

MAXWELL-LORENTZ DYNAMICS

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The Maxwell-Lorentz Equations of Motion

The System of Non-Linear Partial Differential Equations

for a system of N non-rotating charge densities with form factor $\rho \in \mathcal{C}_c^\infty$:

$$\dot{\mathbf{q}}_{t,i} = \frac{\sigma_i \mathbf{p}_{t,i}}{\sqrt{m_i^2 + \mathbf{p}_{t,i}^2}} \quad =: \mathbf{v}_{t,i}(\mathbf{p}_{t,i})$$

$$\dot{\mathbf{p}}_{t,i} = \int d^3x e_i \rho(\mathbf{x} - \mathbf{q}_{t,i}) (\mathbf{E}_t(\mathbf{x}) + \mathbf{v}_{t,i}(\mathbf{p}_{t,i}) \wedge \mathbf{B}_t(\mathbf{x}))$$

$$\dot{\mathbf{E}}_t = \nabla \wedge \mathbf{B}_t - 4\pi \sum_{i=1}^N \mathbf{v}_{t,i}(\mathbf{p}_{t,i}) e_i \rho(\cdot - \mathbf{q}_{t,i})$$

$$\dot{\mathbf{B}}_t = -\nabla \wedge \mathbf{E}_t$$

for $1 \leq i \leq N$. Furthermore, the Maxwell constraints:

$$\nabla \cdot \mathbf{E}_t = 4\pi \sum_{i=1}^N e_i \rho(\cdot - \mathbf{q}_{t,i}) \quad \nabla \cdot \mathbf{B}_t = 0$$

We may recast these equations of motion in the following notation:

$$\varphi_t = ((\mathbf{q}_{t,i})_{1 \leq i \leq N}, (\mathbf{p}_{t,i})_{1 \leq i \leq N}, \mathbf{E}_t, \mathbf{B}_t) \in \mathbb{R}^{3N} \oplus \mathbb{R}^{3N} \oplus \mathcal{F} \oplus \mathcal{F}$$

with \mathcal{F} being a placeholder for a convenient function space. Furthermore, a linear operator

$$A\varphi_t = (0, 0, \nabla \wedge \mathbf{B}_t, -\nabla \wedge \mathbf{E}_t)$$

and a non-linear operator

$$J(\varphi_t) = \left(\begin{array}{c} \int d^3x \, e_i \rho(\mathbf{x} - \mathbf{q}_{t,i}) \left(\mathbf{E}_t(\mathbf{x}) + \mathbf{v}_{t,i}(\mathbf{p}_{t,i}) \wedge \mathbf{B}_t(\mathbf{x}) \right) \\ -4\pi \sum_{i=1}^N e_i \mathbf{v}_{t,i}(\mathbf{p}_{t,i}) \rho(\cdot - \mathbf{q}_{t,i}) \\ 0 \end{array} \right)_{1 \leq i \leq N}$$

In this notation the equations of motion become:

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t)$$

Existence of Dynamics and Present Results

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t) \quad \text{on} \quad \mathbb{R}^{3N} \oplus \mathbb{R}^{3N} \oplus \mathcal{F} \oplus \mathcal{F}$$

Questions

- 1 What choice of \mathcal{F} is physically relevant?
 - 2 Given such a \mathcal{F} , is the existence of dynamics a well-posed initial value problem?
- Looking at the conserved total amount of energy of the Hamiltonian system:

$$H = \sum_{i=1}^N \sigma_i \sqrt{m_i^2 + \mathbf{p}_i^2} + \frac{1}{8\pi} \int d^3x (\mathbf{E}(\mathbf{x})^2 + \mathbf{B}(\mathbf{x})^2)$$

one obvious choice is: $\mathcal{F} = L^2(\mathbb{R}^3, \mathbb{R}^3)$

- For this setup and the case of $N = 1$ non-rotating smooth charge density, global existence and uniqueness solutions to

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t)$$

was proven for a dense set of initial conditions using different techniques:

- [1] A. Komech and H. Spohn, Comm. Part. Diff. Eq. (2000)
- [2] G. Bauer and D. Dürr, Ann. Henri Poincaré (2001)
- [3] H. Spohn, Dynamics of Charged Particles and their Radiation Field, Cambridge (2004)

$\mathcal{F} = L^2$ Excludes Liénard-Wiechert fields

Returning to the question of physical relevant choices for \mathcal{F} we get another hint by the form of solutions to the inhomogeneous Maxwell equations given a charge trajectory $t \rightarrow (\mathbf{q}_t, \mathbf{p}_t)$.

These are the so-called Liénard-Wiechert fields:

$$\mathbf{E}_t^{LW} := \rho * \left(\frac{e}{4\pi} \left(\frac{(1 - \mathbf{v}^2)(\mathbf{n} - \mathbf{v})}{(1 - \mathbf{v} \cdot \mathbf{n})^3 \|\cdot - \mathbf{q}\|^2} + \frac{\mathbf{n} \wedge [(\mathbf{n} - \mathbf{v}) \wedge \dot{\mathbf{v}}]}{(1 - \mathbf{v} \cdot \mathbf{n})^3 \|\cdot - \mathbf{q}\|} \right) \Big|_{t=t_{adv/ret}} \right)$$

$$\mathbf{B}_t^{LW} := \mathbf{n} \wedge \mathbf{E}_t^{LW}$$

where $t_{adv/ret} := t \pm \|\mathbf{x} - \mathbf{q}(t_{adv/ret})\|$ and $\mathbf{n} := \frac{\mathbf{x} - \mathbf{q}}{\|\mathbf{x} - \mathbf{q}\|}$.

- Although $\mathbf{E}_t^{LW}(\mathbf{x})$ and $\mathbf{B}_t^{LW}(\mathbf{x})$ are not singular, they decay like $O(\frac{1}{1+\|\mathbf{x}\|})$ for $\|\mathbf{x}\| \rightarrow 0, \infty$ and thus are for arbitrary trajectories (e.g. Schild solution) **not in L^2** .

Weighted Function Spaces

- In order to include also the Liénard-Wiechert type solutions in the existence theory we may set \mathcal{F} to be the weighted function space:

$$L_w^2 := \{ \mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \mid \sqrt{w} \mathbf{F} \in L^2 \}$$

E.g. for $w(\mathbf{x}) = \frac{1}{1+\|\mathbf{x}\|^2}$ we find $\mathbf{E}_t^{LW}, \mathbf{B}_t^{LW} \in L_w^2$. With

$$\langle \mathbf{F}, \mathbf{G} \rangle_{L_w^2} := \langle \sqrt{w} \mathbf{F}, \sqrt{w} \mathbf{G} \rangle_{L^2}$$

this is a Hilbert space for a large class of weight functions w .

- The domain of A can then be defined by

$$D_w(A) := \mathbb{R}^{3N} \oplus \mathbb{R}^{3N} \oplus H_w^{curl} \oplus H_w^{curl}$$

where $H_w^{curl} := \{ \mathbf{F} \in L_w^2 \mid \nabla \wedge \mathbf{F} \in L_w^2 \}$ which forms a Hilbert space of its own with $\langle \mathbf{F}, \mathbf{G} \rangle_{H_w^{curl}} := \langle \mathbf{F}, \mathbf{G} \rangle_{L_w^2} + \langle \nabla \wedge \mathbf{F}, \nabla \wedge \mathbf{G} \rangle_{L_w^2}$.

Existence and Uniqueness Including Liénard-Wiechert fields

Extension of the Present Result

With our result we extend the present existence theory in order to cover also Liénard-Wiechert type solutions of the Maxwell-Lorentz equations of motion. We regard the equation

$$\dot{\varphi}_t = A\varphi_t + J(\varphi_t) \quad \text{on} \quad \mathcal{H}_w := \mathbb{R}^{3N} \oplus \mathbb{R}^{3N} \oplus L_w^2 \oplus L_w^2$$

Theorem (global existence and uniqueness in \mathcal{H}_w)

Let $w(\mathbf{x}) := \frac{1}{1+\|\mathbf{x}\|^2}$ for $\mathbf{x} \in \mathbb{R}^3$ and initial value $\varphi_0 \in \mathcal{H}_w$ such that $A^j \varphi_0 \in \mathcal{H}_w$ for $j = 0, \dots, n$. Then the following holds:

- There exists a mapping $t \rightarrow \varphi_t$ in $\mathcal{C}^n(\mathbb{R}, \mathcal{H}_w)$ that solves the above equation with initial value $\varphi_t|_{t=0} = \varphi_0$. Furthermore, for all $t \in \mathbb{R}$, $A^j \frac{d^k}{dt^k} \varphi_t \in \mathcal{H}_w$ for all $k = 0, \dots, n$ and $j = 0, \dots, n - k$.
- If $t \rightarrow \tilde{\varphi}_t$ is a mapping in $\mathcal{C}^1(\Lambda \subset \mathbb{R}, \mathcal{H}_w)$ such that it also solves the above equation for $\tilde{\varphi}_t|_{t=0} = \varphi_0$, then $\tilde{\varphi}_t = \varphi_t$ for all $t \in \Lambda$.

The Applied Strategy

We basically follow the standard functional analysis argument:

- At first, one considers only the linear part of the equations of motion by setting $J = 0$, which results in the free Maxwell equations: $\dot{\varphi}_t = A\varphi_t$. It can be shown that A generates a group $(F_t)_{t \in \mathbb{R}}$ such that $\varphi_t = F_t\varphi_0$ is a solution of the free Maxwell equations for initial value $\varphi_t|_{t=0} = \varphi_0 \in D_w(A)$.
- One then regards the full equation: $\dot{\varphi}_t = A\varphi_t + J(\varphi_t)$ in its integral form:

$$\varphi_t = F_t\varphi_0 + \int_0^t F_{t-s}J(\varphi_s)ds \quad =: S[\varphi_{(\cdot)}](t)$$

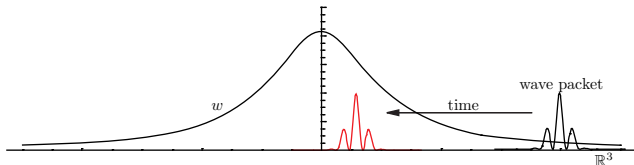
- By restricting S to convenient metric space comprising mappings $\varphi_t : (-T, T) \rightarrow \mathcal{H}_w$, which are w.r.t. the metric close to $t \rightarrow F_t\varphi_0$, one then shows that there exists a $T > 0$ such that S is a contractive self-mapping. Then apply Banach fixed point theorem.

The Generator of the Free Maxwell Dynamics

- Any solution of $\dot{\varphi}_t = A\varphi_t$, i.e. $\varphi_t = F_t\varphi_0$, possessing fields that respect the free Maxwell constraints is also a solution of:

$$\square(\mathbf{E}_t, \mathbf{B}_t) = 0$$

- Incoming waves give rise to an increase of the norm of φ_t :



because the norm of φ_t depends among other terms on:

$$\|\mathbf{E}_t\|_{L_w^2}^2 = \int d^3x w(\mathbf{x}) \|\mathbf{E}_t(\mathbf{x})\|_{\mathbb{R}^3}^2 \quad \text{e.g. } w(\mathbf{x}) := \frac{1}{1 + \|\mathbf{x}\|^2}$$

The operator A will only for the case $w = \text{const}$ generate an unitary group. And only for this case A is self-adjoint.

Smoothness of Solutions

Regularity Theorem

Theorem (Regularity in \mathcal{H}_w)

Let $w(\mathbf{x}) := \frac{1}{1+\|\mathbf{x}\|^2}$ for $\mathbf{x} \in \mathbb{R}^3$ and initial value $\varphi_0 \in \mathcal{H}_w$ such that $A^n \varphi_0 \in \mathcal{H}_w$ for any $n \in \mathbb{N}$.

Then the unique solution

$$t \rightarrow \varphi_t = ((\mathbf{q}_{t,i})_{1 \leq i \leq N}, (\mathbf{q}_{t,i})_{1 \leq i \leq N}, \mathbf{E}_t, \mathbf{B}_t)$$

corresponding to initial value φ_0 possesses fields:

$$(t, \mathbf{x}) \rightarrow \mathbf{E}_t(\mathbf{x}) \quad \text{and} \quad (t, \mathbf{x}) \rightarrow \mathbf{B}_t(\mathbf{x})$$

which are almost equal to functions in $C^\infty(\mathbb{R} \times \mathbb{R}^3, \mathbb{R}^3)$.

Summary

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- The existence theory of the Maxwell-Lorentz equations was extended to cover N non-rotating charged densities as well as Liénard-Wiechert type solutions. For a dense set of initial conditions global existence and uniqueness was proven.
- Also for a dense set of initial conditions the corresponding solutions possess smooth fields (both in time and in space), such that the equation of motion holds in the classical sense.
- The space L^2_w was advertised as a convenient tool for extending L^2 results to cover also functions that behave a little worse than typical L^2 functions, however, without losing the Hilbert space structure. Most of the Sobolev type estimates can be done in a similar fashion than in L^2 . All results actually hold on a large class of weight functions.

Thank you very much listening!