tag{4}$$

Physical role. The physical relevance of POVMs is based on the following *main theorem about POVMs*: *For every quantum physical experiment $\mathcal{E}$ whose possible outcomes lie in a space Ω, there exists a POVM $E(\cdot)$ on Ω such that, whenever the experiment $\mathcal{E}$ is carried out on quantum system with state vector ψ, the random outcome Z has probability distribution given by*

$$\mathbb{P}(Z \in \Delta) = \langle\psi|E(\Delta)|\psi\rangle\,. \tag{5}$$

Observables. When all operators $E(\Delta)$ are *projection operators* then $E(\cdot)$ is also called a *PVM* or *projection-valued measure*. The widespread concept of observables as represented by self-adjoint operators in contained in the concept of POVM as the special case of a PVM on $\Omega = \mathbb{R}$. The self-adjoint operator A usually called the "observable" is obtained from $E(\cdot)$ by setting

$$A = \int_{\mathbb{R}} E(d\lambda)\,\lambda\,. \tag{6}$$

Conversely, given A, the spectral theorem for self-adjoint operators provides the right hand side of this equation, that is, provides the unique PVM $E(\cdot)$ on $\mathbb{R}$ that makes this equation true. Thus, the self-adjoint operator A summarizes the entire information encoded in the PVM $E(\cdot)$ in just one operator.

2 Examples

Observables as represented by self-adjoint operators correspond to the simplest cases of quantum experiments, usually connected with "ideal measurements." POVMs are necessary for more complex experiments.

Time of arrival. Send a particle towards a detector and measure the time at which the detector clicks. As a consequence of the main theorem about POVMs, the statistics of the random result, depending on the initial wave function of the particle, is given by a POVM, i.e., is of the form (5). Since this POVM is a "proper POVM", i.e., not a PVM, there is no self-adjoint operator summarizing it; in other words, there is no "time operator."

Sequence of ideal measurements. Readers familiar with the formalism of ideal quantum measurement of an observable (self-adjoint operator) A may consider a sequence of such measurements, first one corresponding to A_1, then another corresponding to A_2, and so on, up to A_n. To this end, suppose that the A_i have purely discrete spectrum. Note that the operators A_i need not commute with each other, as they are not measured simultaneously, but in a specified order. The sequence of outcomes forms a vector in $\mathbb{R}^n$, whose distribution is given by a POVM $E(\cdot)$ that can be constructed from the PVMs $E_i(\cdot)$ associated by (6) with A_i as follows:

$$E\big\{(\lambda_1,\dots,\lambda_n)\big\} = E_1\{\lambda_1\}^{1/2}\cdots E_n\{\lambda_n\}^{1/2}E_n\{\lambda_n\}^{1/2}\cdots E_1\{\lambda_1\}^{1/2}\,. \tag{7}$$

(The powers 1/2 can be omitted as $P^{1/2} = P$ for every projection P; however, in the above form the equation is still true when the $E_i(\cdot)$ are themselves proper POVMs.)

In case the A_i commute with each other, $E(\cdot)$ is a PVM on $\mathbb{R}^n$. In this sense, a PVM can represent a family of commuting observables. In particular, the three position operators Q_x, Q_y, Q_z of non-relativistic quantum mechanics of a single particle together give rise to the following PVM $P(\cdot) = E(\cdot)$ on $\mathbb{R}^3$:

$$P(\Delta)\psi(x,y,z) = \begin{cases}\psi(x,y,z) & \text{if } (x,y,z)\in\Delta\,,\\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

However, when the A_i do not commute then $E(\cdot)$ is not a PVM but a proper POVM.

To make the setting more general, we can allow that the choice of second observable A_2 depends on the outcome of the first measurement. To take this into account, replace $E_i\{\lambda_i\}$ in (7) by $E_{i,\lambda_1,\dots,\lambda_{i-1}}\{\lambda_i\}$.

Position measurements with constraints. In some cases, not all square-integrable functions on $\mathbb{R}^3$ are possible as physical wave functions, but only those from a suitable subspace $\mathscr{H}_{\text{phys}}$ [2, 4]. For example, photon wave functions are functions $\Psi : \mathbb{R}^3 \to \mathbb{C}^3$ with the constraint $\nabla\cdot\Psi = 0$. As another example, Dirac wave functions $\psi : \mathbb{R}^3 \to \mathbb{C}^4$ are usually regarded as physical only if they consist exclusively of Fourier components with positive energy, in other words, if they lie in the positive spectral subspace $\mathscr{H}_{\text{phys}}$ of the Dirac Hamiltonian. In this case, the usual position operators and the associated PVM as in (8) often map physical state vectors into unphysical ones, and are thus not defined as operators on the physical Hilbert space $\mathscr{H}_{\text{phys}}$. The problem is solved by replacing the "generalized position observable" $P(\cdot)$ with $\tilde{P}(\cdot)$ defined by

$$\tilde{P}(\Delta) := P_{\text{phys}}\,P(\Delta)\,P_{\text{phys}}\,, \tag{9}$$

where P_{phys} denotes the projection to $\mathscr{H}_{\text{phys}}$. Then $\tilde{P}(\Delta)$ is an operator on $\mathscr{H}_{\text{phys}}$, and $\tilde{P}(\cdot)$ is a proper POVM on $\mathbb{R}^3$.

Fuzzy measurements. An ideal detector, when detecting the particle in the region $\Delta\subseteq\mathbb{R}^3$, would collapse the wave function $\psi(x,y,z)$ to the function in (8). Real detectors cut off the wave function in an unsharp way, corresponding to a proper POVM $\tilde{P}(\cdot)$ that arises from the PVM $P(\cdot)$ of (8) by smearing out (convolving) with a "bump function" f (for example a Gaussian):

$$\tilde{P}(\Delta) = \int_{\Delta} d^3x \int_{\mathbb{R}^3} P(d^3y)\, f(y-x)\,. \tag{10}$$

3 The Main Theorem About POVMs

It is not difficult to understand the main theorem; here is a simple argument [3]. Suppose the experiment $\mathcal{E}$ begins at time t_1 and ends at time t_2, and suppose the quantum state of system and apparatus at time t_1 is $\Psi(t_1) = \psi \otimes \phi$. We make three assumptions: (i) The time evolution from t_1 to t_2 is a unitary operator U. (ii) The Born rule, according to which the probability distribution of the configuration Q at time t_2 is given by $\langle \Psi(t_2)|P(\cdot)|\Psi(t_2)\rangle$ with $P(\cdot)$ the position PVM as in (8). (iii) The outcome Z is a function f of the configuration Q at time t_2. Then, for $\Delta \subseteq \Omega$,

$$\mathbb{P}(Z \in \Delta) = \mathbb{P}\big(Q \in f^{-1}(\Delta)\big) = \langle \Psi(t_2)|P(f^{-1}(\Delta))|\Psi(t_2)\rangle \tag{11}$$

$$= \langle \psi \otimes \phi|U^* P(f^{-1}(\Delta))U|\psi \otimes \phi\rangle = \langle \psi|E(\Delta)|\psi\rangle \tag{12}$$

with

$$E(\Delta) = \langle \phi|U^* P(f^{-1}(\Delta))U|\phi\rangle\,, \tag{13}$$

where the scalar product in (13) is a *partial scalar product* in the Hilbert space of the apparatus. It can be shown that (13) defines a POVM.

4 Mathematical Aspects

Definition. The mathematical definition of POVM contains some details we have omitted above. The family of sets Δ for which $E(\Delta)$ is defined is required to be a σ*-algebra*, i.e., closed under the complement operation $\Delta \mapsto \Omega \setminus \Delta$ and under forming countable intersections. A POVM $E(\cdot)$ is further supposed to be σ*-additive*, i.e., additive for any countable union of pairwise disjoint sets $\Delta_1, \Delta_2, \ldots$,

$$E\Big(\bigcup_{i=1}^{\infty} \Delta_i\Big) = \sum_{i=1}^{\infty} E(\Delta_i)\,, \tag{14}$$

where the series on the right hand side is required to converge in the weak operator topology (and then automatically also converges in the strong operator topology).

Integration. Just as integrals can be defined relative to a probability measure $\mathbb{P}$, $\int \mathbb{P}(d\omega)\, f(\omega)$, one can define integrals relative to a POVM. Such integrals have occurred above in (6) and (10). One can define them by

$$\Big\langle \psi \Big| \int E(d\omega)\, f(\omega) \Big| \psi \Big\rangle = \int \langle \psi|E(d\omega)|\psi\rangle\, f(\omega)\,. \tag{15}$$

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