The Mathematical Structure of Quantum Reality

February 2, 2006

Roderich Tumulka
Mathematisches Institut, Eberhard-Karls-Universität,
Tübingen, Germany

In its July 2005 number, *Science* mentions as one of the Top 25 questions that face scientific inquiry over the next quarter-century: *Do Deeper Principles Underlie Quantum Uncertainty and Nonlocality?*

The Traditional “Copenhagen” View is Mysterious

[The consequences of the Copenhagen view are] certainly not contradictory from a purely logical point of view; but hardly anybody would be inclined to take them seriously. (Albert Einstein, 1879–1955)

I can safely say that nobody understands quantum mechanics. (Richard P. Feynman, 1918–1988)

Ordinary quantum mechanics is just fine – for all practical purposes! (John S. Bell, 1928–1990)

Putnam's Table of Views of QM

Hilary Putnam (philosopher, born 1926, now Harvard U) (2005): Kinds of interpretations of quantum mechanics

Collapse	*No collapse*
Produced by something external to the system and not subject to superposition (e.g. Von Neumann)	No hidden variables (Many Worlds)
Spontaneous (e.g. GRW)	Hidden variables (e.g. Bohm)

Von Neumann's Scheme

John von Neumann (1932):

- The state (at time t) of a quantum system (of N particles) is described by $\psi_t : \mathbb{R}^{3N} \to \mathbb{C}$ with $\psi_t \in L^2(\mathbb{R}^{3N})$ and $\|\psi_t\| = 1$.
- When the system is isolated: Schrödinger equation
$$i\hbar \frac{\partial \psi_t}{\partial t} = H\psi_t$$
with $H =$ self-adjoint Hamiltonian operator
$$H = \sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \nabla_k^2 + V(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)$$
with $m_k =$ mass and potential energy $V : \mathbb{R}^{3N} \to \mathbb{R}$.
- When a "measurement" of the "observable" A is made, the possible outcomes are the eigenvalues α of A and occur with probability
$$\langle \psi_t, P_\alpha \psi_t \rangle \,,$$
where P_α is the orthogonal projection in L^2 to the eigenspace of A with eigenvalue α. Then ψ_t collapses to an eigenvector
$$\psi_{t+} = \frac{P_\alpha \psi_{t-}}{\|P_\alpha \psi_{t-}\|} \,.$$

Many Worlds

Everett (1957): ψ_t obeys Schrödinger equation

$$i\hbar \frac{\partial \psi_t}{\partial t} = H \psi_t \,.$$

No collapse.

Different outcomes of an experiment correspond to different parts of ψ_t ("Schrödinger's cat"), which coexist and are regarded as parallel universes.

GRW

ψ_t collapses *spontaneously*: stochastic correction to Schrödinger equation

Ghirardi, Rimini & Weber (1986): Explicit model, Markovian jump process in Hilbert space.

Between the jumps:

$$i\hbar \frac{\partial \psi_t}{\partial t} = H \psi_t \, .$$

After a random waiting time ΔT with rate N/τ,

$$\tau \approx 10^{15}\,\text{sec} \approx 10^{8}\,\text{years},$$

ψ_t collapses:

$$\psi_{t+} = \frac{e^{-(\boldsymbol{q}_k - \boldsymbol{Q})^2/2\sigma^2}\,\psi_{t-}}{\| e^{-(\boldsymbol{q}_k - \boldsymbol{Q})^2/2\sigma^2}\,\psi_{t-} \|}$$

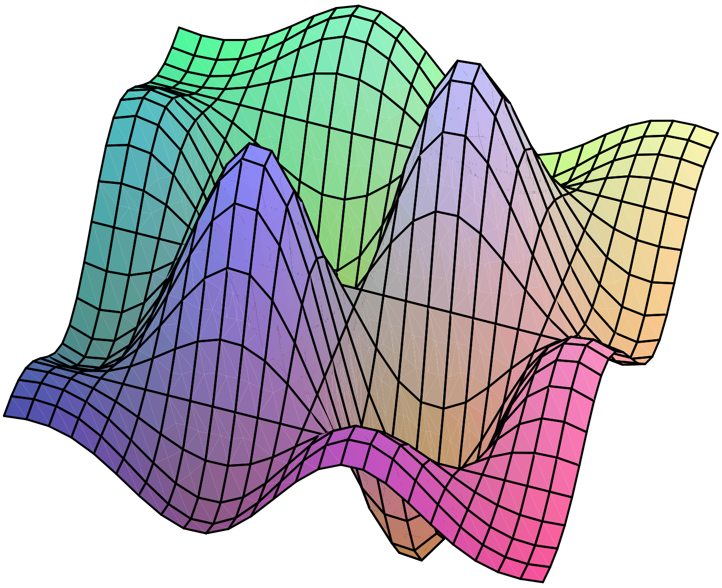
with $\sigma = 10^{-7}\,\text{m}$, $k =$ randomly chosen particle, and $\boldsymbol{Q} =$ random center of the collapse with distribution

$$\mathbb{P}(\boldsymbol{Q} \in d^3\boldsymbol{q}) = \text{const.}\, \| e^{-(\boldsymbol{q}_k - \boldsymbol{q})^2/2\sigma^2}\,\psi_{t-}\|^2 \, d^3\boldsymbol{q} \, .$$

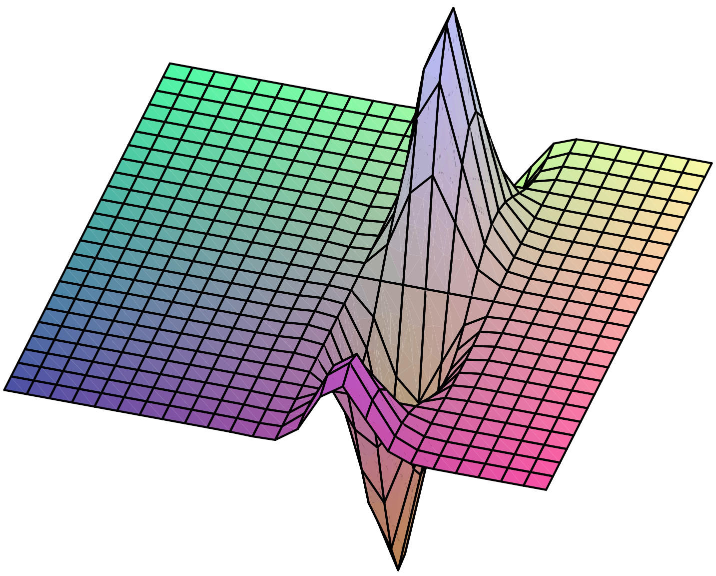
Predictions $\approx$ QM

GRW Collapse

Before:



After:



Bohmian Mechanics

> In 1952 I saw the impossible done. Bell (1982)
>
> [The quantum] paradoxes are simply disposed of by the 1952 theory of Bohm. Bell (1990)

Bohm (1952): A wave function choreographs the motion of N particles. Trajectories $\boldsymbol{Q}_k(t) \in \mathbb{R}^3$ obey

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\boldsymbol{j}_k(Q_t)}{\rho(Q_t)} = \frac{\frac{\hbar}{m_k}\mathrm{Im}\,\psi_t^* \nabla_k \psi_t}{\psi_t^*\,\psi_t}\big(Q_t\big)\,,$$

with $Q_t = \big(\boldsymbol{Q}_1(t), \ldots, \boldsymbol{Q}_N(t)\big)$ the configuration.
$\psi_t : \mathbb{R}^{3N} \to \mathbb{C}$ obeys Schrödinger equation

$$i\hbar\frac{\partial \psi_t}{\partial t} = H\psi_t\,.$$

- No collapse.
- The state (at time t) = $\big(Q_t, \psi_t\big)$.
- Wave–particle duality
- $|\psi|^2$ distribution is equivariant
- Dürr, Goldstein & Zanghì (1992): In a typical Bohmian universe, the configuration of a system with wave function ψ looks random with probability distribution $|\psi|^2$.
- BM $\Rightarrow$ QM formalism

“Which interpretations I think we can rule out” (Putnam 2005)

> [Von Neumann’s] proposal seems less and less attractive to me. That there is something special about macro-observables seems tremendously unlikely... I am now inclined to give up this most *classic* interpretation...
>
> The Many Worlds interpretation now has distinguished advocates... I myself think it is untenable.
>
> Putnam (2005)

> Was the world wave function waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer for some more highly qualified measurer – with a Ph.D.?
>
> Bell (1990)

> Either the wave function, as given by the Schrödinger equation, is not everything, or it is not right.
>
> Bell (1987)

Bohmian Mechanics and Quantum Field Theory

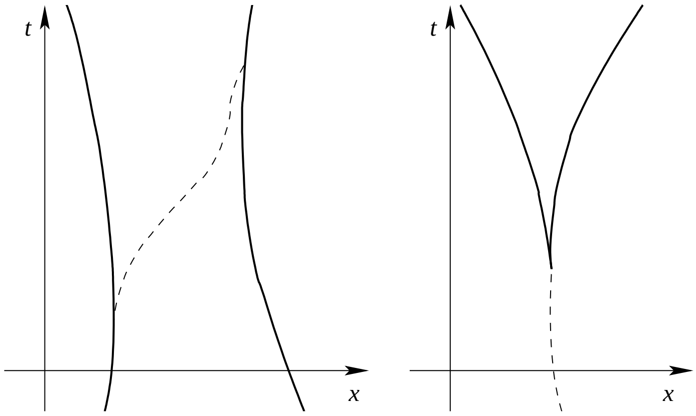
We are left with GRW and Bohm... Why did this conclusion make me unhappy? ... In the case of the Bohm theory, ... it has not yet been extended to a quantum field theory, let alone to a quantum cosmology...

Putnam (2005)

It does not seem possible to extend Bohm's version of quantum mechanics to theories in which particles can be created and destroyed, which includes all known relativistic quantum theories. Steven Weinberg (1996)

- Bohm (1952): field configuration $\phi(\boldsymbol{x})$, guided by $\Psi(\phi)$
- "Bell-type QFT": Particle creation and annihilation in Bohmian mechanics
  - Bell (1986): lattice model.
  - Dürr, Goldstein, Tumulka & Zanghì (2003): Continuum model for an example QFT
  - DGTZ (*Phys. Rev. Lett.* 2004, *Commun. Math. Phys.* 2005, *J. Phys. A* 2005): Bell-type QFT = Particle trajectory model for more or less any given regularized QFT

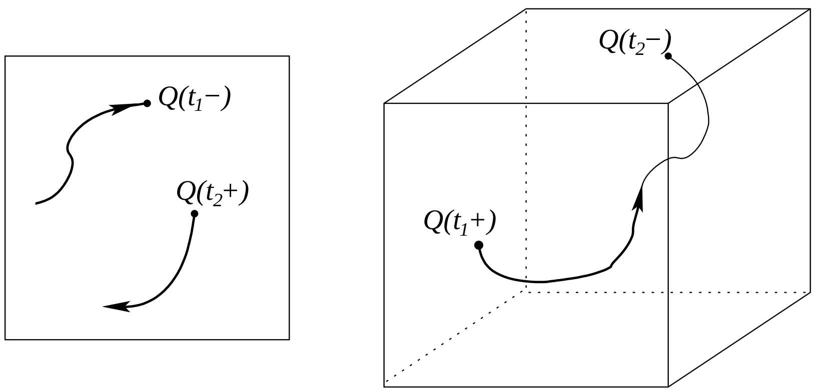
Applies to bosons and fermions; accounts for all predictions of QFT; contains Bohmian mechanics as a special case.



(a) (b)

$$
\begin{aligned}
\mathcal{Q} &= \{\text{all finite subsets of } \mathbb{R}^3\} \\
&= \bigcup_{N=0}^{\infty} \big(\mathbb{R}^{3N} \setminus \text{coincidences}\big)/\text{permutations} \\
&= \text{configuration space for a variable number of particles}
\end{aligned}
$$

(a) (b)



(c) (d)

Mathematical Structure of Bell-Type QFT

$$\Psi \mapsto \mathbb{P}^{\Psi} \text{ measure on path space } \{t \mapsto Q_t\}$$

Configurational history Q_t is a Markovian jump process in $\mathcal{Q}$. Given:

- $\mathscr{H}$ Hilbert space
- $\Psi_t \in \mathscr{H}$, $\|\Psi_t\| = 1$.
- H Hamiltonian
- $\mathcal{Q}$ config. space = countable union of disjoint manifolds
- P "config. observable" = POVM (positive-operator-valued measure) on $\mathcal{Q}$ acting on $\mathscr{H}$

POVM means $P(B) : \mathscr{H} \to \mathscr{H}$ positive for $B \subseteq \mathcal{Q}$,
$P(\cup_n B_n) = \sum_n P(B_n)$ if B_n pairwise disjoint,
$P(\mathcal{Q}) = 1$

$$\text{``}|\Psi_t|^2\text{''} = \langle \Psi_t, P(dq)\, \Psi_t \rangle$$

$$(\mathscr{H}, H, P, \Psi) \mapsto \mathbb{P}$$

$$(\mathscr{H}, H, P, \Psi_t) \mapsto \mathcal{L} \text{ Markov generator}$$

Equivariant distribution

$$\mathbb{P}\big(Q_t \in dq\big) = \langle \Psi_t, P(dq)\, \Psi_t \rangle$$

Construction of Bell-Type QFT

Construct generator $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ from $H = H_0 + H_I$,
H_0 = a differential operator, H_I = an integral operator.

Jump rate from q' to the set $dq \subset \mathcal{Q}$:

$$\sigma^{\Psi_t}(dq|q') = \frac{[J(dq, dq')]^+}{\rho(dq')} = \frac{\left[\frac{2}{\hbar}\,\mathrm{Im}\,\langle \Psi_t, P(dq) H_I P(dq')\, \Psi_t \rangle\right]^+}{\langle \Psi_t, P(dq')\, \Psi_t \rangle}$$

Radon–Nikodým derivative in dq'
$[\cdot]^+$ = positive part of signed measure (Hahn–Jordan decomp.)

Continuous motion (generalization of Bohm's velocity):

$$\frac{df(Q_t)}{dt} = \frac{j \cdot \nabla f(Q_t)}{\rho(Q_t)} = \frac{\mathrm{Re}\,\langle \Psi_t, P(dq) \frac{i}{\hbar}[H_0, f]\, \Psi_t \rangle}{\langle \Psi_t, P(dq)\, \Psi_t \rangle}(q = Q_t)$$

for all $f \in C_0^\infty(\mathcal{Q})$

Rigorous Results on Bell-Type QFT

Global existence in the lattice case:

Theorem (Georgii & Tumulka, *Markov Processes and Related Fields* 2005): Suppose $\mathcal{Q}$ is countable and H is Hilbert–Schmidt. Then there exists a (time-inhomogeneous) Markovian jump process $(Q_t)_{t\in\mathbb{R}}$ in $\mathcal{Q}$ with transition rates $\sigma^{\Psi_t}(q|q')$ and distribution $|\Psi_t|^2$. The process is unique in distribution.

Result includes

- no explosion (∞ly many jumps in finite time)
- the process avoids nodes of Ψ (where jump rate is ill defined).

Remarks on the proof:

- Since $\sigma^{\Psi_t}(q|q')$ is not bounded, this process is not covered by standard criteria for jump processes.
- The proof exploits instead the role of the $|\Psi|^2$ distribution.
- Methods: measure theory, probability theory, analysis

More Rigorous Results on Bell-Type QFT

- DGTZ (*Commun. Math. Phys.* 2005):
  Sufficient conditions for existence of the jump rates σ (and equivariance of $|\Psi|^2$)
  - "H_I = integral operator"
  - Methods: measure theory, operators in Hilbert space, vector bundles
- Teufel & Tumulka (*Commun. Math. Phys.* 2005):
  Sufficient conditions for global existence of almost-all continuous trajectories in the absence of jumps
  - includes Bohmian mechanics
  - improves results of Berndl et al. (1995)
  - possible catastrophes:
    - nodes of Ψ
    - escape to ∞
    - particle coincidences
  - general ODE existence theory for vector fields in some Sobolev space (DiPerna & Lions 1989) not applicable: unbounded divergence of the vector fields
  - Methods: analysis, operators on Hilbert spaces

Relativistic (Lorentz) Invariance

Those paradoxes are simply disposed of by the 1952 theory of Bohm, leaving as the question, the question of Lorentz invariance. So one of my missions in life is to get people to see that if they want to talk about the problems of quantum mechanics – the real problems of quantum mechanics – they must be talking about Lorentz invariance. Bell (1990)

The big question, in my opinion, is which, if either, of these two precise pictures [GRW and Bohm] can be redeveloped in a Lorentz invariant way. Bell (1990)

[N]either of these theories [GRW and Bohm] is Lorentz invariant, and ... it is pretty clear that no theory in either of the classes ... can do without an 'absolute time' parameter. [However,] in my view, *present-day* quantum cosmology *does already* involve a 'background' time parameter. Putnam (2005)

Relativity

- Einstein (1905, 1915): space-time = 4-manifold with Lorentzian (pseudo-Riemannian) metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \quad \text{in a suitable basis}$$

- Lorentz group = $SO(1,3)$.
- Replace Schrödinger equation by Dirac equation, $\psi : (\text{space-time})^N \to (\mathbb{C}^4)^{\otimes N}$

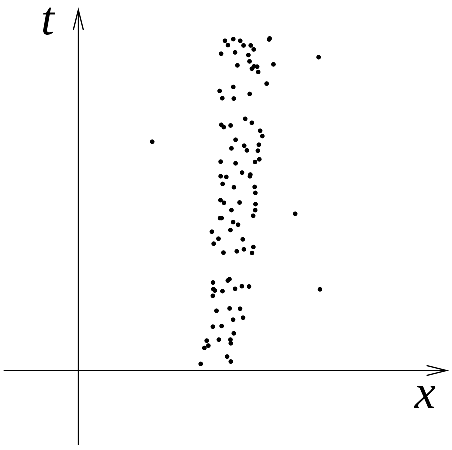
Nonlocality

Bell (1964): QM $\Rightarrow \exists$ faster-than-light influences
(uses violations of "Bell's inequality")
However, messages cannot be transmitted faster than light.

Problems for Bohm and GRW

- Reconcile nonlocality with relativity.
- For any spacelike 3-surface Σ, ψ defines a probability distribution $\rho = |\psi|^2$ and a current j on Σ^N. But there is no 3-surface of *simultaneity*. On which 3-surface should we evaluate $|\psi|^2$ or j?

Relativistic GRW



"Flash" ontology, invented by Bell (1987).
For GRW: put a flash at the center of every collapse.

Tumulka (2004, to appear in *J. Statist. Phys.* 2006):

Previous flash X' defines 3-surface $\Sigma(X', \Delta T) =$
$\{$all points at timelike distance ΔT from X' in the future of $X'\}$.

Choose randomly

- ΔT (proper waiting time) with exp distribution (rate $= 1/\tau$)
- X (the next flash) with $|\psi|^2$ distribution on $\Sigma(X', \Delta T)$

Then multiply ψ by Gaussian on $\Sigma(X', \Delta T)$ centered at X.

Relativistic GRW

Don't focus on the collapse of the wave function, focus on the joint distribution $\mathbb{P}$ of the flashes!

$$\Psi \mapsto \mathbb{P}^\Psi \text{ probability measure on}$$
$$\text{history space } = \{\text{discrete subsets of space-time}\}$$

The process is

- nonlocal (violates Bell's inequality)
- Lorentz invariant
- defined by the (initial) wave function,

  $$\mathbb{P}(\cdot) = \langle \psi_0, E(\cdot)\, \psi_0 \rangle,$$

  with a POVM E
- E involves, for each flash x with predecessor x', a factor (and its adjoint)

  $$e^{-\Delta T/2\tau}\, U_\Sigma^0 \,\text{Gaussian}(\Sigma, x, \sigma)\, U_0^\Sigma$$

  with $\Sigma = \Sigma(x', \Delta T)$,
  $\Delta T = \text{t-dist}(x, x')$,
  U_A^B = unitary Dirac evolution from 3-surface A to B

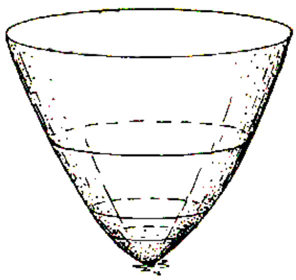
Relativistic Bohmian Mechanics

Two approaches:

- Goldstein & Tumulka (2003):
  Evaluate j^ψ on Σ = future light cone. Law for particle trajectories, *nonlocal and Lorentz invariant*, involves a paradox-free kind of retrocausation.

  Drawback: since $|\psi|^2$ is not equivariant, the theory does not consistently predict probabilities, and thus does not work for explaining QM

- Dürr, Goldstein, Münch-Berndl & Zanghì (1999) (in flat space-time);
  Tumulka (thesis, 2001) (in curved space-time with rigorous proof of equivariance):
  Modify the spirit of relativity by using a preferred foliation $\mathcal{F}$ of space-time into spacelike 3-surfaces. Then Bohmian mechanics has a simple and natural extension to relativistic space-time, evaluating j^ψ on the leaves of $\mathcal{F}$.



For example, let $\mathcal{F}$ be the level sets of the function T : (space-time) $\to \mathbb{R}$, $T(x) = $ timelike-distance(x, big bang).

Drawing: R. Penrose

Conclusions

- Mathematical structure of quantum reality

  $$\Psi \mapsto \mathbb{P}^{\Psi} \text{ probability measure on history space}$$

- Relativistic flashy GRW shows:
  a realistic quantum theory can be consistent with relativity

- Presently we have the alternative between:
  - either a small modification in the predictions of quantum theory (this follows from GRW)
  - or a modification in our understanding of relativity, allowing a preferred foliation (then we have Bohmian trajectories on relativistic space-time)

- Wide open: quantum cosmology.

I've spent much more time thinking about the photon problem [i.e., about quantum theory] than about relativity. Einstein