The dihedral group is the group symmetries of the regular n-gon. It contains $2n$ elements, and one standard way to "present" it is using two generators (a rotation a and a flip f) and three defining relations: $D_n = \langle a, f; f^2 = e,\ afaf = e,\ a^n = e \rangle$.

In the last class (Thursday, February 8, 2007) Tim Lou sketched his proof that for odd n, D_n can already be generated by two relations on the same generators. Here are more details.

Theorem. *For any integer $k \geq 0$, the following two groups are the same:*

$$D_{2k+1} = \langle a, f; f^2 = e,\ afaf = e,\ a^{2k+1} = e \rangle\,,$$

$$\langle a, f; f^2 = e,\ a^{k+1} = fa^k f \rangle.$$

Proof. We show that the two sets of relations imply each other. Assume $f^2 = e$ (since that appears in both sets).

Part I: Assume also $afaf = e$ and $a^{2k+1} = e$. We need to prove that $a^{k+1} = fa^k f$.

Indeed, $a^{-1} = faf$, whence

$$a^{-k} = (faf)^k = (faf)(faf)\dots(faf) = fa^k f$$

(since the in-between $f{\cdot}f$ terms cancel out).

But $a^{-k} = a^{k+1}$ by the relation $a^{2k+1} = e$, proving $a^{k+1} = fa^k f$.

Part II: Now assume $a^{k+1} = fa^k f$. We need to prove that $afaf = e$ and $a^{2k+1} = e$ both hold. Let us start with the two equivalent forms: $a^{k+1} = fa^k f$ and $a^k = fa^{k+1} f$. Raising the first one to the k-th power and the second one to the $(k+1)$-st power (and using the cancellation $f \cdot f = e$ as above), we get $a^{k(k+1)} = fa^{k^2} f$ and $a^{k(k+1)} = fa^{k^2+2k+1} f$.
Hence, $fa^{k^2+2k+1} f = fa^{k^2} f$, proving the claim $a^{2k+1} = e$.

Finally, to prove $afaf = e$, let us square the relation $a^k = fa^{k+1} f$ and use $a^{2k+1} = e$ (which implies that $a^{2k} = a^{-1}$):

$$a^{-1} = a^{2k} = (a^k)^2 = (fa^{k+1} f)^2 = (fa^{k+1} f)(fa^{k+1} f) = fa^{2k+2} f = faf.$$

The obtained relation $a^{-1} = faf$ is equivalent to our claim $afaf = e$.

Remark. The following important observation was used here: if two sets of relations R_1 and R_2 (about the same generators) are equivalent (that is, they mutually imply each other), then they determine the same group.

In fact, the set of all groups (with the given generators) that satisfy R_1 (but perhaps other relations too) is the same as the set of all groups satisfying R_2.

[Writeup courtesy of Prof. Komlós]