Resilience of random graphs

Van H. Vu
Department of Mathematics
Rutgers
vanvu@math.rutgers.edu

(partially joint with J.H. Kim and B. Sudakov)

A typical result in graph theory is of the following form

A graph G (from a certain class) possesses a property P.

We would like to investigate the following general problem

How strongly does G posses P?

To study this question, we define *the resilience of G with respect to P*, which measures how much one should change G in order to destroy P.

There are two natural kinds of resilience: global and local. It is more convenient to first define these quantities with respect to monotone increasing properties.

Definition. (Global resilience) *Let P be an increasing monotone property. The global resilience of G with respect to P is the minimum number r such that one can destroy P by deleting r edges from G.*

The notion of global resilience is not new. In fact, problems about global resilience are popular in extremal graph theory. For example, Turán's theorem gives the answer to the following question.

How many edges should one delete from the complete graph K_n to make it K_k-free ?

We focus on the *local resilience* , which eventually leads to a host of intriguing new questions. To start, let us notice that one can destroy many properties by simple local changes. For instance, to destroy the hamiltonicity, it suffices to delete all edges adjacent to one vertex. This motivates the following notion.

Definition. (Local resilience) Given a monotone increasing property P. The local resilience of a graph G with respect to P is the minimum number r so that one can destroy P by deleting at each vertex of G at most r edges.

Dirac's theorem can be formulated in this language as follows.

The local resilience of K_n with respect to hamiltonicity is $\lfloor n/2 \rfloor$.

To generate new questions, one can match any interesting graph with any natural property and ask for the corresponding resilience. Here is a random example

What is the local resilience of the hypercube with respect to containing a perfect matching ?

In other words, what is the minimum degree of a subgraph of the hypercube containing an edge from every perfect matching ?

The notion of resilience is not restricted to graphs. In fact, one can define it for virtually any structure. For instance, one can consider the resilience of matrices.

What is the global resilience of the Hadamard matrix with respect to being non-singular ?

A relation to the theory of communication.

Players: A sender (Alice), a receiver (Bob) and an eavesdropper (Charles). Alice wants to send a message to Bob. On the other hand, Charles (usually an adversary) can (and want to) intercept and change the message.

Now imagine that the message has a form of a binary matrix, which is the adjacency matrix of a graph. The information Bob wants to learn from the message is hidden in a property of the graph. Can Charles destroy this information ?

The answer depends on how much power Charles possesses and what is the resilience of the given property.

The resilience of random graphs

Fix a property P and consider the set of all (labeled) graphs on the same large vertex set.

Definition Consider random graph $G(n, p)$ and a fixed property P. The local resilience of $G(n, p)$ with respect to P is the minimum number r so that almost surely there is a graph H on $[n]$ with maximum degree at most r such that the graph $G(n, p) \triangle H$ does not have P.

Remark. After one has sampled a graph from the distribution $G(n, p)$, the adversary is allowed to find the *worst* graph H with the smallest possible maximum degree in order to destroy the property P.

Perfect Matching

Let G be a graph on n vertices, where n is even. A classical theorem of Erdős and Rényi (66) shows that for any fixed $\epsilon > 0$ if $p > (1+\epsilon)\log n/n$, then almost surely $G(n,p)$ has a perfect matching.

One natural way to destroy all perfect matchings is the following. Split the vertex set of G into two parts X and Y of size $n/2+1$ and $n/2-1$ respectively. Then delete all edges inside the set X.

In $G(n,p)$ (with p sufficiently large), with high probability all vertices have degree $(1+o(1))np$. So, the local resilience of $G(n,p)$ with respect to having a perfect matching is (a.s.) at most $(1/2+o(1))np$. This is the truth:

Theorem. For even n and $p \gg \log n/n$, the local resilience of $G(n,p)$ with respect to having a perfect matching is a.s. $(1/2+o(1))np$.
(Sudakov-V. 06)

Hamiltonicity

Another classical theorem in random graph theory, by Bollobás (1883 ±) and Komlós and Szemerédi (1983) , shows that if $p > (1+\epsilon)\log n/n$, then almost surely $G(n,p)$ is hamiltonian.

We can remove all hamiltonian cycles of G by splitting the vertex set of G into two parts whose sizes differ by at most two and deleting all the edges inside the larger part.

Another construction is to split the vertex set into two parts whose sizes differ by at most one and delete all edges between them.

Therefore for sufficiently large p, we can again conclude that the local resilience of $G(n,p)$ with respect to being hamiltonian is at most $(1/2+o(1))np$. Again this is the truth

Theorem. For all $p > \log^4 n/n$, the local resilience of $G(n,p)$ with respect to being hamiltonian is $(1/2+o(1))np$. (Sudakov-V. 06)

(Open question: $p = \omega(\log n)$?)

Chromatic number

It was proved by Bollobás (88) for large p and by Łuczak (91) for small p that almost surely $\chi\big(G(n,p)\big) = (1+o(1))\frac{n}{2\log_b(np)}$, where $b = 1/(1-p)$.

The task is to determine the largest r such that for almost all samples G from $G(n,p)$, the following holds. By adding at most r edges to each vertex of G, one cannot increase the chromatic number of $G(n,p)$ by more than $o(\chi(G(n,p))$. Trivially $r \leq (1+\epsilon)\chi(G(n,p))$ for any fixed $\epsilon > 0$, since we can add a clique of that size and the chromatic number of the graph will grow by factor at least $1+\epsilon$. From below, we have the following bound.

Theorem. Let p be a fixed, small positive number. Assume that $n^{-1/3+\epsilon} \leq p \leq 3/4$. The local resilience of $G(n,p)$ with respect to being $(1+o(1))\frac{n}{2\log_b(np)}$-colorable is at least $np^2/\log^5 n$. (Sudakov-V. 06)

Note that for the uniform case when $p = 1/2$, this lower bound is off by only polylogarithmic factor.

For p below $n^{-1/3+\epsilon}$:

Theorem. For every positive integer d and for every $\epsilon > 0$ there is a constant $c = c(d, \epsilon)$ such that the following hold. For any $p > c/n$, almost surely

$$\max_{H, \Delta(H) \leq d} \chi((G(n,p) \cup H) \leq (1+\epsilon)\chi(G(n,p)).$$

(Sudakov-V. 06)

Informally, this result says that for any given d and for most graphs on n vertices with average degree at least c (which is sufficiently large compared to d), adding any graph of maximum degree d has very little impact on the chromatic number.

One may want to compare this theorem against the following deterministic result:

Theorem. Let G and H be two graphs on the same set of points. Then

$$\chi(G \cup H) \leq \chi(G)\chi(H),$$

and there are pairs of G and H such that the equality holds. (Folklore)

Another popular model of random graphs is the model of random regular graphs. Given two parameters d and n, we fix the vertex set $V = \{1, \dots, n\}$ and consider the set of all simple d-regular graphs on V, equipped with the uniform probability; $G(n, d)$.

Annoying fact. It is not know that this model is monotone.

The fact that $G(n, n^{1/4})$ is a.s. hamiltonian does not automatically imply that $G(n, n/4)$ is a.s. hamiltonian. Both facts are true, but with different proofs.

Open Question. (strong monotonicity) Is it true that if P holds for $G(n, d)$ a.s, then it holds a.s. for $G(n, d + 1)$?

True of $d = O(1)$, but for large d

Open Question. (weak monotonicity) Is it true that if P holds for $G(n, d)$ a.s, then it holds a.s. for $G(n, (1 + \epsilon)d)$? (True of $d = O(1)$).

Universality. “Nice“ properties should not depend on the model.

Kim-V. tried to establish universality through coupling:

Conjecture. For $d \ll \log n$, there exists a
$p_1 = (1 + o(1))d/n, p_2 = (1 + o(1)d/n$ and a coupling such that:
$G(n, p_1) \subset G(n, d) \subset G(n, p_2)$.

Theorem. Almost true for $\log n \ll d \leq n^{1/3}$.

$G(n, p_1) \subset G(n, d)$.

$G(n, d) \subset G(n, p_{2)}$ except a few $O(1)$ edges at each vertex. (Kim-V.02)

Corollary. If a property has local resilience $\omega(1)$, then it is monotone in the weak sense.

Erdős and Rényi (63) For $p \geq (1+\epsilon)\log n/n$, $G(n,p)$ is almost surely non-symmetric, i.e., has no non-trivial automorphisms.

What about random regular graphs ?

McKay-Wormald (1984): $G(n,d)$ is a.s non-symmetric for $3 \leq d \leq n^{1/2}$.

What about $n^{1/2} < d$?

Theorem. $G(n,d)$ is a.s non-symmetric for $\log n \ll d \leq n/2$.
(Kim-Sudakov-V. 02)

Define: $I(G) = 1$ if G is non-symmetric and 0 otherwise.

We want to show that with high probability $I(G) = 1$.

View this statement as a sharp concentration result, namely, $I(G)$ is a.s. close to its mean.

However, it is impossible to prove a sharp concentration result for a random variable having only two values close to each other.

The idea here is to "blow up" $I(G)$ using the notion of local resilience . Instead of $I(G)$ we used a function $D(G)$ which is the local resilience of G with respect to being non-symmetric. This function is zero if G is symmetric and rather large otherwise. This gives us room to show that $D(G)$ is strongly concentrated around a large positive value, and from this we can conclude that almost surely the random regular graph is non-symmetric.

Theorem. The local resilience of $G(n,d)$ is a.s. $(2+o(1))(d-\frac{d^2}{n})$.