Crossing Probabilities in the rarefaction front of TASEP

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1 TASEP

Totally asymmetric simple exclusion process $\eta_t \in \{0,1\}^Z$

$$10 \to 01 \quad \text{rate } 1$$

Other transitions are prohibited.

Generator:

$$Lf(\eta) = \sum_j [f(\eta - \delta_j + \delta_{j+1}) - f(\eta)] \tag{1.1}$$

where $\delta_j(x) = 1$ if and only if $x = j$.

Product invariant measures ν_ρ.

Hydrodynamics:

Local equilibrium:

$$\lim_{N\to\infty} \nu^N S(tN)\theta_{[rN]} f = \nu_{u(r,t)} f, \tag{1.2}$$

LLN:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in Z} \phi(x/N)\eta_t(x) = \int \phi(r)u(r,t)dr \text{a.s.}$$

$u(r,t)$ is (weak) solution of inviscid Burgers equation:

$$\partial_t u + \partial_r \Big(u(1-u) \Big) = 0 \tag{1.3}$$

when $\nu^N([rN]) \to u(r,0)$

Rarefaction front: Heaviside initial configuration:

$$\dots 111000 \dots = \hat{1}\hat{0} \quad \text{(notation)}$$

Then $u(r, 0) = 1\{r \le 0\}$ and

$$u(r,t) = \begin{cases} 1, & r \le -t; \\ \frac{t-r}{2t}, & -t < r \le t. \\ 0, & r > t. \end{cases} \tag{1.4}$$

Characteristics Lines that carry the solutions:

$a(r,t) \in \mathbb{R}$

$u(a(r,t),t) = u(r,0)$

$\frac{d}{dt}a(r,t) = 1 - 2(u(a(r,t),t)$

$a(r,t) = r + (1 - 2u(r,0))t$

Infinitely many characteristics in the rarefaction front

Second class particles $\eta_t \in \{0,1,2\}^Z$

$$12 \longrightarrow 21 \quad \text{rate } 1$$

$$20 \longrightarrow 02 \quad \text{rate } 1$$

$$10 \longrightarrow 01 \quad \text{rate } 1$$

Other transitions prohibited.

Initial configuration

$$\ldots 1112000 \ldots$$

$X_2(t)$ position of the (unique) second class particle at time t

$$\lim_{t\to\infty} \frac{X_2(t)}{t} = U \qquad\qquad U \text{ uniform in } [-1,1]$$

almost surely (F-Kipnis, Mountford-Guiol, F-Martin-Pimentel).

Multiclass process Consider $\eta_t \in Z^Z$ with rates

$$k\ell \to \ell k \quad \text{rate } 1 \qquad \text{if } k \prec \ell$$

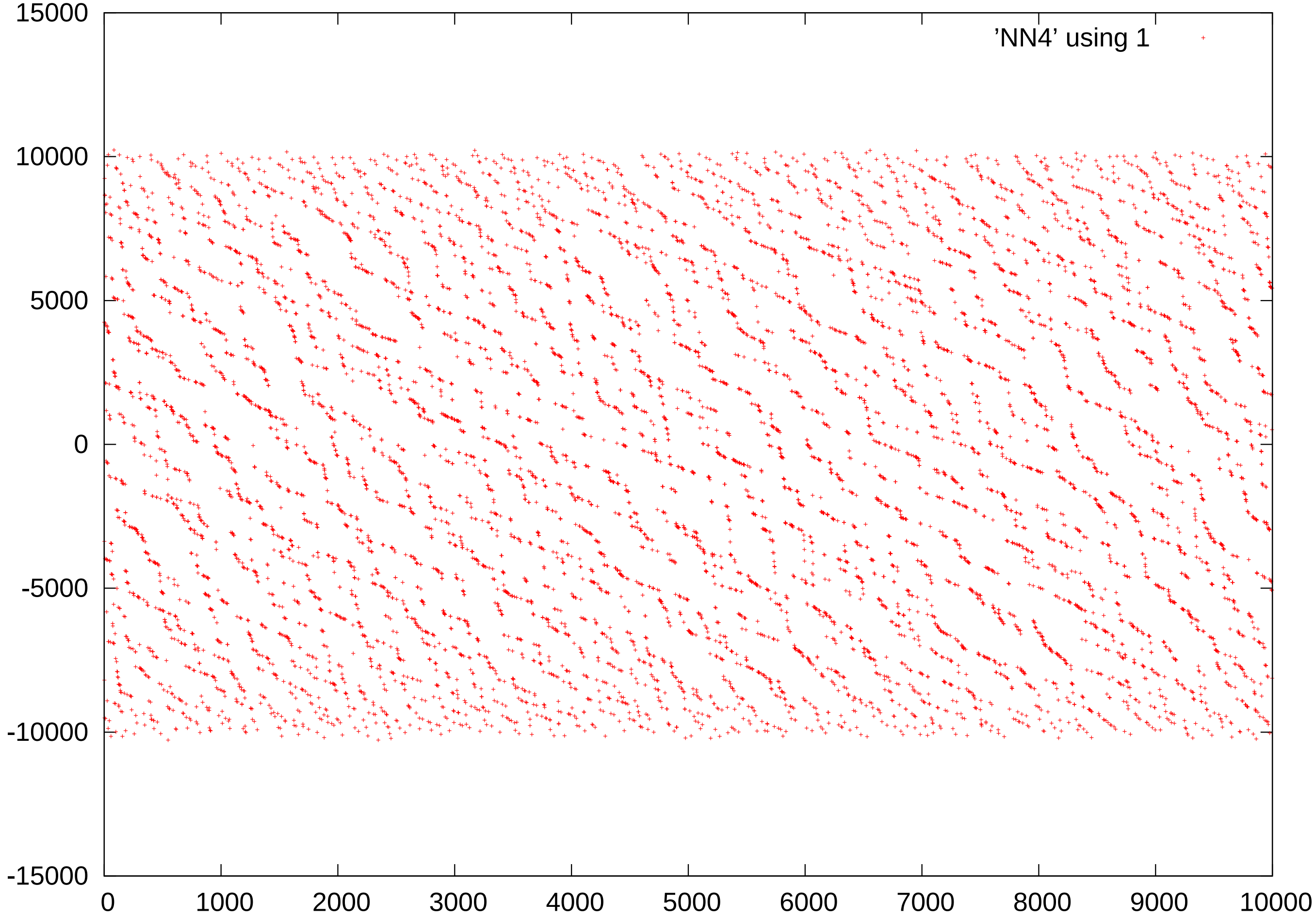
$k \prec 0$ for all k and if $k, \ell \neq 0$, $k \prec \ell$ if $k < \ell$

Initial configuration

$$\ldots(-3)(-2)(-1)123\ldots$$

and ask for the behavior of this.

Simulations: 10000 particles at t=10000. The positions shown are the positions relative to where the particles started; to get absolute positions, you need to add n to the position of the nth particle. As shown, the picture is a snapshot of a stationary (from left to right) process.



'NN4' using 1
15000
10000
5000
0
-5000
-10000
-15000
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000

More modest: second and third class particles Start from

$$\dots 11123000 \dots = \hat{1}23\hat{0}$$

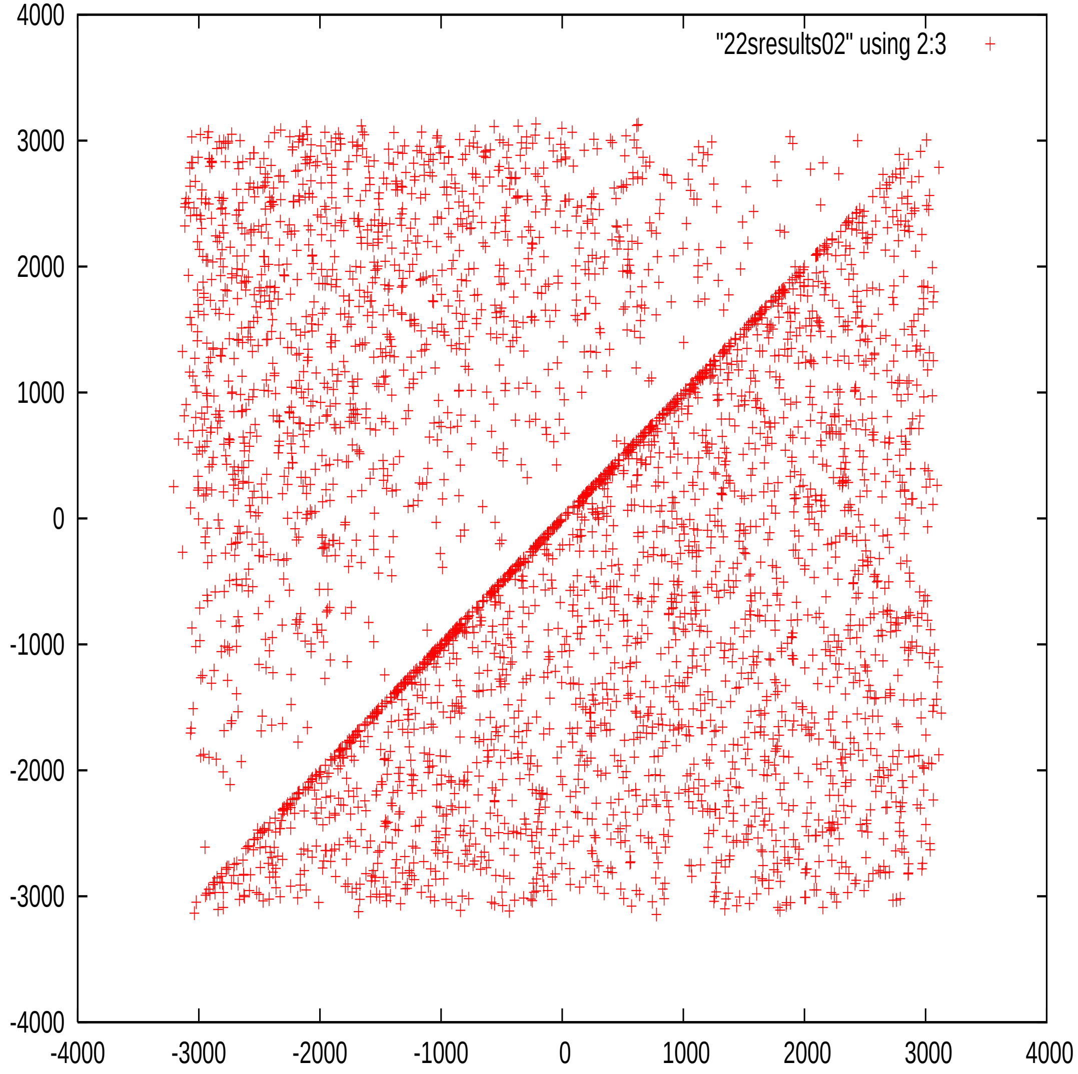
Call $X_2(t)$ and $X_3(t)$ the position of the second class and third class particle respectively.

$X_2(0) = 0$, $X_3(0) = 1$.

Theorem 1.1.

$$\lim_{t \to \infty} P\Big(X_2(t) > X_3(t)\Big) = \frac{2}{3} \tag{1.5}$$

Simulations: Each cross represents one simulation of $(X_2(3000), X_3(3000))$.


"22sresults02" using 2:3
4000
3000
2000
1000
0
-1000
-2000
-3000
-4000
-4000
-3000
-2000
-1000
0
1000
2000
3000
4000

Mean distance between second and third class particles

Theorem 1.2. *Start with* $\hat{1}23\hat{0}$. *Then*

$$\lim_{t\to\infty} \frac{1}{t} E\Big(X_3(t) - X_2(t)\Big)^+ = \frac{2}{3} \tag{1.6}$$

(Compare with two uniforms $E(\text{Max}-\text{Min}) = 1/3$.)

The other limit is open (and intriguing).

Proof of $X_2(t) \to$ Uniform [F-Kipnis]

Want to show:

$$\lim_{N\to\infty} P\Big(\frac{X_2(t)}{t} > r\Big) = \frac{1-r}{2}. \tag{1.7}$$

Define

$$J_t^r(\eta) = \sum_{x\geq r} \eta_t(x) \tag{1.8}$$

We compute in two different ways:

$$E(J_t^r(\hat{1}1\hat{0})) - E(J_t^r(\hat{1}0\hat{0})) \tag{1.9}$$

Couple the processes initially aligned as follows:

$$\begin{array}{c} \hat{1}1\hat{0} \\ \hat{1}0\hat{0} \end{array} \tag{1.10}$$

Couple the processes initially aligned as follows:

$$\begin{array}{c} \hat{1}1\hat{0} \\ \hat{1}0\hat{0} \\ \hat{1}2\hat{0} \end{array} \tag{1.11}$$

Then:

$$E(J^r_{tN}(\hat{1}1\hat{0})) - E(J^r_{tN}(\hat{1}0\hat{0})) \;=\; P(X_2(t) > rt)$$

On the other hand, by coupling $\bar{\eta}_t(x) = \eta_t(x+1)$,

$$E(J_t^r(\hat{1}1\hat{0})) - E(J_t^r(\hat{1}0\hat{0})) \;=\; P(\eta_t([rt]+1) = 1)$$

Then

$$P(X_2(t) > rt) = P(\eta_t([rt]+1) = 1) = \frac{1-r}{2}, \tag{1.12}$$

using convergence to local equilibrium,

Second and third class particles

$X_2(t,\eta)$: position of second class particle for initial configuration η

Define flux of particles thru $X_2(t,\eta)$:

$$J_t^2(\eta) := \sum_{x\geq 1} \eta_t(X_2(t,\eta)+x)) - \eta(X_2(0,\eta)+x). \qquad (1.13)$$

Kolmogorov backwards equation:

$$\begin{aligned} \frac{d}{dt}E\Big(J_t^2(\hat{1}2\hat{0})\Big) &= E\Big(L(J_t^2(\hat{1}2\hat{0}))\Big) \qquad (1.14) \\ &= 1 + E\Big(J_t^2(\hat{1}21\hat{0})\Big) + E\Big(J_t^2(\hat{1}02\hat{0})\Big) - 2E\Big(J_t^2(\hat{1}2\hat{0})\Big)\Big), \end{aligned}$$

So that

$$\frac{d}{dt}E\Big(J_t^2(\hat{1}2\hat{0})\Big) = 1 + E[J_t^2(\hat{1}21\hat{0}) - J_t^2(\hat{1}20\hat{0}) + J_t^2(\hat{1}02\hat{0}) - J_t^2(\hat{1}12\hat{0})]$$

The four initial configurations are aligned as follows:

$$\begin{array}{c} \hat{1}21\hat{0} \\ \hat{1}20\hat{0} \\ \hat{1}02\hat{0} \\ \hat{1}12\hat{0} \end{array} \tag{1.15}$$

The four initial configurations are aligned as follows:

$$\begin{aligned} &\hat{1}21\hat{0} \\ &\hat{1}20\hat{0} \\ &\hat{1}02\hat{0} \\ &\hat{1}12\hat{0} \\ &\hat{1}23\hat{0} \end{aligned} \tag{1.16}$$

And we get

$$\frac{d}{dt} E J_t^2(\hat{1}2\hat{0}) = P(X_2(t) < X_3(t)) \tag{1.17}$$

Since $\{X_2(t) < X_3(t)\}$ are decreasing events,

$$\lim_{t\to\infty} P(X_2(t) < X_3(t)) = \lim_{t\to\infty} \frac{EJ_t^2(\hat{1}2\hat{0})}{t}. \tag{1.18}$$

It was proven by [FMP]:

$$\lim_{t\to\infty} \frac{J_t^2(\hat{1}2\hat{0})}{t} = \Big(\frac{1-\mathcal{U}}{2}\Big)^2 \qquad \text{a.s.}, \tag{1.19}$$

and

$$\lim_{t\to\infty} \frac{EJ_t^2(\hat{1}2\hat{0})}{t} = E\Big(\frac{1-\mathcal{U}}{2}\Big)^2 = \frac{1}{3}, \tag{1.20}$$

Average distance between second and third class particles

Initial condition $\hat{1}32\hat{0}$

Proof of

$$\lim_{t\to\infty}\frac{1}{t}E\Big(X_3(t)-X_2(t)\Big)^+=\frac{2}{3}. \tag{1.21}$$

Let

$$J_t^r(\eta)=\sum_{x\geq rt}\eta_t(x)+\eta([rt])([rt]+1-rt)$$

By the Kolmogorov backwards equation,

$$\frac{d}{dt}E(J_t^r(\hat{1}\hat{0}))=\Big\{E(J_t^r(\hat{1}01\hat{0}))-E(J_t^r(\hat{1}\hat{0}))\Big\}-rE\eta_t([rt]). \tag{1.22}$$

Couple TASEP starting from $\hat{1}01\hat{0}$ and $\hat{1}\hat{0} = \hat{1}10\hat{0}$:

$$\begin{array}{c} \hat{1}01\hat{0} \\ \hat{1}10\hat{0} \end{array} \tag{1.23}$$

Couple TASEP starting from $\hat{1}01\hat{0}$ and $\hat{1}\hat{0} = \hat{1}10\hat{0}$:

$$\begin{array}{c} \hat{1}01\hat{0} \\ \hat{1}10\hat{0} \\ \hat{1}32\hat{0} \end{array} \tag{1.24}$$

Until hitting time:

$$J_t^r(\hat{1}01\hat{0}) - J_t^r(\hat{1}10\hat{0}) = \begin{cases} 1, & X_2(t) \le rt, X_3(t) > rt \\ 0, & X_2(t) \le rt, X_3(t) \le rt \\ 0, & X_2(t) > rt, X_3(t) > rt \end{cases} \tag{1.25}$$

By this last equality, (1.22) equals to

$$\frac{d}{dt} E J_t^r(\hat{1}\hat{0}) = P(X_2(t) \le rt, X_3(t) > rt) - rE\eta_t([rt]) \tag{1.26}$$

Compute the derivative of the current plus local equilibrium:

$$\lim_{t\to\infty} \frac{d}{dt} E J_t^r(\hat{1}\hat{0}) = u(r,1)(1-u(r,1)) - ru(r,1). \tag{1.27}$$

where $u(r,t)$ is given by

$$u(r,t) = \begin{cases} 1, & r \le -t; \\ \frac{t-r}{2t}, & -t < r \le t. \\ 0, & r > t. \end{cases} \tag{1.28}$$

Since

$$E\Big(X_3(t) - X_2(t)\Big)^+ = \sum_y P\Big(X_2(t) \le y < X_3(t)\Big), \tag{1.29}$$

from where

$$\lim_{t\to\infty} \frac{1}{t} E\Big(X_3(t) - X_2(t)\Big)^+ = \int_{-1}^{1} u(r,1)(1-u(r,1))dr$$

□

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