1) [worth 6 points out of 100] Let $f(x) = x\mathcal{U}(1-x)$, where $\mathcal{U}$ is the unit step function, and $g(x) = e^{-x}\sin 2x$ on $[-\pi, \pi]$.
(a) Draw the graph of $f$.
(b) In another diagram, draw the graphs of $g$, $e^{-x}$, and $-e^{-x}$.

2) [5 points] For the complex number $z = 2 - \pi i$, compute $\bar{z}$, $|z|$, $\text{Im} z$, $1/z$, and $e^z$. Simplify where possible.

3) [2 points] Compute $e^{i n \pi}$ for arbitrary integer $n$.

4) [4 points] Is $f$ odd, even, or neither? (No justification required.)
   a) $f(x) = -|x|$  
   b) $f(x) = |x|^3$  
   c) $f(x) = x + |x|$  
   d) $f(x) = x \cosh x + \sinh x$

5) [4 points] Show that if $f$ is an even function, then $f'(x) = df/dx$ is odd.

6) [8 points] Find the eigenvalues $\lambda$ and the corresponding eigenfunctions $X(x)$ of the Sturm–Liouville problem

$$X'' + \lambda X = 0 \text{ with boundary condition } X'(0) = 0, \ X'(L) = 0,$$

where $L$ is a positive constant. Instruction: For every $\lambda$ ($> 0$, $< 0$, or $= 0$) specify the general solution of the differential equation and check the boundary conditions.
7) [12 points] Let \( L > 0 \) be a constant. Consider the function \( f(x) = L^2 - x^2 \) on the interval \([0, L]\).
(a) Compute \( \|f\| \) on \([0, L]\).
(b) Expand \( f \) into a (half-range) cosine series.

8) [10 points] We consider the following boundary value problem for the function \( u(x, y) \) on the square \( 0 \leq x \leq L, 0 \leq y \leq L \):
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < L, \ 0 < y < L \quad (1)
\]
\[
\frac{\partial u}{\partial x}(0, y) = 0, \quad \frac{\partial u}{\partial x}(L, y) = 0 \quad \text{for} \quad 0 < y < L \quad (2)
\]
\[
u(x, 0) = 0, \quad u(x, L) = L^2 - x^2 \quad \text{for} \quad 0 < x < L. \quad (3)
\]
(a) What is the name of equation (1)? What is the name of the type of boundary condition (b.c.) (2)? What is the name of the type of b.c. (3)?
(b) Carry out a separation of variables on the PDE (1), and derive the Sturm–Liouville problem for the \( x \) variable using the b.c. (2).
(c) Using the result of problem 6, specify all product solutions of (1) with b.c. (2).
(d) Using the result of problem 7(b), express the solution of (1) with (2) and (3) as a series.

9) [6 points] Formulate the 1-dimensional wave equation for the vertical displacement \( u(x, t), 0 \leq x \leq L, \) of a string of length \( L \). (Just set up the equation, do not solve it.) Furthermore, formulate a boundary condition expressing that the string is fixed at both ends (as it would be on a guitar or violin). Finally, formulate an example of an initial condition for this wave equation.

10) [4 points] Compute the gradient \( \nabla u \), as well as \( \Delta u \) (where \( \Delta \) is the Laplace operator) of the function \( u(x, y) = xe^{xy}, x, y \in \mathbb{R} \).
11) [8 points] Carry out a separation of variables on the PDE

\[ \frac{\partial u}{\partial t} = 3t^2 \frac{\partial u}{\partial x} - 3t^2 u(x, t) \]

and find all product solutions for \( u(x, t) \).

12) [8 points] For the non-homogeneous ODE

\[ m \frac{d^2 x}{dt^2}(t) + k x(t) = f(t) \]

with \( m = 1 \), \( k = 1 \), and the (non-resonant) 2-periodic driving force

\[ f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cos n \pi t , \]

compute the 2-periodic particular solution \( x_p(t) \) as a Fourier series.

13) [3 points] Suppose that \((f - g, h) = 3 \) and \((h, 2f + g) = -1 \), where \( f, g, h \) are functions and \((\ , \ , \ )\) means the inner product of functions. Find \((f, h)\).

14) [5 points] Expand \( f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases} \) into a complex Fourier series with period 2\( \pi \).

15) [5 points] Compute the (complex) Fourier transform

\[ \hat{f}(\alpha) = C(\alpha) = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, dx \]

of \( f(x) = \begin{cases} x & \text{if } -1 < x < 1 \\ 0 & \text{if } x \leq -1 \text{ or } x \geq 1 \end{cases} \).

Write your result in such a form that one can easily read off the real and imaginary parts.
16) [10 points] We use the following convention for the definition of Fourier transformation:

\[ \mathcal{F}\{f\} = C(\alpha) = \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} \, dx, \quad \mathcal{F}^{-1}\{C\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} C(\alpha) e^{-i\alpha x} \, d\alpha. \]  

(a) Find the Fourier transform of the delta function, \( C(\alpha) = \mathcal{F}\{\delta(x - x_0)\} \), where \( x_0 \) is an arbitrary constant.

(b) Consider the 1-dimensional heat equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]  

on the whole real axis without boundary as the domain of the \( x \) variable. Use Fourier transformation to find \( u(x, t) \) for \( t \geq 0 \) with the initial condition

\[ u(x, 0) = \delta(x). \]  

Hint (for part b): Use part (a), and that the Fourier transform of a Gaussian function is

\[ \mathcal{F}\{e^{-x^2/4p^2}\} = 2\sqrt{\pi} p e^{-p^2\alpha^2}. \]