Borel summability in PDEs.
The 3d Navier-Stokes equation

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Generalized Borel summability.

Generalized Borel summability has been developed into a systematic procedure to solve and study properties general nonlinear ODEs, DEs, some *classes* of PDEs and other problems.

Most differential or difference equations have neither "explicit" nor generally convergent series solutions. We can see this as due to the fact that "explicit" functions or convergent power series are not closed under all operations of interest.

On the other hand, convergence aside, asymptotic expansions can be obtained, in many more problems, rather algorithmically. They may go to all orders and even beyond all orders. Example: formal solution of $f' + f = 1/x$ is

$$\sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} + Ce^{-x}, \quad x \to \infty e^{i\epsilon} \quad (*)$$ a particular case of a transseries, discovered

and studied by Écalle. Transseries generalize asymptotic expansions, include expressions like (*) and in fact general formal solutions of differential or difference equations.

Reason for general applicability: transseries, unlike convergent series *are* closed under "all" operations.

Proposition. Transseries are closed under all algebraic operations, integration, differentiation, solution of generic differential and difference equations etc. At most power-of-factorially divergent ones are closed too.

This means we can solve such problems by means of transseries.

$$f' + f = 1/x \Rightarrow f = \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} + Ce^{-x}, \quad x \to \infty e^{i\epsilon} \qquad (*)$$

But what does it mean? The series is *divergent*. Key remark: divergence, like convergence is a concept relative to a notion of convergence, a topology, or a procedure of summation. The expression (*) is "naively" divergent.

Borel summation relies on a proper combination of Laplace ($\mathcal{L}$) -inverse Laplace pair [for spectral reasons]. Since $\mathcal{L}\mathcal{L}^{-1} = I$ we have

$$\sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} = \mathrm{I} \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} = \mathcal{L}\mathcal{L}^{-1} \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}}$$

$$= \mathcal{L} \sum_{k=0}^{\infty} \mathcal{L}^{-1} \frac{k!}{x^{k+1}} = \mathcal{L} \sum_{k=0}^{\infty} p^k = \int_0^{\infty e^{-i\epsilon}} \frac{e^{-xp}}{1-p} dp = Ei(x) - \pi i e^{-x}$$

This procedure, properly justified, is nothing else but Borel summation. When it succeeds, it is guaranteed to produce objects that have the same properties as the formal expansions, since the procedure is, conceptually at least, the identity.

An essential point is to find the variable ("critical time") for which that the rate of divergence is power-one of the factorial (Gevrey-1). A different ("accelerated") variable can be used, provided appropriate measures are taken to compensate for the substantial problems created by a different variable.

Generalized Borel summation (Écalle) allows for singularities along the Laplace contours and some forms of superexponential growth.

Conjecture. Generalized Borel summable transseries are closed as well.

Proved in *specific settings* such as generic ODEs (Ecalle, Ramis, Balser, Braaksma, OC,...), difference equations (Braaksma and Kuik) and classes of PDEs (OC, Tanveer, Lebowitz)

PDEs

Borel summation is especially useful for PDEs since even existence of solutions is often not answered by classical methods. Formal solutions may exist for many more equations. However, not only formal PDE solutions can be assumed (rough data) nor, if they are, is it always the case that they can be formally calculated.

The natural *adaptation* of the method to PDEs (OC, ST) is that of (C-K) regularizing transformations. If the solutions $\tilde{y}$ of an equation are Borel summable, then $\mathcal{L}^{-1}\tilde{y}$ are convergent series, thus analytic functions. Therefore, if we take an appropriate $\mathcal{L}^{-1}$ of the equation itself, the new equation is likely more regular.

Based on actual/expected divergence with generic analytic ICs, one finds the critical variables (leading to sharp summation and not to over/undersummation), transforms the equation accordingly, the resulting equation has global and regular solutions, and back-transform produces actual solutions.

We have used this approach on the time-harmonic Schrödinger equation to show ionization of the 3d Hydrogen atom (OC, Lebowitz, Tanveer) and other models, in external fields of any strength.

Illustration of (1): The Heat equation $\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \; (*)$

Formal solutions as series in t diverge like $k!$. Critical variable: $1/t$. With $t = 1/T$ and $f(t) = t^{-1/2} g(T)$, *Borel transform in* T and substitution $s = 2i\sqrt{p}$ brings (*) to the *regular* wave equation

$$G_{xx} - G_{ss} = 0 (**)$$

$G(x,s) = F_1(x+s) + F_2(x-s)$. Regularity and decay conditions $\Rightarrow F_1 = F_2 := u$, from which reverting the transformations, we get

$$f(t,x) = t^{-1/2} \int_{-\infty}^{\infty} u(s) e^{-(x-s)^2/4t} ds$$

Nonlinear nth order systems of evolution PDEs

(OC, S. Tanveer 2007 AIHP)

Consider a nonlinear system in sector $\mathbf{x} \in S \subset \mathbb{C}^d$, where C-K conditions fail

$$\mathbf{u}_t + \mathcal{P}(\partial_{\mathbf{x}}^{\mathbf{j}})\mathbf{u} + \mathbf{g}\left(\mathbf{x}, t, \{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\}\right) = 0; \;\; \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_I(\mathbf{x})(**)$$

$\mathbf{u} \in \mathbb{C}^r$, $t \in (0, T)$; $\mathbf{g}$, $\mathbf{u}_I$ and $\mathbf{u}$ analyticity and decaying as $\mathcal{P}$ is subject to a nondegeneracy condition and g is analytic in $\mathbf{x}^{-1/N}$ for some N. Procedure works for small t, or large x or both. Examples of equations:

Kuramoto-Sivashinsky: $u_t + u_{xxxx} + u_{xx} + uu_x = 0$

Modified Harry-Dym: $H_t - H^3 H_{xxx} + H_x + H^3/2 = 0$

Thin film: $\mathbf{h}_t + \nabla \cdot (\mathbf{h}^3 \nabla \Delta \mathbf{h}) = 0$

Summation for large $\mathbf{x}$ in an appropriate sector in $\mathbb{C}^d$.

Theorem. Under these assumptions there is a unique solution of (**) for $t < T$ and

$$\mathbf{u}(\mathbf{x},t) = \int_{\mathbb{R}^{+d}} e^{-\mathbf{p}\cdot\mathbf{x}^{\frac{n}{n-1}}} \mathbf{U}(\mathbf{p},t)d\mathbf{p}$$

Furthermore, $\mathbf{U}$ is exponentially bounded. It is analytic in $\mathbf{p}^{\frac{1}{nN}}$ at $\mathbf{0}$ and for $|\arg p_i| < \frac{\pi}{(n-1)}$ if $\mathbf{u}_0$ is analytic, and $\frac{n}{n-1}$ is optimal ($\mathbf{x}^{\frac{n}{n-1}}$ is the critical variable).

Similar results hold for any $\mathbf{x}$ for small t.

Navier-Stokes

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} + \nabla p = 0 \ \text{ or } \mathbf{f}; \quad \nabla\cdot\mathbf{v} = 0$$

where $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{x} \in \mathbb{R}^3$ or in $\mathbb{T}^3$.

Spatial Fourier transform is convenient; the projection on divergence-free fields (Hodge projection) has the explicit form

$$\mathcal{P} = \left(I - \frac{\mathbf{k}}{k}\mathbf{k}\cdot\right); \quad |\mathbf{k}| = k$$

We let $\hat{*}$ be the ($\mathbf{k}$-space) Fourier convolution, $\check{*}$ the Laplace convolution $(f\check{*}g)(p) = \int_0^p f(s)g(p-s)ds$ and $\overset{*}{*}$ the "double", $\mathbf{k}, p$ convolution. Using the repeated index summation convention, we write

$$\hat{\mathbf{v}}_t + \nu k^2 \hat{\mathbf{v}} = -ik_j \mathcal{P}[\hat{\mathbf{v}}_j \hat{*} \hat{\mathbf{v}}] + \hat{\mathbf{f}}, \quad \hat{\mathbf{v}}(\mathbf{k}, \mathbf{0}) = \hat{\mathbf{v}}_0(\mathbf{k})$$

The forcing $\mathbf{f}$, which of course can be zero, is assumed time-independent. The critical variable is simply $1/t$. After formal inverse Laplace transform, with $\mathbf{V} = \mathcal{L}^{-1}\hat{\mathbf{v}}$ we get a nonlinear convolution equation

$$\mathbf{V}(\mathbf{k}, p) = \mathcal{K}_j \circ \mathcal{N}_j(\mathbf{V})(\mathbf{k}, \cdot) + \mathbf{V}_0(**)$$

where $\mathcal{K}_j$ is an explicit integral operator, with kernel written in terms of Bessel functions

$$K_j(\mathbf{k}, p, p') = i\pi \frac{z'}{z}\Big(J_1(z)Y_1(z') - Y_1(z)J_1(z')\Big)k_j; \quad z = 2k\sqrt{\nu p}$$

$\mathbf{V}_0$ is related to $\mathbf{v_0}$, and $\mathcal{N}_j$ is a quadratic nonlinearity involving $\mathbf{V}$ itself.

It is this equation which has global solution, exponentially bounded in p. Its Laplace transform exists and solves the original N-S.

The **norm** suggested by the expected behavior, and thus adapted to the problem is

$$\|\mathbf{V}\|_{\mu,\beta,\alpha} = \sup_{\mathbf{k},p} \left|(1+k)^{\mu}(1+p^2)e^{-\alpha p+\beta k}\mathbf{V}(\mathbf{k},p)\right| \quad (*)$$

We also define

$$\|\mathbf{g}\|_{\mu,\beta} = \sup_{\mathbf{k}} \left|(1+k)^{\mu}e^{\beta k}\mathbf{g}(\mathbf{k})\right| \quad (*)$$

Assumptions. $\mu > 3, \beta \geq 0$, $\|\mathbf{f}\|_{\mu,\beta} < \infty, \|\mathbf{v}_0\|_{\mu+2,\beta} < \infty$ ($\beta = 0$ allows for nonanylitic data).

Theorem (a) Then $\exists\alpha = \alpha_0$ such that (**) has a unique solution. Therfore, NS has a unique solution, and furthermore $\|\mathbf{v}(.,t)\|_{\mu+2,\beta}$ finite and $\mathbf{v}$ analytic t-plane for $\Re\frac{1}{t} > \alpha_0$. For $\beta > 0$, $\mathbf{v}$ is analytic in x with the same analyticity width as the initial data.

(b) If $\mathbf{v}_0$ is analytic, then $\mathbf{V}(\mathbf{x},\mathbf{p})$ is analytic in $p \in \{0\} \cup \mathbb{R}^+$. In particular the solution exists for $T < T_0 = \alpha^{-1}$; moreover, for $t \ll 1$,

$$\mathbf{v}(\mathbf{x},t) \sim \tilde{\mathbf{v}}(\mathbf{x},\mathrm{t}) = \sum_{m=0}^{\infty} t^m \mathbf{v}_m(\mathbf{x},t); \quad \tilde{\mathbf{v}} \text{ Borel-summable to } \mathbf{v}$$

In particular,local existence follows. The space of initial conditions not identical to the classical ones which use viscosity, but similar. (However, energy methods provide *local* existence, same for any viscosity.)

What is, what can be, and what is not obtained in this way

(1) The dual solution $\mathbf{V}$ *exists* globally, for any IC. Local/global existence in original equation follow from the *properties* of a given $\mathbf{V}$.

(2) One expects global existence, at least for nice enough initial conditions. Then $\mathbf{V}$ or a modification thereof, see below, is expected to decay in p. This can be shown for *specific data* (in principle, of any size) in the following way:

- (a) $\mathbf{V}$ can be calculated rigorously with (in principle) any prescribed accuracy on an *initial interval* $[0, p_0)$: *apriori* estimates $\mathbf{V}$ and all its derivatives are *known*, even though nonoptimally. (At this stage, one can see that the solution can also be calculated numerically with *controlled* errors.)

- (b) These values are used to control the norm of an operator $\mathcal{T}$, which is contractive for *sufficently large* p and the contractivity factor depends on the size of the solution on an initial segment of the p axis.

–This also gives good control of existence in physical space, on an interval $[0, T_0(p_0))$, T_0 increasing with p_0.

Concretely, the relevant constants are ($\|\cdot\| := \|\cdot\|_{\mu,\beta}$):

$$\epsilon = \nu^{-1/2} p_0^{-1/2},\ a = \|\hat{\mathbf{v}}\|, c = \left\| \int_{p_0}^{\infty} \hat{\mathbf{V}} e^{-\alpha_0 p} dp \right\|$$

$$\epsilon_1 = 2\epsilon \left\| \int_0^{p_0} \hat{\mathbf{V}} e^{-\alpha_0 p} dp + \hat{\mathbf{v}}_0 \right\|,\ b = \frac{e^{-\alpha_0 p_0}}{\alpha \sqrt{\nu p_0}} \left\| \int_0^{p_0} (\mathbf{V} \overset{*}{*} \mathbf{V} + \hat{\mathbf{v}}_0 \cdot \mathbf{V}) ds \right\|$$

Theorem Classical solution to NS exists on $[0, T_0)$, where

$T_0^{-1} = \alpha_0 = \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$, where the argument of the $\sqrt{\ }$ is positive.

(3) Exponential growth is an indicator of singularities in the right-half complex t plane. The type of growth (pure, with oscillation, etc) shows where the singuarities are. For IC studied we found them outside the real line. But as $\nu \to 0$ (Euler's equation), for some data, they appear to approach $\mathbb{R}^+$.

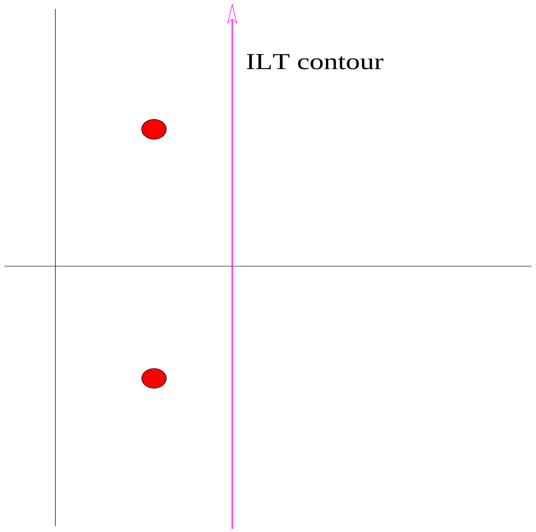

Figure 1: Poles in $1/t \in \mathbb{C}$ are collected by ILT, first step of summation, as complex growing exponentials

(4) *Oscillatory growth* can be eliminated (Tanveer) by *Ecalle acceleration*, the Borel dual to the change of variable $t \mapsto t^{\alpha}$, so that the blow up occurs in the left half plane, implying *decay* again.

Concrete, numerically assisted analysis

We solved by Runge-Kutta the integral equations in p plane, for "Kida" initial conditions, and periodic domain, sometimes with forcing.

Kida: $v_i(\mathbf{x}, 0) = v_0(P^{i-1}(\mathbf{x}))$; P the circular permutation of indices (permutational symmetry is preserved).

$$v_0(x, y, z) = \sin z \Big(\cos 3y \cos z - \cos 3z \cos y \Big)$$

and

$$\mathbf{f} = 0 \quad or \quad \mathbf{f} = \mathbf{f}_1 = \frac{1}{5}\mathbf{v_0}$$

The symmetries make the calculations shorter. Also, some people believe Euler blows up for these ICs. We do see some numerical evidence of that. It is then interesting to see how viscosity intervenes in the picture.

Insofar existence time goes, this relates to the solution in p plane on which we have apriori bounds on the functions and all derivatives; then numerical approach would be rigorous provided roundoff errors were usefully upper-bounded.

Practically, the latter are probably more easily tackled by obtaining a quasi-solution with highest possible (expected) accuracy and then contracting about it.

This would give a computer assisted (instead of numerically assisted) global existence results for the specific data where decay is noted. (It does not provide any good entry into blow-up, when solution *grows*.)

We are in the process of implementing the summation + acceleration procedure.

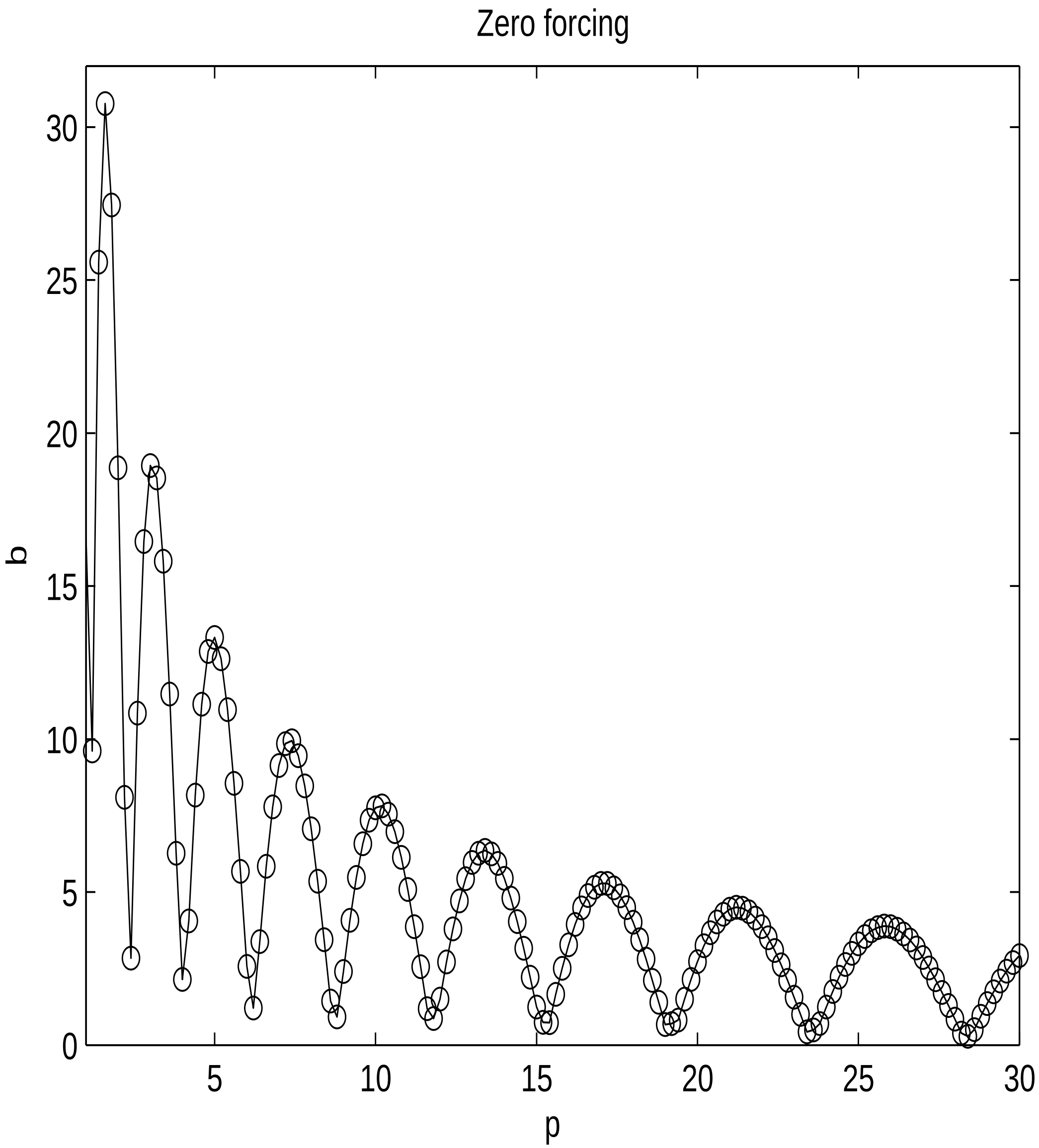

Figure 2: $\|\mathbf{V}\|_{4,0,0}$ as a function of p, for $\mathbf{f} = 0, \nu = 1$.

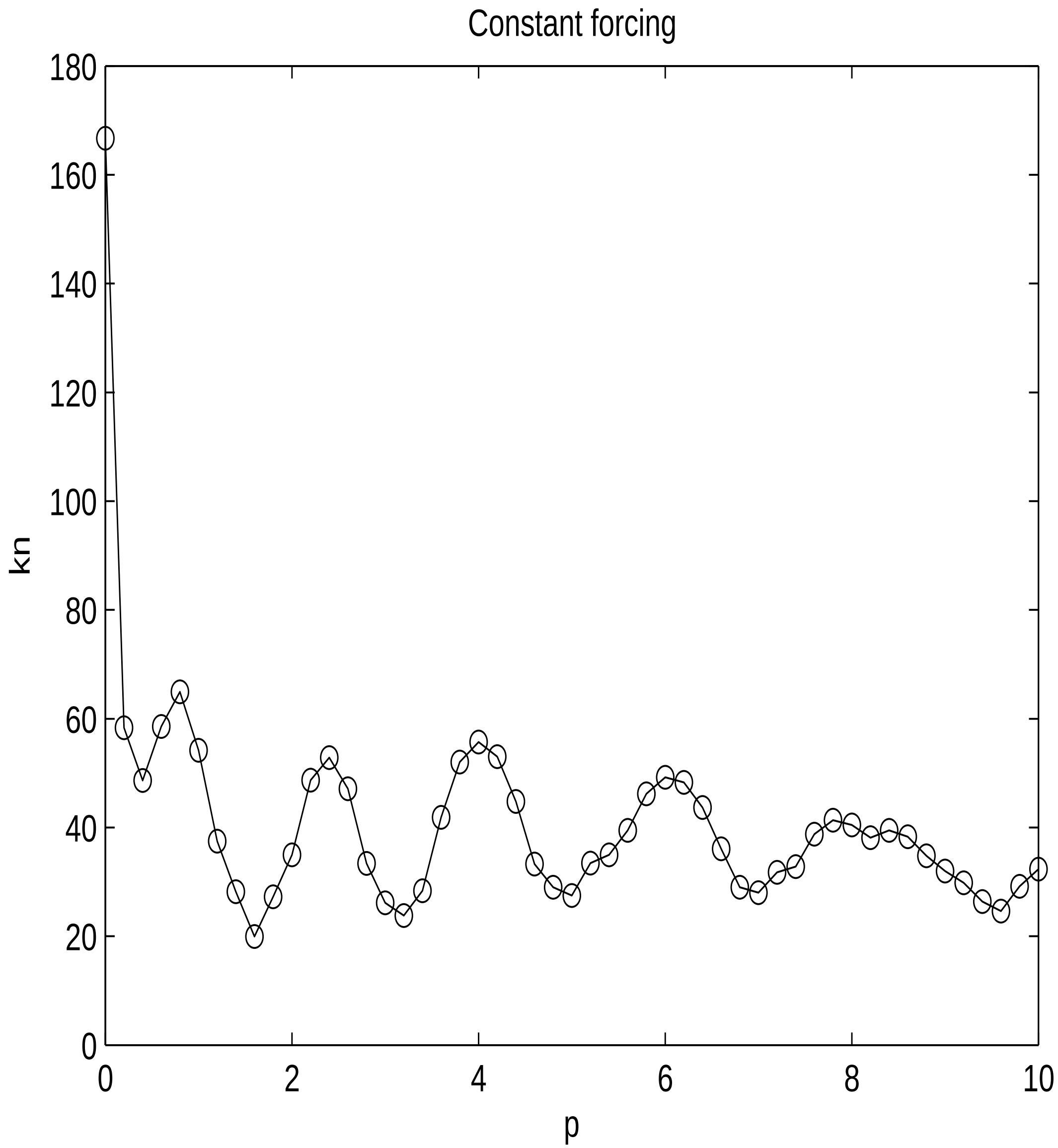

Figure 3: $\|\mathbf{V}\|_{4,0,0}$ as a function of p, for $\mathbf{f} = \mathbf{f}_1, \nu = 0.16$.

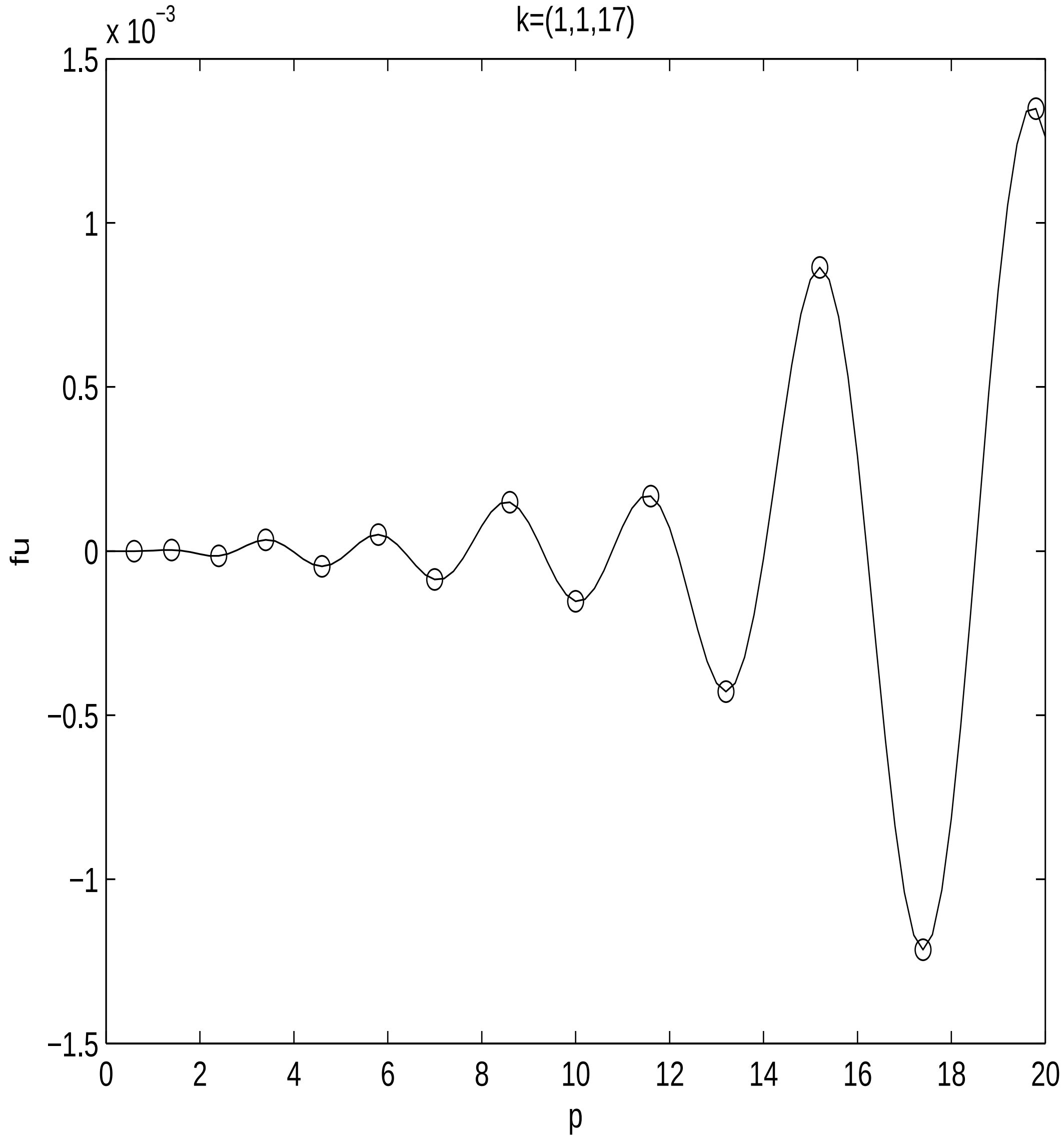

Figure 4: $\mathbf{V}((1,1,17),p)$ as a function of p, for $\mathbf{f} = 0\nu = 0.1$.

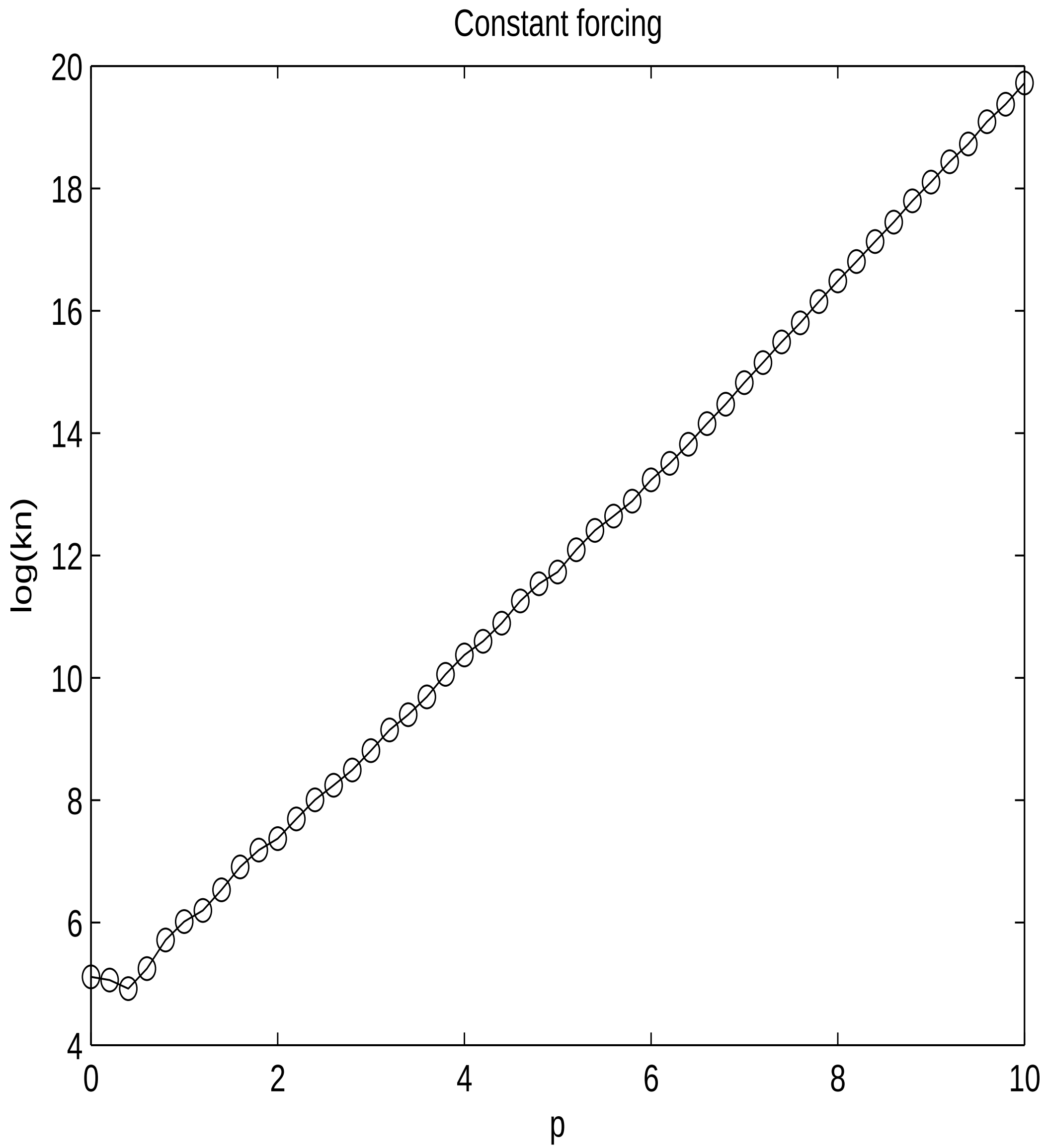

Figure 5: $\ln \|\mathbf{V}\|_{4,0,0}$ as a function of $\ln p$, for $\mathbf{f} = \mathbf{f}_1, \nu = 0.001$.

Lowest frequency of oscillation of various $\mathbf{k}$ and rate of exponential growth modes dictate "neareast" singularity complex t plane singularity, thus the necessary *acceleration*. The proof that there are no singularities even closer at arbitrarily large modes is by showing that acceleration *does work* (if it does).