Shelly: HAPPY BIRTHDAY!

Phase Transition in continuum Potts models

Rutgers, October 2007

Phase transitions in the continuum with several species of particles

Example: the Widom-Rowlinson model.

- Repulsion among particles of different species

- High density $\rightarrow$ separation of species. Existence of several DLR extremal measures, breaking of the symmetry under species exchange.

Classical Peierls argument D. Ruelle PRL, 1971, ...

- Low density $\rightarrow$ unique DLR measure invariant under species exchange.

Goal: behavior at the critical point

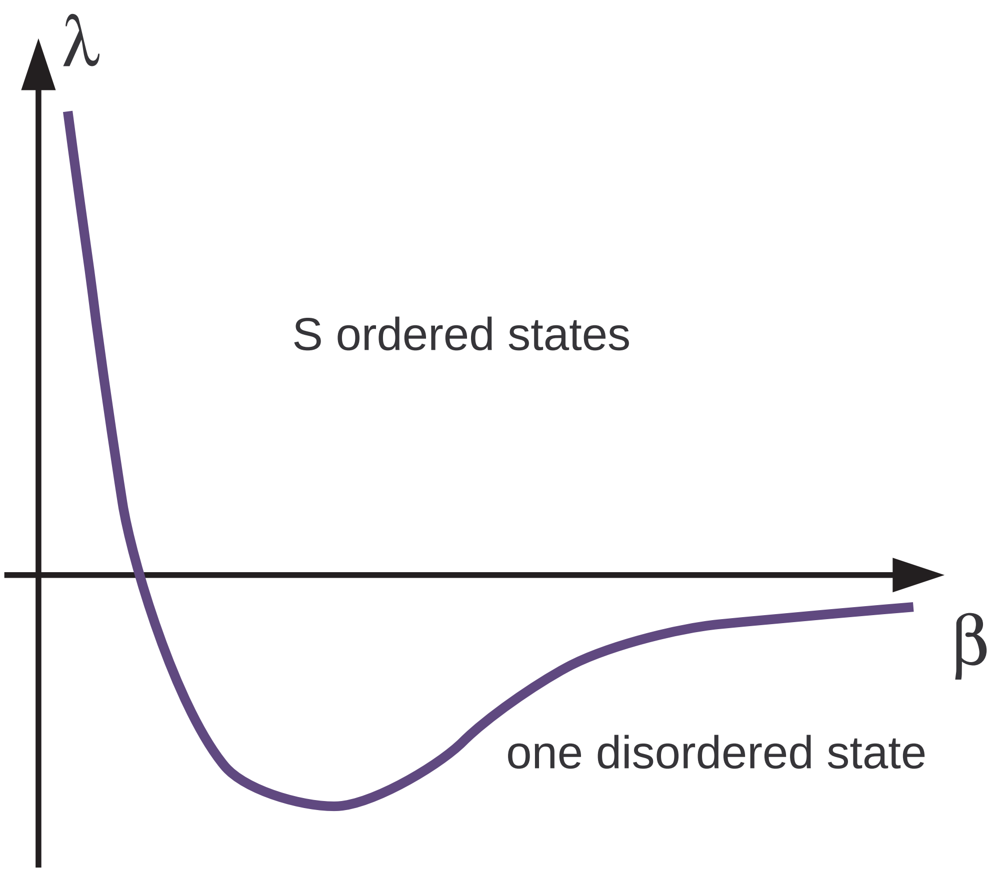
The model

Continuum version of the classical Potts model:

point particles $q = (..., r_i, s_i, ...)$, $r_i \in \mathbf{R}^d$, $d \geq 2$ position, each particle has a spin $s_i \in \{1, .., S\}$, $S > 2$.

particles with different spins repel each other.

Phase diagram: (β, λ)-plane



Behavior at the critical curve

(D.M., I. Merola, E. Presutti, Y. Vignaud)
Kac potential

Roughly speaking the result we have is:

there are $S+1$ **phases**: S "ordered phases" + one disordered phase.

total particles density undergoes a **strictly positive jump across the critical line**

This can be seen as an example of interplay between magnetic and elastic properties and interpreted as a **magneto-striction effect**.

Pirogov-Sinai strategy in the same spirit of the **LMP- model**, (Lebowitz, Mazel, Presutti) (no symmetry between disordered and ordered phases)

- the interaction is given in terms of a **Kac potential**

- **the small parameter is the inverse interaction range** ("perturbing" mean field)

Reduction to **"restricted ensembles"** i.e. configurations locally close to pure phases (local closeness of empirical averages to the mean field values in a pure phase).

NEED: exponentially decay of correlations in the restricted ensembles (cluster expansion, Dobrushin uniqueness..)

Dobrushin uniqueness condition true in the LMP model (in a certain range of the temperature) but not in continuum Potts model.

NEW: We prove

- **a finite size condition** in the Dobrushin-Shlosman spirit (involving some large boxes where self interaction is important)

- use **disagreement percolation** to construct a coupling (van der Berg: CMP (1993) and van der Berg, Maes: Ann. Prob. (1994)) to prove that our finite size condition implies exponential decay of correlations.

Kac interaction

Mean field energy density: $\rho = (\rho_1, ., \rho_S) \in \mathbf{R}_+^S$

$$e_\lambda(\rho) := \frac{1}{2} \sum_{s \neq s'} \rho_s \rho_{s'} - \lambda \sum_s \rho_s$$

Kac potential:

$$J_\gamma(r, r') = \gamma^d J^* \Big(\gamma(r' - r) \Big)$$

$\gamma > 0$ small but fixed. $J^*(r)$ a smooth, symmetric, probability kernel supported by $|r| \leq 1/2$.

Hamiltonian of the Potts model:

$$H_\lambda(q) = \int_{\mathbf{R}^d} e_\lambda(J_\gamma \star q(r)) dr$$

where, $q = (r_1, s_1, \dots, r_i, s_i, \dots)$,

$$J_\gamma \star q(r)_s = \sum_i J_\gamma(r, r_i) \mathbf{1}_{s_i = s}$$

Theorem

For any $d \geq 2$, $S > 2$ and $\beta > 0$ there is $\gamma^* > 0$ so that for any $\gamma \leq \gamma^*$ there exist $\lambda_{\beta,\gamma}$ and $S+1$ mutually distinct, extremal DLR measures at $(\beta, \lambda_{\beta,\gamma})$.

Restricted ensemble

Partition of $\mathbf{R}^d$ in cubes $C_x^{(\ell)}$ of side ℓ, $\ell < \gamma^{-1}$ (=range of the interaction) $x \in \ell\mathbf{Z}^d$

$\mu_\Lambda(\cdot|\bar{\rho}_{\Lambda^c})$= **marginal distribution** of the Gibbs measure **on the density variables**

$$\rho_\Lambda : \ell\mathbf{Z}^d \cap \Lambda \times \{1, \dots, S\} \to [0, \infty)$$

given the b.c. $\bar{\rho}_{\Lambda^c}$

$$\mu_\Lambda(\cdot|\bar{\rho}_{\Lambda^c}) = \frac{1}{Z(\bar{\rho}_{\Lambda^c})} e^{-H_\lambda^{(\mathrm{eff})}(\rho_\Lambda|\bar{\rho}_{\Lambda^c})}$$

In a neighbor of the ground states

$$H_\lambda^{(\mathrm{eff})}(\rho_\Lambda|\bar{\rho}_{\Lambda^c}) \sim H_\lambda^0(\rho_\Lambda|\bar{\rho}_{\Lambda^c})\beta\ell^d$$

$$H^0_\lambda(\rho_\Lambda|\bar\rho_{\Lambda^c}) = -\frac{1}{\beta}\sum_{x,s} I(\rho_\Lambda(x,s))$$
$$+ \sum_{x\in\ell\mathbf{Z}^d} [e_\lambda(\mathbf{J}^{(\ell)}_\gamma \star \rho_\Lambda \cup \bar\rho_{\Lambda^c}(x)) - e_\lambda(\mathbf{J}^{(\ell)}_\gamma \star \bar\rho_{\Lambda^c}(x))]$$

$I(\rho) = \rho[\log\rho - 1]$

$\mathbf{J}^{(\ell)}_\gamma(x_1,x_2)$ constant on the cubes $C^{(\ell)}_{x_i}$, $i = 1,2$

$$\mathbf{J}^{(\ell)}_\gamma(x_1,x_2) = \frac{1}{\ell^{2d}}\int_{C^{(\ell)}_{x_1}}\int_{C^{(\ell)}_{x_2}} J_\gamma(r,r')$$

Pure phases: homogeneous minimizers of H^0_λ: for all β, $\exists\ \lambda_\beta$ such that $H^0_{\lambda_\beta}$ has $S+1$ minimizers: S "ordered phases" + one disordered phase.

Call $\mathbf{u} = (u_1,\dots,u_S)$ one of these minimizers.

GOAL: exponential decay of correlations of

$$\mu_\Lambda^{(\mathbf{u})}(\cdot|\bar\rho_{\Lambda^c}) = \frac{1}{Z(\bar\rho_{\Lambda^c})} e^{-H_\lambda^0(\rho_\Lambda|\bar\rho_{\Lambda^c})} \chi_{\Omega^{(\mathbf{u})}}(\rho_\Lambda)$$

$$\Omega^{(\mathbf{u})} = \{\rho_\Lambda : |\rho_\Lambda(x,s) - u_s| \le \zeta, \forall x, \forall s\}$$

ζ small. (χ_A= characteristic function of A)

• **Variational problem** for λ close to λ_β,

$$\min_{\rho_\Lambda \in \Omega^{(\mathbf{u})}} H_\lambda^0(\rho_\Lambda|\bar\rho_{\Lambda,i}) = H_\lambda^0(\rho^*_{\Lambda,i}|\bar\rho_{\Lambda,i}), \qquad i = 1,2$$

$$|\rho^*_{\Lambda,1}(x,s) - \rho^*_{\Lambda,2}(x,s)| \le c e^{-\gamma\omega \text{dist}(x,\Lambda^c_{\neq})}$$

$$\Lambda^c_{\neq} = \{y \in \Lambda^c : \exists s : \bar\rho_{\Lambda,1}(y,s) \neq \bar\rho_{\Lambda,2}(y,s)\}$$

• **Gaussian approximation**: by Taylor expanding $H_\lambda^0(\rho_\Lambda|\bar\rho_{\Lambda,i})$ we get two gaussians with exponentially close variances and means.

Volume large to have exponential decay but **small** to control the error in the Taylor expansion.