13 Spin and representations of the rotation group $SO(3)$

In this chapter, we leave out most proofs and many details. A full discussion can be found in R. Sexl and H. Urbanke: *Relativity, Groups, Particles* (Springer-Verlag 2001).

$SO(3)$ is the group of rotations in $\mathbb{R}^3$ around the origin. Its elements are $3 \times 3$ matrices $R$ that are orthogonal, $R^t R = RR^t = I$ (where $R^t$ means the transpose of $R$), and have det $R = 1$. $SO(3)$ is a Lie group, i.e., a group and a differentiable manifold such that the group multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$ are $C^\infty$ mappings. $SO(3)$ is compact and has (real) dimension 3.

When rotating the coordinate axes according to $R$, the wave function $\psi \in \mathcal{H} = L^2(\mathbb{R})$ has to be replaced by

$$\psi'(x) = \psi(R^{-1}x). \quad (13.1)$$

The relation $\psi' = U_R \psi$ defines a unitary operator $U_R$ on $\mathcal{H}$, and $R \mapsto U_R$ is a unitary representation of $SO(3)$ on $\mathcal{H}$. Generally, a representation of a group $G$ is a homomorphism $T : G \to GL(V)$, where $V$ is a vector space (called the representation space) and $GL(V)$ is the general linear group of $V$, i.e., the group of invertible endomorphisms of $V$, i.e., invertible linear operators $A : V \to V$. A Hamiltonian $H$ is said to be rotationally symmetric or invariant under the action of $SO(3)$ iff $U_R H(\mathcal{D}) = \mathcal{D}(H)$ and $U_R H U_R^{-1} = H$ for every $R \in SO(3)$.

The transformation law (13.1) can become more complicated in the following way. Consider, instead of $\psi : \mathbb{R}^3 \to \mathbb{C}$, a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$. Then, in the new coordinates, $F$ has to be replaced by

$$F'(x) = RF(R^{-1}x). \quad (13.2)$$

A tensor field $M_{ij}$, i.e., $M : \mathbb{R}^3 \to \mathbb{R}^3 \otimes \mathbb{R}^3 =: \mathbb{R}^{3,3}$ has to be replaced by

$$M'(x) = RM(R^{-1}x)R^t. \quad (13.3)$$

The general pattern here is that

$$f'(x) = T_R(f(R^{-1}x)), \quad (13.4)$$

where $T$ is a representation of $SO(3)$: In (13.1), $T$ was the trivial representation $T_R = 1$; in (13.2), $T$ was the defining representation of $SO(3)$ on $V = \mathbb{R}^3$; in (13.3), $T$ was the inherited representation on tensors (of the appropriate rank), say $V = \mathbb{R}^{3,3}$. For another example, $V$ could be the space of anti-symmetric matrices (a subspace of $\mathbb{R}^{3,3}$ that is invariant under the action of $SO(3)$).

In quantum mechanics, if (13.4) applies with non-trivial $T$, then the particle is said to have spin. All known elementary particles (except the Higgs boson, if it exists and if it is elementary) have spin. The word is not to be taken literally; even in Bohmian mechanics, where the word “particle” is taken literally and particles have positions, they do not spin around their axes. Rather, the wave function is not a scalar; instead, it could
be, e.g., a vector \((\psi : \mathbb{R}^3 \to \mathbb{C}^3)\) or tensor \((\psi : \mathbb{R}^3 \to \mathbb{C}^{3,3})\). In fact, these cases do not occur in nature; instead, other representation spaces \(V\) occur that are called \textit{spin spaces}.

To find representation spaces, it is convenient to use \textit{infinitesimal rotations}. For any Lie group, one defines the elements of the tangent space \(T_I\) at the neutral element \(I\) to be the \textit{infinitesimal generators} of the group; from the group structure it inherits the structure of a \textit{Lie algebra}, i.e., a vector space \(X\) together with a bilinear mapping \([\cdot , \cdot ] : X \times X \to X\) satisfying the \textit{Jacobi identity}

\[
[[A, B], C] + [[C, A], B] + [[B, C], A] = 0 .
\]  

(13.5)

For a Lie group contained in a \(GL(\mathbb{R}^n)\), such as \(SO(3)\), its Lie algebra can be identified with an appropriate space of \(n \times n\) matrices, and the operation \([\cdot , \cdot ]\) is indeed the commutator.

**Proposition 13.1.** \textit{The Lie algebra} \(\mathfrak{so}(3)\) \textit{of} \(SO(3)\) \textit{consists of the anti-symmetric} \(3 \times 3\) \textit{matrices. That is, if} \(R(t)\) \textit{is a smooth curve in} \(SO(3)\) \textit{with} \(R(0) = I\) \textit{then} \(dR/dt(t = 0)\) \textit{is anti-symmetric.}

\textit{Proof.} Differentiate \(I = R^t(t)R(t)\) to obtain \(0 = \dot{R}^tR + R^t\dot{R}\), then set \(t = 0\) to obtain \(0 = \dot{R}^t + \dot{R}\). \(\square\)

It is convenient to use the following basis of \(\mathfrak{so}(3)\):

\[
\Lambda_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \Lambda_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad \Lambda_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .
\]  

(13.6)

Whenever a basis \(B\) of a Lie algebra is given, the coefficients in the relation

\[
[B_i , B_j] = \sum_k c_{ijk}B_k \tag{13.7}
\]

determine the operation \([\cdot , \cdot ]\) completely. For \(\mathfrak{so}(3)\), we have the following fundamental commutation relations for the generators of the rotation group:

\[
[\Lambda_i , \Lambda_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\Lambda_k \tag{13.8}
\]

with \(\varepsilon_{ijk}\) anti-symmetric and \(\varepsilon_{123} = 1\).

When a unitary representation \(U\) of \(SO(3)\) is given, it induces a representation \(u\) of \(\mathfrak{so}(3)\). In fact, for \(A \in \mathfrak{so}(3)\), \(\exp(At)\) is a 1-parameter subgroup of \(SO(3)\), which will be mapped to a 1-parameter group \(U_{\exp(At)} = e^{-iJt}\), whose generator is \(J = u(A)\). The three operators \(J_i := u(\Lambda_i)\) are called the \textit{angular momentum operators} and satisfy the commutation relations

\[
[J_i , J_j] = \sum_{k=1}^{3} \imath \varepsilon_{ijk}J_k \tag{13.9}
\]
as a consequence of (13.8). For example, in the representation $U$ corresponding to (13.1),

$$J_i = \sum_{j,k=1}^3 \varepsilon_{ijk} x_j (-i\hbar \partial_k). \quad (13.10)$$

**Theorem 13.2.** Every continuous unitary representation of a compact Lie group in a Hilbert space is an orthogonal sum of irreducible subrepresentations. Every continuous irreducible representation of a compact Lie group in a Hilbert space is finite-dimensional.


**Proposition 13.3.** There is, up to unitary equivalence, one irreducible representation of $so(3)$ for every dimension $d = \dim \mathcal{H}$; it is called the spin-$s$ representation with $s = \frac{d-1}{2}$.

**Proof.** See the book of Sexl and Urbantke, pages 187–189. \qed

The spin-$\frac{1}{2}$ representation of $so(3)$ is the one that applies to electrons and quarks; $\sigma_i = 2u(\Lambda_i)$ are self-adjoint complex $2 \times 2$ matrices known as the *Pauli spin matrices*,

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$ (13.11)

This representation is called the *spinor representation*, and the elements of the representation space $V_{1/2} \cong \mathbb{C}^2$ are called *spinors* and denoted $\psi_A$, $\phi_B$ or the like.

There is one more twist in the story: The representations of $so(3)$ do not directly correspond to representations of $SO(3)$. That is because $SO(3)$ is not simply connected; it is doubly connected, and its universal covering group is $SU(2)$. The representations of $su(2) = so(3)$ induce unitary representations of $SU(2)$; for integer spin, they induce unitary representations of $SO(3)$, but not for half-odd spin. However, they induce *projective-unitary representations* of $SO(3)$.

In more detail, a rotation $R \in SO(3)$ can be written as $\exp(A)$ for some $A = \sum_i a_i \Lambda_i = \mathbf{a} \cdot \mathbf{A} \in so(3)$; $A$ is unique up to addition of $2\pi nA/\|\mathbf{a}\|$, $n \in \mathbb{Z}$. Since so(3) = su(2), the same expression $\exp(A)$ can be interpreted in $SU(2)$, but there $\exp(A + 2\pi A/\|\mathbf{a}\|) \neq \exp(A)$, only $\exp(A + 4\pi A/\|\mathbf{a}\|) = \exp(A)$. The spin-$\frac{1}{2}$ representation provides a unitary representation of $SU(2)$, but with $R$ it associates two operators on $\mathbb{C}^2$ that differ by a sign:

$$U_R = \pm \exp(\mathbf{a} \cdot \mathbf{\sigma}). \quad (13.12)$$

**13.1 The Pauli equation**

A wave function of $N$ electrons is a function $\psi : \mathbb{R}^{3N} \to (\mathbb{C}^2)^\otimes N$ and has $2^N$ complex components. It evolves according to the so-called *Pauli Hamiltonian*

$$H\psi(x) = \left( \frac{1}{2m} \sum_{k=1}^N \left( \mathbf{\sigma}_{(k)} \cdot (-i\hbar \nabla_k - \mathbf{A}(x_k)) \right)^2 + V(x) \right) \psi(x) \quad (13.13)$$

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with $V(x)$ the electric potential, $A$ the magnetic vector potential, and
\[
\sigma_{(k)} = I \otimes I \otimes \cdots \otimes \underbrace{\sigma \otimes \cdots \otimes 1}_{k\text{-th factor}}.
\] (13.14)

The term
\[
\sum_{k=1}^{N} \left( \sigma_{(k)} \cdot (-i\hbar \nabla_k - A(x_k)) \right)^2
\] (13.15)
in (13.13) can be re-written as
\[
\sum_{k=1}^{N} (-i\hbar \nabla_k - A(x_k))^2 - \sum_{k=1}^{N} \hbar \sigma_{(k)} \cdot B(x_k)
\] (13.16)

with $B = \nabla \times A$ the magnetic field (see, e.g., C. Cohen-Tannoudji, B. Diu, and F. Laloë, Quantum Mechanics, Volume II, Wiley (1977), page 991).