Examples, proofs, and other material from Greenberg's *Euclidean and Non-Euclidean Geometries, Development and History.* W.H. Freeman and Company, 1974.

1 Euclidean Geometry

Geometry existed long before the time of Euclid. People were able to find a number of geometric constructions before any system of axioms existed. It was Euclid, however, who set out to formalize the system of geometry by introducing a set of common starting points from which, without any assumptions, every geometrical observation could be proved. His axioms are the following:

(I) For every point P and for every point Q not equal to P there exists a unique line l that passes through P and Q.

(II) For every segment AB and for every segment CD there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE.

(III) For every point O and every point A not equal to O there exists a circle with center O and radius OA.

(IV) All right angles are congruent to each other.

Before stating Euclid's fifth postulate, it is necessary to state a formal definition.

Definition. Two lines l and m are *parallel* if they do not intersect, i.e., if no point lies on both of them. We denote this by $l \| m$.

We are now ready to state Euclid's fifth postulate, sometimes referred to as the Parallel Postulate.

(V) For every line l and for every point P that does not lie on l there exists a unique line m through P that is parallel to l.

There are many very subtle mathematical oversights in this set of axioms with respect to the propositions (theorems) Euclid provided in his book *Elements.* For example, what is a line? How can a straight line be described, unambiguously and mathematically, such that anyone would be able to reproduce it without a previous inclination for the Euclidean framework? Euclid himself tried to describe a straight line as "that which lies evenly on itself" and a point as "that which has no part". As you can see, these definitions are effectively recursive, and have meaning only to us, who are so familiar with lines and points already. In other words, the only way these make sense is if we are already thinking in terms of the Euclidean geometry to which we are so accustomed.

It is more logical for us to create a list of *undefined* terms, and let the axioms specify the properties of each object *relative to one another*. No matter how we choose to think of these undefined objects, as long as they satisfy the axioms, they will be completely consistent with any familiar theorem which had been derived from our pre-existing notions of Euclidean points, lines, etc. This allows us a great deal of freedom to consider geometry in a more abstract and formal way, without the seemingly necessary bounds of Euclidean geometry. The terms we choose to leave undefined are:

- *point*
- *line*
- *lie on (incide with)*
- *between*
- *congruent*

Another subtlety in Euclid's proofs is that he made a few assumptions based on diagrams. Although most of his assumptions turned out to be consistent, it is better for us to fill in the gaps with some other axioms. For example, consider the proof that the base angles of an isosceles triangle are congruent. The first claim was that the third angle has a bisector that, when extended, intersects the opposite side at a point. Although it may seem obvious, we cannot prove this from any of Euclid's postulates, so we have to set up axioms to describe *betweenness*.

Finally, Euclid himself loathed the fifth postulate, the *parallel postulate*. It was not nearly as elegant or simple as the others, nor could it be proven from any other, but it proved to be eventually necessary to prove certain Euclidean theorems. His distaste is apparent in that he did not use the postulate until his 29th theorem!

2 Incidence Geometry

In an attempt to fill in some of the gaps in Euclid's postulates, we consider these three postulates pertaining to only *incidence*:

I-1 For every point P and for every point Q not equal to P there exists a unique line l that passes through (is incident with) P and Q.

I-2 For every line l there exist at least two distinct points incident with l.

I-3 There exist three distinct points with the property that no line is incident with all three of them.

Now we can derive any theorems we like as long as they follow logically from the axioms. Remember, we are free to *interpret* the undefined terms any way we like, as long as the axioms are satisfied. If they are, the interpretation is a valid *model*.

Example. Consider the set $\mathbf{S} = \{A, B, C\}$. We will call an element $P \in \mathbf{S}$ a *point*. Let a *line* be defined as a subset $\mathbf{L} \subset \mathbf{S}$ such that $\mathbf{L} = \{P, Q\}, P, Q \in \mathbf{S}, P \neq Q$. There are three lines: $\{A, B\}, \{A, C\}$, and $\{B, C\}$. A point P will be interpreted as *incident* with a line $\mathbf{L}$ if $P \in \mathbf{L}$. Thus, A lies on $\{A, B\}$ and $\{A, C\}$, but does not lie on $\{B, C\}$. We verify that this interpretation is a model.

I $P, Q \in \mathbf{S}, P \neq Q$ defines the line $\mathbf{L} = \{P, Q\}$.

II Let $\mathbf{L} = \{P, Q\}$. $P \in \mathbf{L}$ & $Q \in \mathbf{L}$ implies P and Q lie on $\mathbf{L}$.

III $(\exists P, Q, R \in \mathbf{S}) P \neq Q \neq R \Rightarrow (\nexists \mathbf{L} = \{X, Y\}) P, Q, R \in \mathbf{L}$.

Thus, the interpretation is a model, and all theorems derived using the incident postulates will be consistent in the model. If a proof is possible through the axioms, then the proof holds in *all* models.

What is the purpose of models? Remark the contrapositive of the aforementioned statement—If there exists a model in which a proof does not hold, then it is impossible to prove this through the incidence axioms. We can use this approach with the Euclidean parallel postulate. For every line l and every point P not lying on l there exists a unique line through P that is parallel to l. In our model, $(\forall \mathbf{L} = \{Q, R\}, Q \neq R, P \notin \mathbf{L}) \exists! \mathbf{M} = \{P, X\}, \mathbf{M} \bigcap \mathbf{L} = \emptyset$ Since $\mathbf{M} = \{P, Q\}$ or $\mathbf{M} = \{P, R\}, \mathbf{M} \bigcap \mathbf{L}$ is nonempty. Thus, the parallel postulate does not hold, and we have demonstrated that it is impossible to prove it by the incidence axioms. We say that this model has the *elliptic parallel property.* This proves that the Euclidean parallel postulate is *independent* from the three incidence axioms.

The incidence axioms are the first of the set of axioms introduced by David Hilbert, a renowned mathematician of the early 1900s. We would very much like to extend this independence of the parallel postulate to the other axioms in his system. The exact wording of each and every axiom is not very important to us, but the axioms are grouped into those of *incidence, betweenness, congruence, continuity,* and *parallelism.* It would prove useful for us to adopt all these axioms except for the parallel axiom (which turns out to be logically equivalent to Euclid's fifth postulate). How much would we be able to prove without the aid of our parallel postulate?

3 Neutral Geometry

Neutral geometry, sometimes called *absolute* geometry, is the set of theorems provable by Hilbert's axioms, without making a *choice* as to whether or not we consider the parallel postulate to be true. Let us construct some very important proofs in this system. We will not waste space formally defining terms with which most are already familiar. Rigid definitions and proofs can be found in books on the subject.

Theorem 1 (Alternate Interior Angle) *If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.*

While the proof is not complicated, it is fairly lengthy and will be omitted. What is important is that it is possible to proof this without the help of the parallel postulate. This may seem trivial at first, but in fact it yields the following corollaries.

Corollary 2 *Two lines perpendicular to the same line are parallel.*

Corollary 3 *If P is a point not on line l, the line perpendicular to l through P is* unique*.*

The first corollary is evident as a special case of the Alternate Interior Angle Theorem. For the second, consider a case where two lines go through P and are parallel to l. Then the two lines are parallel to each other by the previous corollary, contradicting the fact that they meet at P. One final corollary is one of the most important of neutral geometry.

Corollary 4 *If l is any line and P is any point not on l, there exists at least one line m through P which does not intersect l.*

First, construct n, the perpendicular of l through P. Construct m perpendicular to n. n is parallel to l. With these few elegant corollaries, we have *almost* proved the infamous parallel postulate. However, we have not, and *cannot* prove that the line parallel to l is unique.

While we're on a roll *almost* proving things, let us consider the following theorem.

Theorem 5 (Saccheri-Legendre) *The sum of the degree measures of the three angles in any triangle is less than or equal to 180°.*

The general proof involves assuming the opposite, that there exists a triangle with angle sum $180° + p°$, and arriving at a contradiction. We can do no better than prove this upper bound, for the way we prove the angle sum equal to 180° in Euclidean geometry is by exploiting the parallel postulate!

Thus, in neutral geometry, we can see that a triangle whose angle sum is less than 180° does not yield any contradictions. If such triangles can exist, it is convenient to state the following definition.

Definition. The *defect* of a triangle is 180° minus the angle sum of the triangle.

What we can see is that defect has an additive property. That is, given $\triangle\ ABC$ and a point D between A and B, the defect($\triangle\ ABC$) = defect($\triangle\ ACD$) + defect($\triangle\ BCD$). In order for the defect of $\triangle\ ABC$ to be zero, the defect of any triangle that makes a part of △ ABC must also be zero (defect can never be negative by the Saccheri-Legendre Theorem). This gives us the remarkable theorem:

Theorem 6 *If a triangle exists whose angle sum is 180°, then* every *triangle has an angle sum equal to 180°.*

From any triangle with defect zero, we can construct a rectangle. We can then "lay out" the rectangle arbitrarily many times, so that any triangle can be "embedded" into it. Since defect is additive, *any* triangle must have defect zero! This theorem can be written in the contrapositive:

Theorem 7 *If there exists a triangle with positive defect, then* every *triangle has a positive defect.*

Remark that we have not stated whether or not there exists a triangle with positive defect or zero defect— the best we can do is form conditionals. However, we know from high school geometry that it can be proven that a triangle's angle sum is 180°. In this proof, however, we utilized the parallel postulate. What we will demonstrate in the next section is that the defect of a triangle (and therefore all triangles) is dependent upon the existence of the parallel postulate.

4 Non-Euclidean Geometry

In the previous section, we considered a geometry without the parallel postulate. Now we will consider a geometry under the *negation* of the parallel postulate.

Hyperbolic Axiom There exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P.

We will now show that under this axiom, there exists a triangle whose angle sum is less than 180°. Given a line l and a point P not on l, we construct one parallel in the familiar way. Draw the unique perpendicular to l through P. Let it meet l at Q. Now we draw the perpendicular to $\overrightarrow{PQ}$ at P. Call this new perpendicular m. m is parallel to l. Draw n, another parallel to l through P. Now, if we draw another line through P and intersecting l at R, we realize m must *always* be on the exterior of $\angle QPR$. n will make a larger angle with $\overrightarrow{PQ}$ than will $\overrightarrow{PR}$. Also, we can extend R far enough so that the angle R of $\triangle\ PQR$ will be arbitrarily small. These inequalities together give us $\measuredangle QRP + \measuredangle RPQ < 90°$. Thus, $\triangle\ PQR$ is a triangle with a positive defect.

We now know that with a different choice for a parallel postulate, every triangle has a positive defect, and rectangles do not exist! It can be shown that for *every* line l and *every* point P, there exist at least two distinct lines parallel to l through P. We will now prove an even more surprising theorem.

Theorem 8 *In hyperbolic geometry if two triangles are similar, then they are congruent, i.e. AAA is a valid test for congruency.*

The proof for this, in a nutshell, is to assume similar triangles exist. Then we can "fit" the smaller triangle inside the larger one such that one corresponding angle coincides. Then the two opposite sides (from each triangle) are parallel. From supplementary angles, a convex quadrilateral with angle sum = 360° is determined. This can be cut up into smaller triangles, each of which have no defect. Thus, the assumption leads to a contradiction, and similar triangles cannot exist!

Another way of stating this theorem is that given three angles of a triangle, the sides are determined uniquely. The angle measure of an equilateral triangle determines the length of the sides. This means that in hyperbolic geometry there exists an *absolute* length, as there exists an absolute angle measure (360°) in Euclidean geometry. As an example, if one observes a length to be two meters, one cannot at will let it represent two kilometers. If you think about it, the only reason we can pull these tricks in Euclidean geometry is due to our familiarity with rectilinear coordinates. Rectangles do not exist in hyperbolic geometry, so we must be very careful about our intuition.

How can we interpret the notion of area in hyperbolic geometry? First, it would be a good idea to list a few essential properties that we would like area to exhibit. First, the area of any triangle must be non-negative. Second, congruent triangles should have equivalent areas. Third, area should be additive, in that the area of a triangle should be the sum of the areas of any constituent triangles. These properties sound very similar to those of defect! In fact, we can see that the area of a triangle is proportional to its defect. The only other factor is a proportionality constant K.

$$area(\triangle\, ABC) = K \times defect(\triangle\, ABC).$$

Since the defect of a triangle must be less than 180°, the maximum area any triangle can have is *bounded*.