Theorem: A finite group of isometries in $\mathbb{R}^d$ must have a common fixed point.

Proof (Courtesy of Professor Roe Goodman).

Let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite group of isometries in $\mathbb{R}^d$. Let $p \in \mathbb{R}^d$ be arbitrary, and let $\bar{p}$ be the centroid (barycenter) of the orbit of p:

$$\bar{p} = \frac{1}{n} \sum_{i=1}^{n} g_i(p).$$

Claim: $\bar{p}$ is a common fixed point for all isometries in G, that is, $g_i(\bar{p}) = \bar{p}$ for all i.

We will use the known fact that if f is an isometry fixing the origin, then f is a linear operator. Hence, if g is an arbitrary isometry, then $f(x) = g(x) - g(0)$ is linear.

Let $h \in G$ be any one of the isometries. We need to show that $h(\bar{p}) = \bar{p}$.

Indeed, writing $f(x) = h(x) - h(0)$,

$$\begin{aligned} h(\bar{p}) = h(0) + f(\bar{p}) = h(0) + f\left(\frac{1}{n}\sum_{i=1}^{n} g_i(p)\right) &= h(0) + \frac{1}{n}\sum_{i=1}^{n} f(g_i(p)) \\ = \frac{1}{n}\sum_{i=1}^{n}[h(0) + f(g_i(p))] &= \frac{1}{n}\sum_{i=1}^{n} h(g_i(p)) \end{aligned}$$

since f is linear.

Now, the functions $h \circ g_1, h \circ g_2, \ldots, h \circ g_n$ are the elements of G in a different order ($g \mapsto h \circ g$ is a bijection on G), hence

$$h(\bar{p}) = \frac{1}{n}\sum_{i=1}^{n} h(g_i(p)) = \frac{1}{n}\sum_{i=1}^{n}(h \circ g_i)(p) = \frac{1}{n}\sum_{g \in G} g(p) = \bar{p}$$

as claimed.