

The general principle is this: <<Every large system contains a subsystem that is much more organized than the original one.>>

A classical statement of this type is the Bolzano-Weierstrass theorem of calculus:

Theorem 1 (Bernard Bolzano-Karl Weierstrass). *Every bounded (real) sequence contains a convergent subsequence.*

This is really just a fancy corollary of the pigeonhole principle, so it's not a true Ramsey-type statement; the term Ramsey theory is reserved for a body of statements that are "two-sided" in nature, having thus a strong similarity to the celebrated theorem of Frank Ramsey (see below). They are usually of the following form: *If the small parts (points, or pairs of points, or triples, . . .) of a very large system is subdivided into two classes ("two-colored"), then some still fairly large part of the system has all its small parts in the same class ("monochromatic").* [Two-coloring can be replaced by coloring with a few colors.]

The following calculus theorem is thus a true Ramsey theorem:

Theorem 2. *Every real sequence contains a monotone subsequence.*

[Note that "sequence" in analysis/calculus always means an infinite sequence.] Indeed, monotone comes in two varieties: increasing and decreasing, and hence the statement is a simple corollary of Ramsey's theorem:

Theorem 3 (Frank Ramsey 1930). *Let us two-color the pairs of an infinite set V . Then V has an infinite subset with all pairs of the same color. More generally, two-colored here can be replaced by finitely colored and pairs by k -sets for any fixed k .*

[Graph theory language: Every infinite graph contains *either* an infinite clique, *or* an infinite independent set. This *either . . . or* is a typical feature of Ramsey theory.]

Here are some classical Ramsey-type theorems:

Theorem 4 (Issai Schur 1916). *If \mathbb{N} is finitely colored, then there are x, y, z of the same color with $x + y = z$.*

Theorem 5 (Bartel van der Waerden 19??). *If we finitely color the integers, then one of the color-classes will contain arbitrarily long arithmetic progressions.*

More generally:

Theorem 6 (Tibor Gallai 19??). *Let S be a finite subset of \mathbb{R}^d , and let the vertices of \mathbb{R}^d be finitely colored. Then there is a monochromatic set $S' \in \mathbb{R}^d$ homothetic to S .*

One get get from (the easier) infinite statements to (the harder) finite ones by using compactness:

Theorem 7 (Compactness). *Let \mathcal{H} be a hypergraph with all edges finite. If every finite subhypergraph of \mathcal{H} has chromatic number at most k , then so does \mathcal{H} itself.*
Dénes König's version: An infinite rooted tree in which each node has a finite degree necessarily contains an infinite path.

Here is a finite version of Theorem 2.

Theorem 8 (Pál Erdős-György Szekeres 1935). *Every real sequence of length n contains a monotone subsequence of length at least \sqrt{n} . More generally, every real sequence of length greater than pq contains either an increasing subsequence of length greater than p or a non-increasing subsequence of length greater than q .*

In the following two geometry statements the points are in the plane in general position (no three are collinear).

Lemma 9 (Eszter Klein 1935).

(a) *Any five points contain a convex quadrilateral.*

(b) *If every 4-subset of a point-set is in convex position, then the whole is.*

Theorem 10 (Pál Erdős-György Szekeres 1935). *For every k there is a finite n such that from any n points in the plane some k are in convex position. If $f(k)$ is the smallest such n , then $f(k) \leq \binom{2n-4}{n-2} + 1$. [Conjecture: $f(k) = 2^{n-2} + 1$.]*

Theorem 11 (Robert Dilworth 1950). *Every partial order of size greater than pq contains either a chain of size greater than p or an antichain of size greater than q .*

Theorem 12 (Tic-Tac-Toe: A. Hales-R. Jewett 1963). *For every k there is a d_0 such that if $d \geq d_0$ and $[k]^d$ is finitely colored, then there exists a monochromatic line.*

The next one is a trivial induction:

Theorem 13 (Pál Erdős-Leo Moser 19??). *Every tournament of order 2^{k-1} contains a transitive subtournament of order k . [Order k means: on k vertices.]*

Easy homeworks: Every tournament contains a (directed) Hamilton path. Every strongly connected tournament contains a (directed) Hamilton cycle.

Glossary:

Let A and B be subsets of \mathbb{R}^d (or of \mathbb{Z}^d or of any linear space). We say that they are **homothetic** [or B is a homothetic copy of A] if B can be obtained from A by dilation and translation, that is, if there are constants (scalars) c, t such that $c \neq 0$ and $B = cA + t := \{cx + t : x \in A\}$ [“similarity without rotation and reflection”].

A **tournament** is a directed graph (without loops) in which for each pair $\{a, b\}$ of distinct vertices exactly one of (a, b) and (b, a) is an edge [either a beats b or b beats a — no draw is possible]. A tournament is **transitive** if ‘ a beats b ’ and ‘ b beats c ’ imply ‘ a beats c ’. Thus, a transitive tournament is really a total order.

A **Hamilton path (cycle)** in a graph is a simple path (cycle) that contains all vertices. A Hamilton path (cycle) in a digraph is a simple directed path (directed cycle) that contains all vertices. [Simple means that no vertex is visited twice.] **Strongly connected** means that one can get from anywhere to anywhere with a directed path.

A **hypergraph** is a pair (V, E) (called vertices and edges) where E is a family of some non-empty subsets of V . [When $E \subset \binom{V}{2}$, that is, all edges are pairs, we end up with an ordinary (undirected) graph.] The **chromatic number** of a (hyper)graph is the least positive integer k such that the vertices can be k -colored in such a way that no edge is monochromatic.