Novel Type of Hamiltonians Without Ultraviolet Divergence for Quantum Field Theories

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Regensburg, 1 October 2014

Joint work with Stefan Teufel, Julian Schmidt, and Jonas Lampart

Supported by the John Templeton Foundation
Features of the novel approach

Problem:
- Hamiltonian involving particle creation and annihilation is usually UV divergent, and thus ill defined

New approach:
- IBC = interior–boundary condition
- allows a new way of defining a Hamiltonian $H_{IBC}$
- provides rigorous definition of a self-adjoint $H_{IBC}$, at least for some scenarios (and we hope in many)
- no need for discretizing space, smearing out particles over positive radius, or other UV cut-off
- no need for renormalization, or taking limit of removing the UV cut-off
- makes use of particle–position representation
Particle–position representation of a Fock space vector

Configuration space of a variable number of particles:

\[ Q = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n} = \bigcup_{n=0}^{\infty} Q^{(n)} \]

Fock space:

\[ \mathcal{F}^\pm = \bigoplus_{n=0}^{\infty} S^\pm \mathcal{H}_1^\otimes n \]

with \( S_+ = \text{symmetrizer} \), \( S_- = \text{anti-symmetrizer} \), \( \mathcal{H}_1 = 1\)-particle Hilbert space = \( L^2(\mathbb{R}^3, \mathbb{C}^k) \)

\( \psi \in \mathcal{F} \rightarrow \psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots) \)

\( \psi: Q \rightarrow S \) with \( S = \text{value space} = \bigcup_{n=0}^{\infty}(\mathbb{C}^k)^\otimes n \)

\( \psi \) is an (anti-)symmetric function

here \( d = 1 \), \( n = 0, 1, 2, 3 \)
There are issues with the particle–position representation in relativistic QFT...

- Some QFTs lead to \( \infty \) number of particles [cf. Deckert, Merkl et al.]
  BUT perhaps one can get along with that

- Photons are believed not to have a good position representation
  BUT photon wave functions are believed to be mathematically equivalent to (complexified) classical Maxwell fields, and that may be good enough

- \( Q = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n} \) depends on a choice of hypersurface ("\( \mathbb{R}^{3} \)"") in space-time \( \mathcal{M} \)
  BUT multi-time wave function may be defined on (the spacelike subset of) \( \bigcup_{n=0}^{\infty} \mathcal{M}^{n} \) [Petrat & Tumulka Ann. Phys., J. Math. Phys. 2014]

...BUT they do not seem fatal.
There are two species of particles, x-particles and y-particles.

x-particles can emit and absorb y-particles.

Configuration space \( \mathcal{Q} = \bigcup_{m,n=0}^{\infty} (\mathbb{R}^3_x)^m \times (\mathbb{R}^3_y)^n \)

Hilbert space \( \mathcal{H} = \mathcal{F}_x^- \otimes \mathcal{F}_y^+ \)

\( \psi : \mathcal{Q} \rightarrow \mathbb{C}, \psi = \psi(x^m, y^n), \) where \( x^m \) is any x-configuration with \( m \) particles

As always, \( i\partial_t \psi = H\psi \) Schrödinger eq
The original Hamiltonian is UV divergent.

Naive original Hamiltonian:

\[
(H_{\text{orig}} \psi)(x^m, y^n) = -\frac{\hbar^2}{2m_x} \sum_{i=1}^{m} \nabla^2_{x_i} \psi(x^m, y^n) - \frac{\hbar^2}{2m_y} \sum_{j=1}^{n} \nabla^2_{y_j} \psi(x^m, y^n) + \\
+ nE_0 \psi(x^m, y^n) + \\
+ g \sqrt{n+1} \sum_{i=1}^{m} \psi(x^m, (y^n, x_i)) + \\
+ \frac{g}{\sqrt{n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta^3(x_i - y_j) \psi(x^m, y^n \setminus y_j),
\]

with \( g = \) coupling constant, \( E_0 = \) rest energy, \( y^n \setminus y_j = \) leave out \( y_j \).

\( H_{\text{orig}} \) is ill-defined because the wave funct of the newly created \( y \)-particle, \( \delta^3(x - y) \), does not lie in \( L^2(\mathbb{R}^3) \) (or, has infinite energy).
Well-defined, regularized version of $H$

**UV cut-off $\varphi \in L^2(\mathbb{R}^3)$:**

\[
(H_{\text{cutoff}} \psi)(x^m, y^n) = -\frac{\hbar^2}{2m_x} \sum_{i=1}^{m} \nabla^2_{x_i} \psi(x^m, y^n) - \frac{\hbar^2}{2m_y} \sum_{j=1}^{n} \nabla^2_{y_j} \psi(x^m, y^n) + \\
+ nE_0 \psi(x^m, y^n) + \\
+ g\sqrt{n+1} \sum_{i=1}^{m} \int_{\mathbb{R}^3} d^3y \, \varphi^*(x_i - y) \psi(x^m, (y^n, y)) + \\
+ g\frac{\sqrt{n}}{\sqrt{n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi(x_i - y_j) \psi(x^m, y^n \setminus y_j)
\]

“smearing out” the $x$-particle with “charge distribution” $\varphi(\cdot - x)$
Novel idea: Interior–boundary condition

**Interior–boundary condition (IBC)**

For every \( \omega \in \mathbb{S}^2 \) and \((x^m, y^n) \in Q\) with \( x^m \cap y^n = \emptyset\),

\[
\lim_{r \downarrow 0} r \psi(x^m, (y^n, x_i + r \omega)) = \frac{g m_y}{2 \pi \hbar^2 \sqrt{n + 1}} \psi(x^m, y^n)
\]  

Here, “boundary” = diagonal; boundary config: where \( x_i = y_j \) (or \( r = 0 \)); interior config: one \( y \)-particle removed

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Hamiltonians Without Ultraviolet Divergence
Interior–boundary condition (IBC)

For every $\omega \in S^2$ and $(x^m, y^n) \in Q$ with $x^m \cap y^n = \emptyset$,

$$\lim_{r \searrow 0} r \psi(x^m, (y^n, x_i + r \omega)) = \frac{g m_y}{2\pi \hbar^2 \sqrt{n + 1}} \psi(x^m, y^n)$$  \hspace{1cm} (1)

IBC (1) $\Rightarrow \psi$ typically diverges like $1/r$ on the diagonal
Interior–boundary condition and rigorous Hamiltonian

Interior–boundary condition (IBC)

For every $\omega \in \mathbb{S}^2$ and $(x^m, y^n) \in \mathcal{Q}$ with $x^m \cap y^n = \emptyset$,

$$\lim_{r \downarrow 0} r \psi(x^m, (y^n, x_i + r \omega)) = \frac{g}{2\pi\hbar^2 \sqrt{n+1}} m_y \psi(x^m, y^n)$$  \hspace{1cm} (1)

Hamiltonians $H = H_{IBC}$

$$(H\psi)(x^m, y^n) = -\frac{\hbar^2}{2m_x} \sum_{i=1}^{m} \nabla^2_{x_i} \psi - \frac{\hbar^2}{2m_y} \sum_{j=1}^{n} \nabla^2_{y_j} \psi + nE_0 \psi$$

$$+ \frac{g}{4\pi} \sqrt{n+1} \sum_{i=1}^{m} \int_{\mathbb{S}^2} d^2 \omega \lim_{r \downarrow 0} \frac{\partial}{\partial r} \left( r \psi(x^m, (y^n, x_i + r \omega)) \right)$$

$$+ \frac{g}{\sqrt{n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta^3(x_i - y_j) \psi(x^m, y^n \setminus y_j)$$  \hspace{1cm} (2)
Comparison $H_{\text{orig}}, H_{\text{IBC}}$

\[
H_{\text{orig}} \psi = H_{\text{free}} \psi + g \sqrt{n + 1} \sum_{i=1}^{m} \psi(x^m, (y^n, x_i)) + \\
+ \frac{g}{\sqrt{n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta^3(x_i - y_j) \psi(x^m, y^n \setminus y_j),
\]

\[
H_{\text{IBC}} \psi = H_{\text{free}} \psi + \frac{g \sqrt{n + 1}}{4\pi} \sum_{i=1}^{m} \int_{S^2} d^2\omega \lim_{r \to 0} \frac{\partial}{\partial r} \left( r \psi(x^m, (y^n, x_i + r\omega)) \right) \\
+ \frac{g}{\sqrt{n}} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta^3(x_i - y_j) \psi(x^m, y^n \setminus y_j),
\] (2)
- There is only 1 \( x \)-particle, and it is fixed at the origin. \( \mathcal{H} = \mathcal{F}_y^+ \)

- configuration space \( Q = \bigcup_{n=0}^{\infty} \mathbb{R}^{3n} \)

- boundary config: any particle at the origin

- IBC \( \lim_{r \downarrow 0} r \psi(y^n, r\omega) = \frac{g m_y}{2\pi \hbar^2 \sqrt{n + 1}} \psi(y^n) \) \hspace{1cm} (3)

- \( H_{IBC} \psi = H_y,\text{free}\psi + \frac{g\sqrt{n + 1}}{4\pi} \int_{S^2} d^2\omega \lim_{r \downarrow 0} \frac{\partial}{\partial r} \left( r \psi(y^n, r\omega) \right) \]

\[ + \frac{g}{\sqrt{n}} \sum_{j=1}^{n} \delta^3(y_j) \psi(y^n \setminus y_j) \] \hspace{1cm} (4)

**Theorem (LSTT 2014)**

On a suitable dense domain \( \mathcal{D}_{IBC} \) of \( \psi \)s in \( \mathcal{F}_y^+ \) satisfying the IBC (3), \( H_{IBC} \) is well defined, self-adjoint, and positive.
Why it works: flux of probability into a point

- probability current \( \mathbf{j}_{y_j}(y^n) = \frac{\hbar}{m_y} \text{Im} \psi^* \nabla_{y_j} \psi \)

- \[ \frac{\partial |\psi(y^n)|^2}{\partial t} = - \sum_{j=1}^{n} \nabla_{y_j} \cdot \mathbf{j}_{y_j} + (n + 1) \lim_{r \searrow 0} r^2 \int_{\mathbb{S}^2} d^2 \omega \omega \cdot \mathbf{j}_{y_{n+1}}(y^n, r\omega) \]

  flux into 0 on \((n + 1)\)-sector

- motion towards 0 \(\Rightarrow\)
  \( \rho \sim 1/r \) as \( r \rightarrow 0 \)
Bohmian picture

- $t \mapsto Q(t) \in Q$ piecewise continuous, jumps between $Q^{(n)}$ and $Q^{(n+1)}$
- within $Q^{(n)}$, Bohm's law of motion

$$\frac{dQ}{dt} = \frac{\hbar}{m_B} \text{Im} \frac{\nabla \psi^{(n)}}{\psi^{(n)}} (Q(t))$$

- with IBC:
  - when $Q(t) \in Q^{(n)}$ reaches $y_j = 0$, it jumps to $(y^n \setminus y_j) \in Q^{(n-1)}$
  - emission of new $y$-particle at 0 at random time with random direction
- with UV cut-off:
  - emission and absorption occurs anywhere in a ball around 0 (= in the support of $\varphi$)
Note that $H_{IBC}$ cannot be decomposed into a sum of two self-adjoint operators $H_{\text{free}} + H_{\text{interaction}}$.

That is because the domain $\mathcal{D}_{IBC}$ is different from the free domain $\mathcal{D}_{\text{free}}$.

The Laplacian is not self-adjoint on $\mathcal{D}_{IBC}$ (i.e., does not conserve probability) because it allows a nonzero flux of probability into the boundary

$$\partial Q^{(n+1)} = Q^{(n)} \times \{0\} \cup \text{(permutations thereof)}.$$  

The additional terms in $H_{IBC}$ compensate that flux (by adding it to $Q^{(n)}$).
Comparison of IBC to a known boundary condition

Interior–boundary conditions have not been considered before, but other boundary conditions have. In particular, in order to define on $H = L^2(\mathbb{R}^3, \mathbb{C})$ a Schrödinger equation with $\delta$-potential,

$$H = -\hbar^2 \frac{\nabla^2}{2m} + \delta^3(x),$$

one employs the

Bethe–Peierls boundary condition [1935]

$$\lim_{r \to 0} \left( \frac{\partial}{\partial r} \psi(r \omega) - \alpha \psi(r \omega) \right) = 0 \quad \forall \omega \in \mathbb{S}^2 \text{ with known constant } \alpha \in \mathbb{R}$$

which leads to 0 current into the origin. For comparison,

IBC

$$\lim_{r \to 0} r \psi(y^n, r \omega) = \alpha_n \psi(y^n) \quad \forall \omega \in \mathbb{S}^2 \text{ with suitable constant } \alpha_n \in \mathbb{R}$$

which leads to nonzero current into $Q^{(n)} \times \{0\}$. 
Again, 1 x-particle fixed at the origin, $H = \mathcal{F}_y^+$.

**Theorem (LSTT 2014)**

For $E_0 > 0$, $H_{IBC}$ possesses a non-degenerate ground state $\psi_0$, which is

$$
\psi_0(y_1, \ldots, y_n) = \mathcal{N} \frac{(-g)^n}{(4\pi)^n \sqrt{n}} \prod_{j=1}^{n} \frac{e^{-\sqrt{2mE_0}|y_j|/\hbar}}{|y_j|}
$$

with eigenvalue $E = g^2 m \sqrt{2mE_0}/\pi \hbar^3$.

That is, the x-particle is dressed with a cloud of y-particles.
Effective potential between $x$-particles

To compute effective interaction between $x$-particles by exchange of $y$-particles, consider

- 2 $x$-particles fixed at $x_1 = (0, 0, 0)$ and $x_2 = (R, 0, 0)$, $\mathcal{H} = \mathcal{F}_y^+$
- 2 IBCs, one at $x_1$ and one at $x_2$
- 2 creation and annihilation terms in $H_{IBC}$
- Presumably, the ground state is

$$\psi_0 = c_n \prod_{j=1}^{n} \sum_{i=1}^{2} \frac{e^{-\sqrt{2mE_0}|y_j-x_i|/\hbar}}{|y_j - x_i|}$$

with eigenvalue

$$E = \frac{2g^2 m}{\pi \hbar^2 \left( \frac{\sqrt{2mE_0}}{\hbar} - \frac{e^{-\sqrt{2mE_0}R/\hbar}}{R} \right)}$$

- That is, $x$-particles effectively interact through an attractive Yukawa potential
Consider again the scenario with 1 x-particle fixed at the origin, $\mathcal{H} = \mathcal{H}_y^+$. Consider $H_{\text{cutoff}} = H_\varphi$ with $\varphi = \delta^3$, limit $\varphi \to \delta^3$. Then there exist constants $E_\varphi \to \infty$ and a self-adjoint operator $H_{\infty}$ such that $H_\varphi - E_\varphi \to H_{\infty}$.

[van Hove 1952, Nelson 1964, see also Dereziński 2003]

**Theorem (LSTT 2014)**

$H_{\infty} = H_{IBC} + \text{const}$
Neumann vs. Dirichlet conditions

Usually, Neumann \( \frac{\partial u}{\partial n} \bigg|_{\partial Q} = 0 \), Dirichlet \( u \bigg|_{\partial Q} = 0 \), Robin \( (\alpha \frac{\partial u}{\partial n} + \beta u) \bigg|_{\partial Q} = 0 \).

In the previous example: Dirichlet-type IBC

\[
\lim_{r \searrow 0} r \psi(y^n, r\omega) = \alpha_n \psi(y^n) \quad \text{(IBC)} \quad \text{and, for } y^n \in (\mathbb{R}^3 \setminus \{0\})^n,
\]

\[
H_{IBC}\psi(y^n) = H_{\text{free}}\psi + \frac{g \sqrt{n + 1}}{4\pi} \int_{S^2} d^2\omega \lim_{r \searrow 0} \frac{\partial}{\partial r} \left( r \psi(y^n, r\omega) \right)
\]

One can also use instead: a Neumann-type IBC. Leads to a different \( H \).

\[
\lim_{r \searrow 0} \frac{\partial}{\partial r} \left( r \psi(y^n, r\omega) \right) = \alpha_n \psi(y^n) \quad \text{(IBC)} \quad \text{and, for } y^n \in (\mathbb{R}^3 \setminus \{0\})^n,
\]

\[
H_{IBC}\psi(y^n) = H_{\text{free}}\psi + \frac{g \sqrt{n + 1}}{4\pi} \int_{S^2} d^2\omega \lim_{r \searrow 0} r \psi(y^n, r\omega)
\]

Similarly, one can use Robin-type conditions.
Now Dirac operators instead of $-\nabla^2$

- Now suppose that $y$-particles are relativistic and have spin $\frac{1}{2}$.
- A free $y$-particle is described by the Dirac equation.
- Again, 1 $x$-particle is fixed at the origin. Add Coulomb potential.
- Let us restrict ourselves to $0 \leq n \leq 1$, i.e., $\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$
  with $\mathcal{H}^{(0)} = \mathbb{C}^4$ and $\mathcal{H}^{(1)} = L^2(\mathbb{R}^3, \mathbb{C}^4)$.

**Theorem (LSTT 2014)**

Suppose $|q_x q_y| > 1$. Then there is a self-adjoint operator $H_{IBC}$ on a dense domain $\mathcal{D}$ in $\mathcal{H} = \mathbb{C}^4 \oplus L^2(\mathbb{R}^4, \mathbb{C}^4)$ such that:

- For any $\psi = (0, \psi^{(1)})$ with $\psi^{(1)} \in H^1(\mathbb{R}^3, \mathbb{C}^4)$ (first Sobolev space) and $\psi^{(1)}(x) = 0$ on a neighborhood of the origin,

$$
\psi \in \mathcal{D} \text{ and } H\psi = \left(0, \left(-i\hbar \alpha \cdot \nabla + \beta m + \frac{q_x q_y}{|y|}\right)\psi^{(1)} \right).
$$

- $H_{IBC}$ does not conserve particle number, i.e., $H_{IBC}$ is not of block-diagonal form w.r.t. $\mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$. 

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Hamiltonians Without Ultraviolet Divergence
Formulate quantum electrodynamics in terms of IBCs, building on

**Quantenelektrodynamik im Konfigurationsraum.**

Von L. Landau und R. Peierls in Zürich.

(Eingegangen am 12. Februar 1930.)


(QED in the particle–position representation)
Thank you for your attention