Happy Birthday

Shelly

and many more returns !

Shelly Fest
Rutgers 8/10/07

Correlation Decay
for the Weakly Nonlinear Schrödinger Equation

joint work with

Jani Lukkarinen

Hamiltonian dynamics of interacting particles

- kinetic theory

free motion $\cong$ interaction

1975 Lanford's proof

- hard spheres at low density
- Boltzmann equation for $|t| \leq t_0$

// What about weakly interacting quantum fluids ? //

Ho, Landau 1997

Benedetto, Castella, Esposito, Pulvirenti, 2004, ...

How to simplify ?

particles $\Longleftrightarrow$ waves

$\leadsto$ weakly nonlinear wave equation

TAKE:

nonlinear Schrödinger equation

‖space dimension 3‖

$$\psi_t : \mathbb{R}^3 \to \mathbb{C}$$

$$i \frac{\partial}{\partial t} \psi_t(x) = -\Delta \psi_t(x) + \lambda \int dy \, |\psi_t(y)|^2 V(x-y) \psi_t(x)$$

$\lambda > 0$, λ small

Hamiltonian *

$$H(\psi) = \frac{1}{2}\int dx\, |\nabla\psi(x)|^2 + \frac{1}{2}\lambda \int dx \int dy\, |\psi(x)|^2 V(x-y) |\psi(y)|^2$$

$\mathrm{Re}\,\psi(x)$, $\mathrm{Im}\,\psi(x)$ are canonically conjugate

- H is conserved
- $\|\psi_t\|$ is conserved
- gauge invariance ψ_t solution, so $e^{-i\vartheta}\psi_t$
- $e^{-\beta H(\psi)}$ is ultraviolet divergent

$\Longrightarrow$ substitute $\mathbb{Z}^3$ for $\mathbb{R}^3$ //

existence of time evolution (infinite energy)

Lanford, Lebowitz, Lieb (1974)

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* Canonical quantization yields

N quantum particles interacting by λV

N arbitrary, conserved

initial conditions

e.g.: translation invariant Gaussian measure

$\langle \psi \rangle = 0, \quad \langle \psi \psi \rangle = 0$

$\langle \psi(x)^* \psi(y) \rangle = C(x-y) \qquad , x, y \in \mathbb{Z}^3$

fast decay

Erdös, Yau (2000)

$i \frac{\partial}{\partial t} \psi_t(x) = -\Delta \psi_t(x) + \lambda V(x) \psi_t(x)$

random, Gaussian

NO multi-point Wigner functions

Duhamel expansion

$$i \frac{d}{dt} a_t = \omega a_t + \lambda (a_t)^3 \qquad a_t \in \mathbb{C}$$

$$i \frac{d}{dt} (a_t)^n = n\omega (a_t)^n + \lambda n (a_t)^{n+2}$$

$$a_t = e^{-i\omega t} a_0 - i\lambda \int_0^t ds\, e^{-i\omega(t-s)} \underbrace{a(s)^3}$$

SUBSTITUTE

etc.

$$a(t) = \sum_{n=0}^{\infty} c_n(t) (a_0)^n$$

convergence ?

non linear

number of terms at order n : $n!$

Gaussian pairings: $n!$

time integration: $\dfrac{t^n}{n!}$

zero radius of convergence //

one has to cut the series at N_0

choose: $\lambda^2 (N_0 !) = 1$

error term ?

$$\psi(t) = \psi_{\text{main}}(t) + \psi_{\text{error}}(t)$$

Duhamel expanded   full time evolution

$\Rightarrow$ stationarity

choose initial data according to

$$\frac{1}{Z} e^{-2H(\psi)} d\psi$$

mass $\omega(k) \geq \omega_0 > 0$

$$i \frac{\partial}{\partial t} \psi_t = -\Delta \psi_t + \psi_t$$

$\rightsquigarrow$ Gibbs state is ℓ_1-clustering | Procacia, Scoppola

$\| \psi(x,t)$ stationary in $x,t \|$

$f \in \ell_2$ NOT SCALED

$$|\mathbb{E}(\langle f, \psi_{error}(t)\rangle \langle \psi, f\rangle)|^2$$

Duhamel cut

$$\le \mathbb{E}\left(\left|\int_0^t ds \, \langle f, \mathcal{A}(t-s)\underbrace{\psi(s)}_{\text{full}}^{\otimes(2n+1)}\rangle\right|^2\right) \times \mathbb{E}(|\langle \psi, f\rangle|^2)$$

Stationarity

$$\le t \int_0^t ds \, \mathbb{E}(|\langle f, \mathcal{A}(s)\underbrace{\psi}_{\text{time } 0}^{\otimes(2n+1)}\rangle|^2) \underbrace{\mathbb{E}(|\langle \psi, f\rangle|^2)}_{\mathcal{O}(1)}$$

- partial time integration (smooth cut off)

// $(n!)^2$ can be balanced //

$\Rightarrow$ large, finite n analysis result

$\lambda = 0$

$$\mathbb{E}\left(\hat{\psi}(k,t)\, \hat{\psi}(k')^* \right) = \frac{1}{\omega(k)}\, e^{-i\omega(k)t}\, \delta(k-k')$$

$\lambda > 0$, $t \geq 0$

$$\lim_{\lambda \to 0} e^{i\omega(k)t/\lambda^2}\, e^{iR_0 t/\lambda}\, \mathbb{E}\left(\hat{\psi}(k, t/\lambda^2)\, \hat{\psi}(k')^* \right)$$

$$= e^{-iR_1 t}\, \frac{1}{\omega(k)}\, e^{-\nu(k)t}\, \delta(k-k')$$

$R_0 = \int dk \frac{2}{\omega}$, $R_1 = \int dk \frac{2}{\omega^2} \int dk \frac{2}{\omega}$

$$\nu(k_1) = -\int_0^\infty dt \int dk_2\, dk_3\, dk_4\, \delta(k_1+k_2-k_3-k_4) \times e^{it(\omega_1+\omega_2-\omega_3-\omega_4)}\, \frac{\omega_2-\omega_3-\omega_4}{\omega_2\omega_3\omega_4}$$

$\omega_j = \omega(k_j)$

$\|\ \mathrm{Re}\,\nu(k) > 0\ \|$ DECAY

physical space

$x, y \in \mathbb{Z}^3$

- up to fastly oscillating factors

$\nu_0, D \in \mathbb{C}$ $\qquad \nu(k) = \nu_0 + \frac{1}{2} D k^2$

$\mathrm{Re}\, \nu_0 > 0, \quad \mathrm{Re}\, D > 0$

$t \geq 0$

$$\mathbb{E}\left(\psi(x, t/\lambda^2)\, \psi(y,0)^* \right)$$

$$\cong e^{-\nu_0 t} \frac{1}{(2Dt)^{3/2}} e^{-(x-y)^2/2Dt}$$

Remarks

- There are conditions on the dispersion relation ω

- Limit holds only for small kinetic times

  The main contribution to the error term goes as $(t)^n$,

  thus $t < 1$ (as Lanford)

- Iterative scheme to estimate high dimensional oscillatory integrals

- $t, f \mapsto \langle f, \psi(t/\lambda^2)\rangle$ converges to a Gaussian process OU process

- energy density

$$E\left(\int dk\,\omega(k)\,|\hat{\psi}(k, t/\lambda^2)|^2 \int dk'\,\omega(k')\,|\hat{\psi}(k')|^2\right) \quad ?$$