Existence, uniqueness and long time behavior of a CSL equation

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The Equation

We want to analyze the following stochastic differential equation (SDE) in the Hilbert space $\mathcal{H} \equiv \mathcal{L}^2(\mathbb{R})$

$$d\psi_t = \left[-\frac{i}{\hbar} \frac{p^2}{2m} \, dt + \sqrt{\lambda} \, (q - \langle q \rangle_t) \, dW_t - \frac{\lambda}{2} (q - \langle q \rangle_t)^2 \, dt \right] \psi_t, \qquad \psi_0 = \psi$$

where **p** is the momentum operator, **q** the position operator, **W_t** is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\langle q \rangle_t = \langle \psi_t | q | \psi_t \rangle$.

The equation describes the evolution of the wave function of a free quantum “particle”, subject to random and spontaneous localizations in space. It is one of the fundamental equations of collapse models.

m is a mass of the particle and **λ** is a positive constant which sets the strength of the collapse process.

Three questions

There are three relevant questions related to this equation:

1. **Existence and uniqueness of solutions:** which precise statement can be made?
2. **Explicit solution**: the equation is simple enough that it should be possible to find the explicit solution.
3. **Long time behavior of the solution:** several conjectures indicate that the solution of the equation has a well defined asymptotic behavior, namely that it converges asymptotically to wave function having a Gaussian shape. Which precise statement can be made?

1. Existence and uniqueness of solutions

Theorem: Given T > 0, there exists a unique strong solution of the equation for any t in [0,T].

The stochastic process ψ_t is a *strong solution* when

1. it is a.s. continuous.
2. it is measurable with respect to the filtration generated by the Wiener process.
3. with probability 1 it satisfies

$$\psi_t = \psi - \frac{i}{\hbar}\int_0^t \frac{p^2}{2m}\psi_s ds + \sqrt{\lambda}\int_0^t (q - \langle q\rangle_s)\psi_s dW_s - \frac{\lambda}{2}\int_0^t (q - \langle q\rangle_s)^2 \psi_s ds$$

With t in [0,T].

D. Gatarek and N. Gisin, *Journ. Math. Phys.* **32**, 2152 (1991).

A.S. Holevo, *Prob. Theory and Relat. Fields* **104**, 483 (1996).

2. The explicit solution

Step 1: linearization of the equation. This is a well known procedure in the literature.

Consider the linear equation

$$d\phi_t \;=\; \left[-\frac{i}{\hbar}\frac{p^2}{2m}\,dt \;+\; \sqrt{\lambda}\,q\,d\xi_t \;-\; \frac{\lambda}{2}\,q^2\,dt \right]\phi_t, \qquad \phi_0 = \psi$$

where ξ_t is a Wiener process with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

The square norm $\|\phi_t\|^2$ is a martingale; choose $\mathbb{Q}$ such that $\mathrm{d}\mathbb{P} = \|\phi_t\|^2 \mathrm{d}\mathbb{Q}$.

Define:

$$\psi_t \;=\; \begin{cases} \phi_t / \|\phi_t\| & \text{if: } \|\phi_t\| \neq 0, \\ 0 & \text{otherwise;} \end{cases}$$

Then ψ_t solves the original non-linear equation, where the two noises are related by the Girsonov transformation

$$dW_t \;=\; d\xi_t \;-\; 2\sqrt{\lambda}\langle q\rangle_t\,dt$$

The explicit solution

Step 2: reduction to a deterministic equation. The idea is to perform a sort of “interaction picture” transformation in order to remove the stochastic part of the evolution.

Consider the operators

$$\begin{array}{cccc} \mathcal{Q}_a : & \mathcal{D}(\mathcal{Q}_a) \subseteq \mathcal{L}^2(\mathbb{R}) & \rightarrow & \mathcal{L}^2(\mathbb{R}) \\ & \phi(x) & \rightarrow & e^{ax}\phi(x) \end{array} \qquad \begin{array}{cccc} \mathcal{P}_a : & \mathcal{D}(\mathcal{P}_a) \subseteq \mathcal{L}^2(\mathbb{R}) & \rightarrow & \mathcal{L}^2(\mathbb{R}) \\ & \phi(x) & \rightarrow & \phi(x+a) \end{array}$$

Define

$$\boxed{\phi_t \; = \; \exp\left(i\vartheta_t\hbar\right)\, \mathcal{Q}_{\sqrt{\lambda}\xi_t + (ic_t/\hbar)}\, \mathcal{P}_{-b_t}\, \varphi_t}$$

with

$$\begin{array}{rcll} m\dot{b}_t & = & c_t - i\hbar\sqrt{\lambda}\,\xi_t & \qquad b_0 = 0 \\ \dot{c}_t & = & 2i\,\hbar\,\lambda\, b_t & \qquad c_0 = 0 \\ \dot{\vartheta}_t & = & -\,i\hbar\,\lambda\, b_t^2 - \frac{1}{2m}\, c_t^2 + \frac{i\hbar}{m}\sqrt{\lambda}\,\xi_t\, c_t + \frac{\lambda\hbar^2}{2m}\,\xi_t^2, & \qquad \theta_0 = 0 \end{array}$$

The explicit solution

Then φ_t solves the Schrödinger equation for the **non self-adjoint (NSA) Harmonic oscillator**

$$i\hbar \frac{d}{dt}\varphi_t = \left[\frac{p^2}{2m} - i\hbar\lambda\, q^2\right]\varphi_t, \qquad\qquad \varphi_0 = \psi$$

The Green's function is:

$$g_t(x,y) = \sqrt{\frac{\lambda}{\upsilon\pi\sinh\upsilon t}}\exp\left[-\frac{\lambda}{\upsilon}(x^2+y^2)\coth\upsilon t + 2\,\frac{\lambda}{\upsilon}\,x\,y\sinh^{-1}\upsilon t\right]$$

with $\upsilon = (1+i)\omega/2$ and $\omega = 2\sqrt{\hbar\lambda/m}$. Define:

$$\begin{aligned}\mathcal{T}_t: \quad \mathcal{D}(\mathcal{T}_t) \subseteq \mathcal{L}^2(\mathbb{R}) &\quad\to\quad \mathcal{L}^2(\mathbb{R}) \\ \phi(x) &\quad\to\quad \int dy\, g_t(x,y)\phi(y),\end{aligned}$$

The explicit solution

To summarize, the solution of the linear SDE can be written has

$$\phi_t \;=\; \exp\left(i\vartheta_t\hbar\right)\, \mathcal{Q}_{\sqrt{\lambda}\xi_t+(ic_t/\hbar)}\, \mathcal{P}_{-b_t}\, \mathcal{T}_t\, \psi$$

Theorem: With probability 1, and for any T > 0, the following statements are true for t in [0,T]:

$$\begin{array}{llcl}
1. & \psi \in \mathcal{L}^2(\mathbb{R}) & \Rightarrow & \phi_t \in \mathcal{L}^2(\mathbb{R}) \\
2. & \psi \in \mathcal{L}^2_B(\mathbb{R}) & \Rightarrow & \phi_t \text{ solves the linear SDE} \\
3. & \psi \in \mathcal{L}^2_C(\mathbb{R}) & \Rightarrow & \lim_{t\to 0} \|\phi_t - \psi\|_{\text{sup}} = 0.
\end{array}$$

Where $\mathcal{L}^2_B(\mathbb{R})$ and $\mathcal{L}^2_C(\mathbb{R})$ are, respectively, the subspace of all bounded and the subspace of all bounded continuous functions of $\mathcal{L}^2(\mathbb{R})$ and $\|\cdot\|_{\text{sup}}$ is the supremum norm.

3. The asymptotic behavior

The **idea** is the following. The **eigenvalues** of the NSA harmonic oscillator

$$H = \frac{p^2}{2m} - i\hbar\lambda\, q^2$$

are:

$$\lambda_n = \frac{1-i}{2}\hbar\omega_n, \qquad \omega_n = \left(n + \frac{1}{2}\right)\omega$$

The imaginary part contributes with a damping factor to the evolution, which is the bigger, the higher the eigenvalue. Accordingly, one expects only the ground state (which is a Gaussian) to survive, as t approaches infinity.

Conjecture: "every initial state converges asymptotically to a wave function having a Gaussian shape".

There have been several proofs of this conjecture, none of which is fully rigorous.

The asymptotic behavior

The **eigenstates** are (like for the SA harmonic oscillator)

$$\phi_n(x) \;=\; \sqrt{a}\, e^{-a^2x^2/2} H_n(ax), \qquad a^4 = -2i\lambda m/\hbar$$

and $H_n(z)$ is the Hermite polynomial of degree n.

The situation looks very similar to the standard SA harmonic oscillator, with the only difference that the argument of the Hermite polynomials is complex rather than real.

The states ϕ_n form a complete set which is also minimal, and are linearly independent. However, they **do not** form a (Schauder) basis.

E.B. Davies, *Proc. Roy. Soc. London. A* **455**, 585 (1999).

E.B. Davies, *J. London Math. Soc.* **70**, 420 (2004).

E.B. Davies, *Linear Operators and their Spectra,* Cambridge University Press (2007).

The asymptotic behavior

However, with some caution one can still work with these eigenstates.

Define

$$P_n \phi \;\equiv\; \langle \phi | \phi_n^* \rangle \, \phi_n \quad \Rightarrow \quad P_n P_m = \delta_{n,m} P_n \quad \text{and} \quad \| P_n \| \;\simeq\; e^{2\alpha n}$$

One has:

$$\mathcal{T}_t \;=\; \sum_{n=0}^{\infty} e^{-(1+i)\omega_n t/2} P_n \qquad \text{for } t > t_1$$

Moving back to the linear SDE, one has:

$$e^{i\vartheta_t \hbar}\, \mathcal{Q}_{\sqrt{\lambda}\xi_t + (ic_t/\hbar)}\, \mathcal{P}_{-b_t}\, \mathcal{T}_t\, \phi(x) \;=\; \sum_{n=0}^{\infty} A_n(t) e^{i\overline{k}_t x + \gamma_t} e^{-(1+i)\omega_n t/2} \phi_n(a(x - \overline{x}_t))$$

where $A_n(t), \overline{k}_t, \gamma_t$ and $\overline{x}_t$ are stochastic processes. The series converges for $t > t_2$

The asymptotic behavior

What remains to be proven is that, given

$$\psi_t \quad = \quad \frac{\phi_t}{\|\phi_t\|} \quad = \quad \sum_{n=0}^{\infty} \psi_t^{(n)}$$

then with probability 1 with respect to the measure $\mathbb{P}$ one has

$$\lim_{t \to \infty} \left\| \psi_t - \psi_t^{(0)} \right\| \quad = \quad 0$$

Work in progress.