Isometries in $\mathbb{R}^n$

Definitions. The space $\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$ is equipped with the usual **inner product**, **metric**, and (Euclidean) **distance:** if $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are two points in $\mathbb{R}^n$, then the inner product (or dot product) is defined as $\mathbf{x} \cdot \mathbf{y} := \sum_i x_i y_i$, the length as $\|\mathbf{x}\| := \sqrt{\sum_i x_i^2}$, and the distance as $d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_i (x_i - y_i)^2}$.

A **hyperplane** in $\mathbb{R}^n$ is an $(n-1)$-dimensional plane with equation $\mathbf{v} \cdot \mathbf{x} = c$. A hyperplane is usually specified by one of its points $\mathbf{p} = (p_1, \ldots, p_n)$ and a nonzero vector $\mathbf{v} = (v_1, \ldots, v_n)$ - called a normal vector of the hyperplane - as the set of points $\mathbf{x} = (x_1, \ldots, x_n)$ satisfying $(\mathbf{x} - \mathbf{p})\mathbf{v} = 0$, that is, the set $\{(x_1, \ldots, x_n) : \sum_i (x_i - p_i) v_i = 0\}$.

If $\mathbf{P}$ and $\mathbf{Q}$ are two distinct points in $\mathbb{R}^n$, then the set $\{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{P}) = d(\mathbf{x}, \mathbf{Q})\}$ is a hyperplane, called the *perpendicular bisector* of the segment $\mathbf{PQ}$.

Given a hyperplane H and a point $\mathbf{P}$ not on H, there's a unique point $\mathbf{P}'$ such that H is the perpendicular bisector of $\mathbf{PP}'$; the map that assigns to each point $\mathbf{P}$ this corresponding $\mathbf{P}'$ if $\mathbf{P} \notin H$ and assigns $\mathbf{P}$ to itself if $\mathbf{P} \in H$ is called the **reflection** *about* H (or *on* H).

We say that $n+1$ points in $\mathbb{R}^n$ are **in general position** if there is no hyperplane which contains all of them. Thus, two points in $\mathbb{R}^1$ are in general position if they are different, three points in $\mathbb{R}^2$ are in general position if they are not collinear, and four points in $\mathbb{R}^3$ are in general position if they are not coplanar.

An **isometry** in $\mathbb{R}^n$ is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ which preserves distances, that is, for which

$$d(f(\mathbf{x}), f(\mathbf{y})) = d(\mathbf{x}, \mathbf{y}) \quad \text{for all points } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Clearly, the *composition* of isometries is an isometry; in fact, the isometries of $\mathbb{R}^n$ form a (non-Abelian) *group* with respect to composition.

Lemma. *An isometry of $\mathbb{R}^n$ is a bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$.*

(It is obvious that an isometry is one-to-one, but it is harder to see that it is also onto. The latter follows from the theorem at the bottom of the page since reflections are bijections.)

Lemma. *Let the points $\mathbf{B}_1, \ldots, \mathbf{B}_{n+1}$ be in general position in $\mathbb{R}^n$, and let f be an isometry on $\mathbb{R}^n$. Then the images $f(\mathbf{B}_1), \ldots, f(\mathbf{B}_{n+1})$ are also in general position.*

Given a function $f : \mathbb{R}^n \to \mathbb{R}^n$, we say that a point $\mathbf{x}$ is a **fixed point** for f (or "f fixes the point $\mathbf{x}$") if $f(\mathbf{x}) = \mathbf{x}$.

Theorem (Global Positioning). *Let the points $\mathbf{B}_1, \ldots, \mathbf{B}_{n+1}$ be in general position in $\mathbb{R}^n$. Then, the distances from these $\mathbf{B}_i$-s uniquely determine any point, that is, if $\mathbf{P}$ and $\mathbf{Q}$ are two points of $\mathbb{R}^n$ such that $d(\mathbf{P}, \mathbf{B}_i) = d(\mathbf{Q}, \mathbf{B}_i)$ for $i = 1, 2, \ldots, n+1$, then $\mathbf{P} = \mathbf{Q}$.*

Corollary. *In particular, if f is an isometry that fixes some $n+1$ points which are in general position, then f fixes all points of $\mathbb{R}^n$, that is, f is the identity map.*

Corollary. *Let f and g be two isometries of $\mathbb{R}^n$, and let the points $\mathbf{B}_1, \ldots, \mathbf{B}_{n+1}$ be in general position. If f and g agree on all of $\mathbf{B}_1, \ldots, \mathbf{B}_{n+1}$, then $f = g$ everywhere.*

The following important theorem can be proved by (backward) induction on the number of fixed points an isometry has.

Theorem. *Every isometry of $\mathbb{R}^n$ can be obtained as the composition of at most $n+1$ reflections.*