7 Unitary 1-parameter groups and self-adjoint operators

Since $L^2 \subset \mathcal{S}'$, the distributional solution of the free Schrödinger equation just defined provides, in particular, a solution $\psi(t)$ for every initial datum $\psi_0 \in L^2$. Since $\mathcal{F}$ on $\mathcal{S}'$ extends $\mathcal{F}$ on $L^2$ (and so do multiplication operators),

$$\psi(t) = \mathcal{F}^{-1}e^{-i\omega t}\mathcal{F}\psi_0. \quad (7.1)$$

(Recall that $\omega = \omega(k) = \sum_{j=1}^{d} \hbar k_j^2/2m_j$.) The operator mapping $\psi_0$ to $\psi(t)$,

$$U_t = \mathcal{F}^{-1}e^{-i\omega t}\mathcal{F}, \quad U_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad (7.2)$$

is unitary (as it is the composition of 3 unitary operators) and known as the (free) propagator. In fact, the $U_t, t \in \mathbb{R}$ form a unitary 1-parameter group, i.e., a 1-parameter subgroup of the group of all unitary operators $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. To see this, we note the homomorphism property of $t \mapsto U_t$:

$$U_0 = I, \quad U_sU_t = U_{s+t}. \quad (7.3)$$

The last equation is almost obvious from the meaning of $U_t$ as “evolving by $t$ time units” (and the time translation invariance, i.e., the fact that the evolution from time $s$ to time $s+t$ is the same as from $0$ to $t$), but it can also be checked explicitly:

$$\mathcal{F}^{-1}e^{-i\omega s}\mathcal{F}\mathcal{F}^{-1}e^{-i\omega t}\mathcal{F} = \mathcal{F}^{-1}e^{-i\omega(s+t)}\mathcal{F}. \quad (7.4)$$

A question we need to ask about $\psi(t) = U_t\psi_0$ is: In which case is $t \mapsto \psi(t)$ actually differentiable, and what is its time derivative? We first note that it is continuous, as

$$\|\psi(t) - \psi(t_0)\|^2 = \|(U_t - U_{t_0})\psi_0\|^2 = \int_{\mathbb{R}^d} \left|e^{-i\omega t} - e^{-i\omega t_0}\right|^2 |\hat{\psi}_0(k)|^2 dk \xrightarrow{t \to t_0} 0 \quad (7.5)$$

by the dominated convergence theorem. The statement $\frac{d\psi}{dt}(t_0) = \phi$ means that the following expression tends to zero as $t \to t_0$:

$$\left\|\frac{\psi(t) - \psi(t_0)}{t-t_0} - \phi\right\|^2 = \int_{\mathbb{R}^d} \left|\frac{e^{-i\omega t} - e^{-i\omega t_0}}{t-t_0} - \frac{\hat{\phi}(k)}{\hat{\psi}_0(k)}\right|^2 |\hat{\psi}_0(k)|^2 dk. \quad (7.6)$$

This is the case if and only if

$$\hat{\phi}(k) = -i\omega e^{-i\omega t_0}\hat{\psi}_0(k) \quad (7.7)$$

for almost every $k$ and, in addition, $\omega \hat{\psi}_0 \in L^2$. In this case, $\phi$ actually is

$$\phi = i \sum_j \frac{\hbar}{2m_j} \frac{\partial^2 \psi_t}{\partial x^2_j} = -\frac{i}{\hbar} H \psi_t \quad (7.8)$$

with weak derivatives. That is, if $\omega \hat{\psi}_0 \in L^2$ then $t \mapsto \psi_t$ is differentiable at all times, $H\psi_t$ (with weak derivatives) exists, lies in $L^2$, and $i\hbar \frac{d\psi_t}{dt} = H\psi_t$. One says that under this condition $\psi_t$ solves $i\hbar \frac{d\psi_t}{dt} = H\psi_t$ in the $L^2$ sense.
**Definition 7.1.** For $m \in \mathbb{Z}$, the $m$-th Sobolev space $H^m(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ is the set of those $f \in \mathcal{S}'$ for which $f$ is a measurable function and

$$(1 + |k|^2)^{m/2} \hat{f} \in L^2(\mathbb{R}^d).$$

(7.9)

For $m \geq 0$, $H^m \subset L^2$.

So the initial conditions for which $\psi_t$ solves the Schrödinger equation in the $L^2$ sense are those in the second Sobolev space $H^2$.

**Lemma 7.2.** (Sobolev’s lemma) Let $\ell \in \mathbb{N}_0$ and $f \in H^m(\mathbb{R}^d)$ with $m > \ell + \frac{d}{2}$. Then $f \in C^\ell(\mathbb{R}^d)$ and

$$\partial^\alpha f \in C^\infty(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) : \lim_{R \to \infty} \sup_{|x| > R} |f(x)| = 0 \right\}$$

(7.10)

for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq \ell$.

**Proof.** We show that $k^\alpha \hat{f}(k) \in L^1$ for all $\alpha$ with $|\alpha| \leq \ell$. Then we use the Lemma of Riemann–Lebesgue to conclude $\partial^\alpha f \in C^\infty$.

**Lemma 7.3.** (Lemma of Riemann–Lebesgue) For $f \in L^1(\mathbb{R}^d)$, $\hat{f} \in C^\infty(\mathbb{R}^d)$.

**Proof.** For $f \in \mathcal{S}$ we know $\hat{f} \in \mathcal{S} \subset C^\infty$. Obviously, $\|\hat{f}\|_\infty \leq (2\pi)^{-d/2}\|f\|_{L^1}$. Thus, Fourier transformation is a bounded linear map from a dense set in $L^1$ to $C^\infty$. By Theorem 3.13, it has a unique bounded linear extension $L^1 \to C^\infty$ (which must be $\mathcal{S}$).

We now prove Sobolev’s lemma. Since $f \in H^m$, $(1 + |k|^2)^{m/2} \hat{f}(k) \in L^2$ and thus, for every $\alpha$ with $|\alpha| \leq \ell$,

$$\int_{\mathbb{R}^d} |k^\alpha \hat{f}(k)| \, dk \leq \int (1 + |k|^2)^{\ell/2} |\hat{f}(k)| \, dk$$

(7.11)

$$= \int (1 + |k|^2)^{m/2} |\hat{f}(k)| \left(1 + |k|^2\right)^{(\ell-m)/2} \, dk$$

(7.12)

$$\leq \left\| (1 + |k|^2)^{m/2} \hat{f}(k) \right\|_{L^2} \left( \int \frac{dk}{(1 + |k|^2)^{m-\ell}} \right)^{1/2},$$

(7.13)

using the Cauchy–Schwarz inequality. The last integral is finite iff $2(m - \ell) > d$. □

One may consider, instead of $t \mapsto \psi_t$, $t \mapsto U_t$. That is a mapping $\mathbb{R} \to \mathcal{B}(L^2)$, where $\mathcal{B}(L^2)$ is the space of bounded operators on $L^2$. (For any normed spaces $X, Y$, the space $\mathcal{B}(X, Y)$ of bounded linear operators $X \to Y$ is again a normed space with the operator norm. If $Y$ is complete (i.e., a Banach space) then so is $\mathcal{B}(X, Y)$.) However, $t \mapsto U_t$ is not continuous:

$$\|U_t - U_{t_0}\| = \left\| e^{-i\omega t} - e^{-i\omega t_0} \right\| = \sup_{k \in \mathbb{R}^d} \left| e^{-i\omega t} - e^{-i\omega t_0} \right| = 2 \quad \forall t \neq t_0.$$
7.1 Unitary 1-parameter groups

Definition 7.4. Let $A \in \mathcal{B}(\mathcal{H})$ and $A_n$ be a sequence in $\mathcal{B}(\mathcal{H})$.

(i) $A_n$ converges in norm to $A$ ($\lim A_n = A$, $A_n \to A$) iff
\[ \|A_n - A\| \to 0. \] (7.15)

(ii) $A_n$ converges strongly to $A$ ($s\lim A_n = A$, $A_n \rightharpoonup A$) iff
\[ \|A_n\psi - A\psi\| \to 0 \quad \text{for all } \psi \in \mathcal{H}. \] (7.16)

(iii) $A_n$ converges weakly to $A$ ($w\lim A_n = A$, $A_n \rightharpoonup A$) iff
\[ \langle \psi | (A_n - A)\phi \rangle \to 0 \quad \text{for all } \psi, \phi \in \mathcal{H}. \] (7.17)

The following implications hold:

\[ \text{norm convergence} \Rightarrow \text{strong convergence} \Rightarrow \text{weak convergence} \] (7.18)

The converse implications are not generally valid.

So, $U_t$ is a strongly continuous unitary 1-parameter group.

Definition 7.5. A densely defined operator $H$ with domain $\mathcal{D}(H) \subseteq \mathcal{H}$ is the generator of a strongly continuous unitary 1-parameter group $U_t$ iff

(i) $\mathcal{D}(H) = \{ \psi \in \mathcal{H} : t \mapsto U_t\psi \text{ is differentiable} \}$

(ii) $ih\, dU_t\psi/\,dt = HU_t\psi$ for $\psi \in \mathcal{D}(H)$.

The generator of the physical time evolution is called the Hamiltonian. The Hamiltonian of the free Schrödinger is essentially the Laplace operator; more precisely, it is
\[ H = \mathcal{F}^{-1}h\omega \mathcal{F}, \quad \text{with } \mathcal{D}(H) = H^2(\mathbb{R}^d). \] (7.19)

We will see soon that exactly the self-adjoint operators are generators of strongly continuous unitary 1-parameter groups.

Proposition 7.6. Let $H$ be a generator of a strongly continuous unitary 1-parameter group $U_t$.

(i) $\mathcal{D}(H)$ is invariant under $U_t$, i.e., $U_t\mathcal{D}(H) = \mathcal{D}(H)$ for all $t \in \mathbb{R}$.

(ii) $H$ commutes with $U_t$, i.e.,
\[ [H, U_t]\psi = HU_t\psi - U_tH\psi = 0 \quad \text{for all } \psi \in \mathcal{D}(H). \] (7.20)

(iii) $H$ is symmetric, i.e.,
\[ \langle H\psi | \phi \rangle = \langle \psi | H\phi \rangle \quad \text{for all } \psi, \phi \in \mathcal{D}(H). \] (7.21)
(iv) $U$ is uniquely determined by $H$.
(v) $H$ is uniquely determined by $U$.

Proof.  
(i) $s \mapsto U_s U_t \psi = U_{s + t} \psi$ is differentiable iff $s \mapsto U_s \psi = U_{-t} U_{s + t} \psi$ is.
(ii) For $\psi \in \mathcal{D}(H)$,
\[ U_t H \psi = U_t \left( i \hbar \frac{d}{ds} U_s \psi \right) \bigg|_{s=0} = i \hbar \frac{d}{ds} U_s \psi \bigg|_{s=0} = i \hbar \frac{d}{ds} U_s U_t \psi \bigg|_{s=0} = HU_t \psi . \quad (7.22) \]
(iii) This follows from the unitarity: For $\psi, \phi \in \mathcal{D}(H),$
\[ 0 = \frac{d}{dt} \langle \psi | \phi \rangle = \frac{d}{dt} \langle U_t \psi | U_t \phi \rangle = \left\{ -i \frac{\hbar}{2} HU_t \psi | U_t \phi \right\} + \langle U_t \psi | -i \frac{\hbar}{2} HU_t \phi \rangle \quad (7.23) \]
\[ = \frac{i}{\hbar} \langle U_t H \psi | U_t \phi \rangle - \frac{i}{\hbar} \langle U_t \psi | H \phi \rangle = \frac{i}{2} \left( \langle H \psi | \phi \rangle - \langle \psi | H \phi \rangle \right) . \quad (7.24) \]
(iv) Suppose that $H$ is also a generator of $\tilde{U}_t$. Then, by the symmetry of $H$,
\[ \frac{d}{dt} \left\| (U_t - \tilde{U}_t) \psi \right\|^2 = 2 \frac{d}{dt} \left( \| \psi \|^2 - \Re \langle U_t \psi | \tilde{U}_t \psi \rangle \right) \]
\[ = -2 \Re \left( -i \frac{\hbar}{2} HU_t \psi | \tilde{U}_t \psi \right) + \langle U_t \psi | -i \frac{\hbar}{2} HU_t \phi \rangle \quad (7.25) \]
\[ = -2 \Re \left( i \frac{\hbar}{2} \langle HU_t \psi | \tilde{U}_t \psi \rangle - i \frac{\hbar}{2} \langle U_t \psi | H \tilde{U}_t \psi \rangle \right) \quad (7.26) \]
\[ = 0 \quad (7.27) \]
for all $\psi \in \mathcal{D}(H)$. From $(U_0 - \tilde{U}_0) \psi = 0$ we can conclude $\tilde{U} |_{\mathcal{D}(H)} = U |_{\mathcal{D}(H)}$; since $\mathcal{D}(H) = \mathcal{H}$, we obtain $\tilde{U} = U$ on all of $\mathcal{H}$.

(v) is immediate from the definition of $H$. \hfill \Box

Example 7.7. Let $T_t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the group of translations, $T_t \psi(x) = \psi(x - t)$. It is strongly continuous and generated by $H = -i \hbar \frac{d}{dx}$, defined on $\mathcal{D}(H) = H^1(\mathbb{R})$.

7.2 Adjoint of a bounded operator

If $A : X \to Y$ is continuous, where $X$ and $Y$ are normed spaces, then the adjoint $A' : Y' \to X'$ is defined by
\[ (A' y')(x) = y'(Ax) \quad (7.29) \]
for $x \in X$ and $y' \in Y'$, and is continuous, too (with $\|A'\| \leq \|A\|$ because $\|A' y'\| \leq \|y'\| \|A\|$). We mention the fact that $X'$ and $Y'$ are automatically Banach spaces.
We will now explain that any Hilbert space $\mathcal{H}$ can be regarded as its own dual space; this will allow us to regard the dual $A'$ of an operator $A$ on $\mathcal{H}$ again as an operator $A^*$ on $\mathcal{H}$.

For finite-dimensional spaces $X$, an inner product defines an identification between $X$ and its dual space $X^D = X'$, $J : X \to X^D$, by $J\psi = \langle \psi | \cdot \rangle$. We will now see that a Hilbert space can be identified with its continuous dual space in the same way.

**Theorem 7.8.** *(Riesz representation theorem)* Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{H}'$. Then there is a unique $\psi_T \in \mathcal{H}$ such that

$$
T(\phi) = \langle \psi_T | \phi \rangle \quad \forall \phi \in \mathcal{H}.
$$

**Proof.** Existence. Let $T \in \mathcal{H}'$ and $M$ the kernel of $T$. (Note that if $\psi_T$ exists then $M = \psi_T^\perp$.) If $M = \mathcal{H}$ then $T = 0$ and $\psi_T = 0$ does what was claimed. Now suppose $M \neq \mathcal{H}$. Then we want to show that $M^\perp$ is one-dimensional. To see this, note that for any $\psi_0, \psi_1 \in M^\perp \setminus \{0\}$, by setting $\alpha = T(\psi_0)/T(\psi_1)$, we have that

$$
T(\psi_0 - \alpha \psi_1) = T(\psi_0) - \alpha T(\psi_1) = 0
$$

and thus

$$
\psi_0 - \alpha \psi_1 \in M \cap M^\perp = \{0\}
$$

or $\psi_0 = \alpha \psi_1$. Thus, $M^\perp$ is 1-dimensional. Since $T$ is continuous, $M$ is closed. By the projection theorem, every $\phi \in \mathcal{H}$ can be written uniquely as

$$
\phi = \phi_M + \phi_{M^\perp} = \phi_M + \frac{\langle \psi_0 | \phi \rangle}{\| \psi_0 \|^2} \psi_0.
$$

Now set $\psi_T = \frac{T(\psi_0)^*}{\| \psi_0 \|^2} \psi_0$ and obtain

$$
T(\phi) = T(\phi_M) + \frac{\langle \psi_0 | \phi \rangle}{\| \psi_0 \|^2} T(\psi_0) = \langle \psi_T | \phi \rangle.
$$

Uniqueness. This follows from the definiteness of the inner product.

**Corollary 7.9.** *(Self-duality)* The mapping

$$
J\psi = \langle \psi | \cdot \rangle
$$

is a bijection between $\mathcal{H}$ and $\mathcal{H}'$. It is anti-linear, continuous, and isometric (i.e., preserves norms).

**Proof.** The range of $J$ lies in $\mathcal{H}'$ because $\langle \cdot | \cdot \rangle$ is continuous. By the Riesz theorem, $J$ is surjective. By the Cauchy–Schwarz inequality, $J$ is isometric and therefore injective and continuous.

**Definition 7.10.** For bounded $A : \mathcal{H} \to \mathcal{H}$, its Hilbert-space-adjoint $A^*$ is defined by $J^{-1}A'J$. $A$ is called self-adjoint if $A^* = A$.  

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Proposition 7.11. For $A \in \mathcal{B}(\mathcal{H})$,
\[
\langle \psi | A \phi \rangle = \langle A^* \psi | \phi \rangle \quad \forall \psi, \phi \in \mathcal{H}.
\] (7.36)

$A^*$ is uniquely determined by this property. A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is self-adjoint iff it is symmetric, i.e.,
\[
\langle \psi | A \phi \rangle = \langle A \psi | \phi \rangle \quad \forall \psi, \phi \in \mathcal{H}.
\] (7.37)

Proof. By definition of $A^*$,
\[
\langle \psi | A \phi \rangle = (J \psi)(A \phi) = A'(J \psi)(\phi) = J J^{-1} A' J \phi = J A^* \phi = \langle A^* \psi | \phi \rangle.
\] (7.38)

Uniqueness: Since the mapping $\phi \mapsto \langle \psi | A \phi \rangle$ is continuous and linear, the Riesz representation theorem guarantees the uniqueness of $\chi \in \mathcal{H}$ with $\langle \psi | A \phi \rangle = \langle \chi | \phi \rangle$.

The statement about self-adjointness now follows. We note already that, for unbounded operators $A$, being symmetric does not in general imply being self-adjoint.

Example 7.12. Orthogonal projections $P$ are self-adjoint operators. They are bounded, $\|P\| = 1$ (except $P = 0$), and satisfy $\langle P \psi | \phi \rangle = \langle \psi | P \phi \rangle$, see (3.17). Also, any finite linear combination of self-adjoint bounded operators with real coefficients is self-adjoint and bounded.

Theorem 7.13. (Properties of the adjoint) Let $A, B \in \mathcal{B}(\mathcal{H})$, $\lambda \in \mathbb{C}$. Then

(i) $(A + B)^* = A^* + B^*$ and $(\lambda A)^* = \lambda A^*$.

(ii) $(AB)^* = B^* A^*$

(iii) $\|A^*\| = \|A\|

(iv) $A^{**} = A$

(v) $\|A A^*\| = \|A^* A\| = \|A\|^2$

(vi) $\ker A = (\text{im } A^*)^\perp$ and $\ker A^* = (\text{im } A)^\perp$.

Proof. (i)–(iii) follow from the corresponding properties of $A'$ ($\|A'\| = \|A\|$ follows from the Hahn–Banach theorem), and (iv) from
\[
\langle \psi | A \phi \rangle = \langle A^* \psi | \phi \rangle = \langle \phi | A^* \psi \rangle^* = \langle A^{**} \phi | \psi \rangle^* = \langle \psi | A^{**} \phi \rangle
\] (7.39)
for all $\psi, \phi \in \mathcal{H}$. Concerning (v) we observe that
\[
\|A \phi \|^2 = \langle A \phi | A \phi \rangle = \langle \phi | A^* A \phi \rangle \leq \|\phi\|^2 \|A^* A\|
\] (7.40)
and conclude
\[
\|A\|^2 = \sup_{\|\phi\|=1} \|A \phi \|^2 \leq \|A^* A\| \leq \|A^*\| \|A\| = \|A\|^2.
\] (7.41)
Concerning (vi), $\phi \in \ker A$ iff $A \phi = 0$ iff $\langle \phi | A \phi \rangle = 0 \forall \psi$ iff $\langle A^* \psi | \phi \rangle = 0 \forall \psi$ iff $\phi \in (\text{im } A^*)^\perp$. 

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Example 7.14. Let $L : \ell^2 \to \ell^2$ and $R : \ell^2 \to \ell^2$ be the left shift and right shift operators:
\[ L(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots), \quad R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots). \] (7.42)
They are adjoints of each other, $R^* = L$ and $L^* = R$:
\[ \langle x | R y \rangle = \sum_{n=2}^{\infty} x_n y_{n-1} = \sum_{n=1}^{\infty} x_{n+1} y_n = \langle L x | y \rangle. \] (7.43)
Note that $R$ is not unitary, although it is isometric: $L R = I$ but $R L \neq I$.

Proposition 7.15. $U \in \mathcal{B}(\mathcal{H})$ is unitary iff $UU^* = I = U^* U$.

Proof. If $U$ is unitary then
\[ \langle U^* U \psi - \psi | \phi \rangle = \langle U \psi | U \phi \rangle - \langle \psi | \phi \rangle = 0 \quad \text{for all } \psi, \phi \in \mathcal{H} \] (7.44)
and thus $U^* U = I$. It follows further that $UU^* U = U$ and, since $U$ is surjective, $UU^* = I$.

Conversely, let $UU^* = I = U^* U$. Then $U$ is surjective and
\[ \langle U \psi | U \phi \rangle = \langle U^* U \psi | \phi \rangle = \langle \psi | \phi \rangle \quad \text{for all } \psi, \phi \in \mathcal{H}. \] (7.45)
\[ \square \]

Proposition 7.16. Let $H : \mathcal{H} \to \mathcal{H}$ be bounded and self-adjoint. Then
\[ U_t = e^{-iHt/\hbar} = \sum_{n=0}^{\infty} \frac{(-iHt/\hbar)^n}{n!} \] (7.46)
defines a strongly continuous (even norm continuous) unitary 1-parameter group whose generator is $H$ with $\mathcal{D}(H) = \mathcal{H}$. (It is even true that $\mathbb{R} \to \mathcal{B}(\mathcal{H}) : t \mapsto e^{-iHt/\hbar}$ is differentiable.)

Proof. The series converges in the operator norm because it is Cauchy and $\mathcal{B}(\mathcal{H})$ is a Banach space. The claims can be verified in much the same way as for the ordinary exponential function. First check that when $A, B \in \mathcal{B}(\mathcal{H})$ and $AB = BA$ then $e^{A+B} = e^A e^B = e^B e^A$. This implies the group property. Then check that $e^{A^*} = (e^A)^*$. This implies $U_{-t} = U_t^*$ and thus $U_t^* U_t = I = U_t U_t^*$, so $U_t$ is unitary. $\|e^{tA} - I\| \leq \sum_{n=1}^{\infty} |t^n| \|A^n\|/n! = e^{\|A\|} - 1 \to 0$ as $t \to 0$. Concerning $dU_t/dt$, $\|e^{tA} - I\|/t - A\| = \| \sum_{n=2}^{\infty} t^{n-1} A^n/n! \| \leq \sum_{n=2}^{\infty} |t|^{n-1} \|A^n\|/n! = [(e^{\|A\|} - 1)/|t| - \|A\|] \to 0$ as $t \to 0$. \[ \square \]
7.3 Adjoint of an unbounded operator

Recall that an unbounded operator is a linear mapping $A : \mathcal{D}(A) \to \mathcal{H}$, $\mathcal{D}(A) \subseteq \mathcal{H}$. If $\mathcal{D}(A)$ is dense in $\mathcal{H}$ then one says that $A$ is densely defined. $A$ is called symmetric iff

$$\langle A\psi | \phi \rangle = \langle \psi | A\phi \rangle \quad \text{for all } \psi, \phi \in \mathcal{D}(A). \quad (7.47)$$

(Example: the free Hamiltonian on the second Sobolev space is a densely defined symmetric operator.) $B : \mathcal{D}(B) \to \mathcal{H}$ is called an extension of $A$ if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$; in this case we write $A \subseteq B$.

**Definition 7.17.** The adjoint operator $A^*$ of the densely defined operator $A : \mathcal{D}(A) \to \mathcal{H}$ has the domain

$$\mathcal{D}(A^*) = \{ \psi \in \mathcal{H} : \exists \chi \in \mathcal{H} \forall \phi \in \mathcal{D}(A) : \langle \psi | A\phi \rangle = \langle \chi | \phi \rangle \} \quad (7.48)$$

$$= \{ \psi \in \mathcal{H} : \phi \mapsto \langle \psi | A\phi \rangle \text{ is continuous on } \mathcal{D}(A) \} \quad (7.49)$$

and is on this domain defined by the relation

$$\langle \psi | A\phi \rangle = \langle A^*\psi | \phi \rangle \quad (7.50)$$

for all $\psi \in \mathcal{D}(A^*)$ and $\phi \in \mathcal{D}(A)$. (If $\mathcal{D}(A)$ were not dense then this relation would not uniquely determine $A^*$. $A^*$ is a linear operator but not necessarily densely defined.) If $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $A = A^*$ then $A$ is called self-adjoint.

**Theorem 7.18.** Every strongly continuous unitary 1-parameter group has a generator, which is densely defined and self-adjoint (Stone’s theorem). Conversely, a densely defined operator $H$ is the generator of a strongly continuous unitary 1-parameter group iff it is self-adjoint.

We will not prove this theorem. The second statement follows from the spectral theorem for self-adjoint operators, which we will describe (but not prove) in the next chapter. As we will explain, the group generated by $H$ can be written as $e^{-iHt/\hbar}$ also for unbounded $H$.

**Example 7.19.** (Multiplication operators) Let $V : \mathbb{R}^d \to \mathbb{C}$ be measurable, and let $M_V$ be the multiplication operator $M_V : \mathcal{D}(M_V) \to L^2(\mathbb{R}^d)$,

$$(M_V\psi)(x) = V(x) \psi(x) \quad (7.51)$$

deﬁned on

$$\mathcal{D}(M_V) = \left\{ \psi \in L^2(\mathbb{R}^d) : V\psi \in L^2(\mathbb{R}^d) \right\}. \quad (7.52)$$

$\mathcal{D}(M_V)$ is always dense in $L^2$ and the adjoint operator $M_V^*$ is given by

$$(M_V^*\psi)(x) = V(x)^* \psi(x), \quad \text{i.e., } M_V^* = M_V^* \quad (7.53)$$
on $\mathcal{D}(M_V^*) = \mathcal{D}(M_V)$. If $V$ is real-valued then $M_V$ is self-adjoint.

**Example 7.20.** The free Hamiltonian $H = -\sum_{j=1}^d \frac{\hbar^2}{2m_j} \partial_j^2$ on $H^2(\mathbb{R}^d)$ is self-adjoint because it is unitarily equivalent (via $\mathcal{F}$) to the multiplication operator $M_{\hbar^2}$ on its maximal domain.