1 Euclidean Constructible Numbers

Theorem: The set of numbers, $\mathbb{E}$, that can be constructed using Euclidean construction rules forms a field.

1.1 Introduction to Fields

Reminder: Let $*$ be an associative binary operation on a nonempty set G. We call $(G, *)$ a group if G has an identity and each element of G has an inverse.

Definition: A field is a set F with two commutative binary operations $+$ and $*$ (addition and multiplication) such that:

(i) F is a group under $+$

(ii) $F \setminus \{0\}$ is a group under $*$

(iii) distributive law: $(\forall a, b, c \in F)\ a * (b + c) = a * b + a * c$

Which are fields under the usual addition and multiplication operations?

- $\mathbb{Q}$
- $\mathbb{R}$
- $\mathbb{Z}_5$
- $\mathbb{C}$
- $\mathbb{Z}$
- $\mathbb{Z}_6$
- More generally, for which n is $\mathbb{Z}_n$ a field?

Definition: Suppose F,K are fields. Then $F \preceq K$ means F is a subfield of K, that is, $F \subseteq K$ and F is a field with the *same* $+$ and $*$ of K.

- $\mathbb{Q} \preceq \mathbb{R} \preceq \mathbb{C}$
- $\mathbb{Z}_5 \not\preceq \mathbb{Z}_7$
- $\mathbb{Z}_5 \not\preceq \mathbb{R}$

1.2 Some facts about Euclidean Geometry

- A collapsing compass is equivalent to the modern or noncollapsing compass. That is, even with the collapsing compass we can copy given distances to other locations.
- Given a line segment $\overline{AB}$ along line L, we can construct a perpendicular bisector.
- We can construct a circle with a given diameter.
- Given a line L and a point P, we can construct a line through P that is perpendicular to L. (two cases: P may or may not be on L)
- Given a line L and a point P not on L, we can construct a line through P that is parallel to L.

1.3 Constructible Numbers

Once some distances x and y are constructed, we can construct new ones from them:

- $(\forall x, y \in \mathbb{E}) x + y \in \mathbb{E}$
- $(\forall x, y \in \mathbb{E}) x - y \in \mathbb{E}$
- $(\forall x, y \in \mathbb{E}) x * y \in \mathbb{E}$
- $(\forall x, y \in \mathbb{E})$ if $y \neq 0$, then $x/y \in \mathbb{E}$

These four properties show that $\mathbb{E}$ is a field, containing all elements of $\mathbb{Q}$:

$$\mathbb{Q} \preceq \mathbb{E} \preceq \mathbb{R} \preceq \mathbb{C}$$

However, there is an extra property of $\mathbb{E}$ that shows that it is larger than $\mathbb{Q}$, namely that $\mathbb{E}$ is closed under taking square root:

- $(\forall x \in \mathbb{E})$ if $x \geq 0$, then $\sqrt{x} \in \mathbb{E}$

Thus, $\mathbb{E}$, the set of constructible numbers, is strictly larger than the set of rational numbers since $\mathbb{E}$ is also closed under taking square root (of positive elements); an example of a number which is in $\mathbb{E}$ but not in $\mathbb{Q}$ is $\sqrt{2}$.