# Codimension, Multiplicity and Integral Extensions

Aron Simis*
Departamento de Matemática
Universidade Federal de Pernambuco
50740-540 Recife, PE, Brazil
aron@dmat.ufpe.br

Bernd Ulrich†
Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA
ulrich@math.msu.edu

Wolmer V. Vasconcelos‡
Department of Mathematics
Rutgers University
Piscataway, NJ 08854-8019, USA
vasconce@math.rutgers.edu

**Abstract**

Let $A \subset B$ be a homogeneous inclusion of standard graded algebras with $A_0 = B_0$. To relate properties of $A$ and $B$ we intermediate with another algebra, the associated graded ring $G = \mathrm{gr}_{A_1 B}(B)$. We give criteria as to when the extension $A \subset B$ is integral or birational in terms of the codimension of certain modules associated to $G$. We also introduce a series of multiplicities associated to the extension $A \subset B$. There are applications to the extension of two Rees algebras of modules and to estimating the (ordinary) multiplicity of $A$ in terms of that of $B$ and of related rings. Many earlier results by several authors are recovered quickly.

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# 1 Introduction

Let $A \subset B$ be a homogeneous inclusion of standard graded Noetherian rings with $A_0 = B_0 = R$. Our primeval goal is to give a unified treatment of criteria for the integrality (resp. birationality) of the extension $A \subset B$ which would include many of the earlier results that dealt with special cases of this general setup. On one side, the guiding principle has been to profit from the intertwining between the gradings by means of a third graded algebra, namely, the associated graded ring $G = \mathrm{gr}_{A_1 B}(B)$. Thus, whether the extension $A \subset B$ is integral or birational can be expressed in terms of the codimension of certain modules over $G$. This idea has been exploited on several occasions by D. Rees, D. Kirby, S. Kleiman and A. Thorup ([10], [13], [21], [22]) and, closer in spirit to the approach followed here, by D. Katz ([8]). A first instance of such an extension is provided when two submodules $F \subset E \subset R^e$ of a free module over a Noetherian ring $R$ are given. In the symmetric algebra $S_R(R^e) \simeq R[T_1, \ldots, T_e]$ of $R^e$, the forms defined by $F$ and $E$ generate, respectively, subalgebras $\mathcal{R}(F) \subset \mathcal{R}(E) \subset S_R(R^e) = \mathcal{R}(R^e)$, called the *Rees algebras* of $F$ and $E$, respectively. Another setting is that of a homogeneous inclusion $A \subset B$ of standard graded algebras over a field, such as those that occur in morphisms of projective varieties. Stretching our considerations, we introduce an infinite (but eventually stable) series of relative multiplicities associated to the extension $A \subset B$. As it turns out, the two "extreme" multiplicities of the series have been studied before by other authors, but the idea of a whole series of such multiplicities seems to be novel. At any rate, these multiplicities are well fitted to detect birationality and integrality.

Besides the previous approach, there is a second alternative for detecting integrality, which has more of a traditional flavor. These other criteria are strongly based on the behaviour of the extension $A \subset B$ locally at primes in $A$ of low codimension. As the first approach it too leads quite readily to proofs of several previous results of Rees, Kleiman, Thorup and others.

We now describe the contents of each section in further detail.

Section 2 sets up some of our criteria for integrality. Typically they will in the above setup translate the integrality (or the integrality and birationality) of the extension $A \subset B$ into terms of the positivity of the codimension of the annihilator $0 :_G B_1 G$ or of its stable value $0 :_G (B_1 G)^\infty$. This formulation provides an algorithmic pathway to integrality since the calculation of codimensions can be approached in various manners.

In Section 3 we exploit this further if the length $\lambda_R(B_1/A_1)$ is finite, by introducing a series of relative multiplicities $e_t(A, B)$, one for each $t \geq 1$, associated to the numerical functions $\lambda_R(B_n/A_{n-t+1}B_{t-1})$. We characterize the integrality of the extension $A \subset B$ and its generic reduction number $t-1$ in terms of the vanishing of the relative multiplicity $e_t(A, B)$. We also show that $e_t(A, B)$ has a stable value $e_\infty(A, B)$. The relative multiplicity $e_1(A, B)$ has been considered before by Kirby and Rees ([10, p. 237–238]), whereas $e_\infty(A, B)$ was introduced by Kleiman and Thorup ([13, 5.7]). It turns out that the vanishing of $e_\infty(A, B)$ is related to the integrality of the extension, whereas $e_1(A, B)$ detects both integrality and birationality. We prove that the same birationality condition is responsible for the equality of the above two relative multiplicities, thus shedding extra light on their meaning in the various quoted sources.

In Section 4 we give a general criterion for integrality in a different direction, perhaps more "classical". The principal result here is to the effect that if $A$ is a locally equidimensional and universally catenary Noetherian ring, then the integrality of a ring extension $A \subset B$ is implied by the conditions that every minimal prime of $B$ contracts to a minimal prime of $A$ and that the extension $A_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ is integral for every prime $\mathfrak{p}$ of $A$ with $\dim A_{\mathfrak{p}} \leq 1$. The proof is obtained through a sequence of reduction steps and eventually makes use of the well known fact that a normal domain is the intersection of its localizations at height one primes. This sort of argument has of course been used before in varied context (cf., e.g., [8], [18], [23]). We then apply this criterion to the case of a homogenous extension $A \subset B$ to get a souped-up version of the criterion in terms of the local integrality along primes $p$ of $R = A_0 = B_0$ whose extensions $pA_p$ have height at most one. We are also able to derive a theorem by Kleiman and Thorup ([14, 1.1.2]) to the effect that if $R$ is local, equidimensional and universally catenary, $B$ is a polynomial ring and $\mathfrak{a} = \mathrm{ann}_R B_1/A_1$ has positive height, then $A/\mathfrak{a}A$ is equidimensional.

Section 5 is devoted to the case of an extension of Rees algebras of two modules. Most results will be straightforward applications of the previous sections, which we find convenient to expose as they have been established before in different ways, by several authors, while we proceed to obtaining them in a uniform fashion. We recover, in particular, results of Böger ([2]), Buchsbaum–Rim ([5]), Katz ([8]), Kirby–Rees ([10]), Kleiman–Thorup ([13]), and McAdam ([18]). A significant notion related to the material of this section is that of the Buchsbaum–Rim multiplicity of a torsionfree module (in a given embedding). We establish the connection between this numerical invariant and reductions of modules and close the section with a curious result giving an upper bound for the reduction number of a zero-dimensional ideal $I$ in a Cohen–Macaulay local ring $R$ in terms of the first coefficient in the Hilbert–Samuel polynomial of $I$.

In Section 6 we study a homogeneous embedding $A \subset B$ of standard graded algebras defined over an Artinian local ring (mostly a field). One uses the previous tools to relate the multiplicity of $B$ to the multiplicities of $A$ and of the special fiber $B/A_1B$ or its components of maximal dimension. Naturally, the relevant case is when the extension $A \subset B$ is not integral, but still $\dim A = \dim B$. The main result in this section gives a mixed upper bound for $e(A)$ – additive with respect to the various multiplicities coming from $B$, and multiplicative with respect to the rank of $B$ over $A$. Such inequalities become equalities (essentially) in the presence of the condition that $B/A_1B$ is one-dimensional. We use them to estimate the multiplicity of the special fiber ring $k \otimes_R \mathcal{R}(I)$, where $I \subset R$ is an ideal generated by forms of the same degree in a standard graded $k$-algebra $R$. This in turn leads to bounds for the reduction number of an ideal, and the degree of the image of a rational map between projective varieties. We also recover the Plücker–Teissier formula for the degree (class) of the dual variety of a hypersurface with non-deficient dual (cf. [12], [15], [20], [24]).

## 2 Codimension and integrality

In this section we develop the technical tools to study integrality of an extension $A \subset B$. of algebras. We first fix the basic set-up.

**Notation 2.1** Let $A \subset B$ be a homogeneous inclusion of standard graded Noetherian rings with $A_0 = B_0$. Write $\mathcal{R} = \mathcal{R}_B(A_1B) \subset B[T]$ and $G = \mathrm{gr}_{A_1B}(B)$ for the Rees algebra and the associated graded ring of the $B$–ideal $A_1B$, respectively. We endow $\mathcal{R}$ with a bigrading by assigning bidegree $(1,0)$ to the elements of $B_1$ and bidegree $(0,1)$ to the elements of $A_1T$. Thus $G$ becomes a bigraded ring as well and one has

$$G/B_1G \simeq \mathcal{R}/B_1\mathcal{R} \simeq \bigoplus_{i\geq 0} \mathcal{R}_{(0,i)} = \bigoplus_{i\geq 0} A_iT^i \simeq A,$$

which yields an identification $V(B_1G) = \operatorname{Spec}(A)$.

**Lemma 2.2** *We use the setting of* (2.1).

(a) $\dim A \leq \dim B = \dim G$

(b) *Suppose that $A_0 = B_0$ is local and $B$ is equidimensional and universally catenary. Then*

  (i) *$G$ is equidimensional;*

  (ii) *If $B_{\mathfrak{q}\cap A}$ is integral over $A_{\mathfrak{q}\cap A}$ for every minimal prime $\mathfrak{q}$ of $B$ then every minimal prime of $B$ contracts to a minimal prime of $A$ and, moreover, $A$ is equidimensional with* $\dim A = \dim B$.

**Proof.** (a) Since $A = G/B_1G$, one has $\dim A \leq \dim G$, and as there exists a maximal ideal of $B$ of maximal height containing $B_1B$ it is well known that $\dim G = \dim B$.

(b) To deal with (i), since $G$ is bigraded, one may replace $B$ by the localization at its homogeneous maximal ideal. But then the extended Rees algebra $\mathcal{R}[T^{-1}]$ is an equidimensional and catenary $\mathbb{Z}$-graded ring with a unique maximal homogeneous ideal. As $G \simeq \mathcal{R}[T^{-1}]/(T^{-1})$ with $T^{-1}$ a homogeneous non zerodivisor, it follows that $G$ is equidimensional.

As for (ii), notice that every minimal prime $\mathfrak{p}$ of $A$ is a contraction of a minimal prime of $B$ as can be seen by localizing at $\mathfrak{p}$. The assertion now follows from the dimension formula for positively graded Noetherian domains (see, e.g., [25, 1.2.2]). $\square$

**Theorem 2.3** *We use the setting of* (2.1). *Let $t > 0$, $s \geq 0$ be integers.*

(a) $V((B_1G + 0:_G B_tG)/B_1G) = \operatorname{Supp}_A(B/\sum_{i=0}^{t-1} B_iA)$*; thus $(B_1G + 0:_G B_tG)/B_1G$ has height greater than $s$ if and only if $B_\mathfrak{p} = \sum_{i=0}^{t-1} B_iA_\mathfrak{p}$ for every prime $\mathfrak{p}$ of $A$ with* $\dim A_\mathfrak{p} \leq s$.

(b) height $0:_G B_tG > s$ *if and only if $B$ is integral over $A$ and $B_\mathfrak{p} = \sum_{i=0}^{t-1} B_iA_\mathfrak{p}$ for every prime $\mathfrak{p}$ of $A$ with* $\dim A_\mathfrak{p} \leq s$.

**Proof.** (a) One has

$$V((B_1G + 0:_G B_tG)/B_1G) = \operatorname{Supp}_{G/B_1G}(B_tG/B_{t+1}G) = \operatorname{Supp}_A(B_tG/B_{t+1}G),$$

where the first equality follows from Nakayama Lemma and the equality $(B_1G)(B_tG) = B_{t+1}G$, while the second one uses the identification $G/B_1G \simeq A$. Looking at the graded components of $G$, one sees that there is an isomorphism of $A$–modules

$$B_tG/B_{t+1}G \simeq \sum_{i=0}^{t} B_iA/\sum_{i=0}^{t-1} B_iA,$$

which yields $\operatorname{Supp}_A(B_tG/B_{t+1}G) = \operatorname{Supp}_A(\sum_{i=0}^{t} B_iA/\sum_{i=0}^{t-1} B_iA)$. Finally, one has

$$\operatorname{Supp}_A(\sum_{i=0}^{t} B_iA/\sum_{i=0}^{t-1} B_iA) = \operatorname{Supp}_A(B/\sum_{i=0}^{t-1} B_iA)$$

since $B$ is standard graded.

(b) If height $0:_G B_tG > s \geq 0$, then for every minimal prime $\mathfrak{q}$ of $G$, $(B_tG)_\mathfrak{q} = 0$, hence $B_tG \subset \mathfrak{q}$, which implies $B_1G \subset \sqrt{0}$. On the other hand, $B_1G \subset \sqrt{0}$ if and only if $B_1 \subset \sqrt{A_1B}$, which is equivalent to the integrality of $B$ over $A$.

Thus we may assume in either case that $B_1G \subset \sqrt{0}$ and that $B$ is integral over $A$. Now the asserted equivalence follows from part (a). □

**Corollary 2.4** *We use the setting of* (2.1). *For a given integer $t > 0$,* height $0:_G B_tG > 0$ *if and only if $B$ is integral over $A$ and $B_\mathfrak{p} = \sum_{i=0}^{t-1} B_iA_\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ of $A$.*

Notice that, in the terminology of reductions, the equality $B_\mathfrak{p} = \sum_{i=0}^{t-1} B_iA_\mathfrak{p}$ over all minimal primes $\mathfrak{p}$ of $A$ means that the generic reduction number of $A \subset B$ is at most $t-1$.

**Corollary 2.5** *We use the setting of* (2.1).

(a) height $0:_G B_1G > 0$ *if and only if $B$ is integral over $A$ and $B_\mathfrak{p} = A_\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ of $A$.*

(b) height $0:_G (B_1G)^\infty > 0$ *if and only if $B$ is integral over $A$.*

**Proof.** Apply Corollary 2.4 noticing that $0:_G (B_1G)^\infty = 0:_G B_tG$ for $t \gg 0$. □

**Remark 2.6** Corollary 2.5(b) can also be obtained in a more direct way. Indeed, to say that $0:_G (B_1G)^\infty$ has positive height means that $B_1G \subset \sqrt{0}$, which in turn is equivalent to the integrality of $B$ over $A$.

Combining Lemma 2.2(b.i) and Corollary 2.4, we have:

**Corollary 2.7** *In addition to the assumptions of* (2.1) *suppose that $A_0 = B_0$ is local and $B$ is equidimensional and universally catenary. For a given integer $t > 0$, the following are equivalent:*

(i) $\dim B_tG < \dim G$;

(ii) height $0:_G B_tG > 0$;

(iii) *$B$ is integral over $A$ and $B_\mathfrak{p} = \sum_{i=0}^{t-1} B_iA_\mathfrak{p}$ for every minimal prime $\mathfrak{p}$ of $A$.*

## 3 Rees criteria

We collect here several criteria based on multiplicities, similar to the ones in [10] and [13], and derive them via application of the above results.

Let $S$ be a Noetherian ring and $M$ a finitely generated $S$–module. If $(S, \mathfrak{m})$ is local and $\mathfrak{a}$ is an $\mathfrak{m}$–primary ideal, one writes $e_{\mathfrak{a}}(M)$ for the multiplicity of $M$ with respect to $\mathfrak{a}$ and $e(M)$ for $e_{\mathfrak{m}}(M)$. If on the other hand $S$ is standard graded with $S_0$ Artinian local and $M$ is graded, $e(M)$ denotes the multiplicity of $M$. In this case $e(M) = e(M_{\mathfrak{m}})$, where $\mathfrak{m}$ is the homogeneous maximal ideal of $S$. We refer to [4, Chapter 4] for basic properties of multiplicities.

We will in this section stick to the following setup.

**Notation 3.1** Let $A \subset B$ be a homogeneous inclusion of standard graded Noetherian rings with $R = A_0 = B_0$ local and $\lambda(B_1/A_1) < \infty$, where $\lambda(\cdot) = \lambda_R(\cdot)$ denotes length. Write $d = \dim B$ and $\mathfrak{m}$ for the maximal ideal of $R$. The bigraded ring $G = \mathrm{gr}_{A_1 B}(B)$ introduced in (2.1) admits also a standard grading as an $R$–algebra given by total degree. If $t > 0$ is an integer and $\mathfrak{a} = \mathrm{ann}_R\, B_1/A_1$, then $B_t G$ is a finitely generated graded module over $G/0{:}_G\, B_1 G$, the latter being a standard graded ring with $[G/0{:}_G\, B_1 G]_0 = R/\mathfrak{a}$ Artinian local.

In the next proposition we introduce a family of "relative" multiplicities $e_t(A, B)$.

**Proposition 3.2** *With the assumptions of* (3.1) *one has:*

(a) *For every $n \geq t-1$, $\lambda(B_n/A_{n-t+1}B_{t-1}) = \lambda([B_t G]_n)$.*

(b) *For $n \gg 0$, $\lambda(B_n/A_{n-t+1}B_{t-1})$ is a polynomial function $f_t(n)$ of degree*

$$\dim B_t G - 1 = \dim G/0{:}_G\, B_t G - 1 \leq \dim G - 1 = d - 1.$$

(c) *The polynomial $f_t(n)$ is of the form*

$$f_t(n) = \frac{e_t(A, B)}{(d-1)!} n^{d-1} + \textit{lower terms},$$

*where $e_t(A, B) = 0$ (if $\dim G/0{:}_G\, B_t G < d$) or $e_t(A, B) = e(B_t G)$ (if $\dim G/0{:}_G\, B_t G = d$).*

**Proof.** Notice that

$$[B_t G]_n = \bigoplus_{j=1}^{n-t+1} A_{j-1}B_{n-j+1}/A_j B_{n-j},$$

which gives (a). The remaining assertions follow from (a) because $B_t G$ is a finitely generated graded module over a ring that is standard graded over an Artinian local ring. $\square$

We easily obtain the following general characterization.

**Theorem 3.3** *In the setting of* (3.1) *consider the conditions*

(i) *$B$ is integral over $A$ and $B_{\mathfrak{p}} = \sum_{i=0}^{t-1} B_i A_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ of $A$;*

(ii) *$B$ is integral over $A$ and $B_{\mathfrak{p}} = \sum_{i=0}^{t-1} B_i A_{\mathfrak{p}}$ for every prime $\mathfrak{p} \in V(\mathfrak{m}A)$ with $\dim A/\mathfrak{p} = d$;*

(iii) *$e_t(A,B) = 0$, i.e., $f_t(n)$ has degree $< d-1$.*

*The implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *hold in general and all three conditions are equivalent if $B$ is equidimensional and universally catenary.*

**Proof.** The implication (i) $\Rightarrow$ (ii) is clear since $\dim A \le d$ by Lemma 2.2(a). If $B$ is integral over $A$ then $B_1 G \subset \sqrt{0}$. Thus, by Theorem 2.3(a), (ii) implies that $\dim G/0{:_G} B_t G < d$, which is equivalent to (iii) according to Proposition 3.2(c). Finally, if $B$ is equidimensional and universally catenary then the inequality $\dim G/0{:_G} B_t G < d$ implies (i), as shown in Corollary 2.7. □

The sequence of multiplicities $e_t(A,B)$ is obviously non-increasing. The next proposition shows that it stabilizes eventually.

**Proposition 3.4** *We use the setting of* (3.1). *If $\dim G/0{:_G} (B_1G)^\infty = d$ and $0{:_G} B_t G = 0{:_G} (B_1 G)^\infty$ then $e_t(A,B) = e(G/0{:_G} (B_1G)^\infty)$. In particular, $e_t(A,B)$ is constant for $t \gg 0$.*

**Proof.** In the light of Proposition 3.2(c) we only need to show that $e(B_tG) = e(G/0{:_G} B_tG)$. By the associativity formula for multiplicities it suffices to check this locally at every minimal prime $\mathfrak{q}$ of the $G$-ideal $0{:_G} B_tG$. However, $B_tG \not\subset \mathfrak{q}$ since $0{:_G} B_tG = 0{:_G} (B_1G)^\infty$ by assumption. Therefore, $(B_tG)_{\mathfrak{q}} \simeq (G/0{:_G} B_tG)_{\mathfrak{q}}$. □

In the setting of Proposition 3.2 we will write $e(A,B) = e_1(A,B)$ and $e_\infty(A,B) = e_t(A,B)$ for $t \gg 0$. Notice that the polynomial function defining $e(A,B)$ simplifies to

$$\lambda(B_n/A_n) = \frac{e(A,B)}{(d-1)!} n^{d-1} + \text{lower terms},\ n \gg 0.$$

Part (b) of the next result, which is a special case of our Theorem 3.3, was also proved by Kleiman and Thorup (cf. [13, 6.3]).

**Corollary 3.5** *With the assumptions of* (3.1) *one has:*

(a) *If $B$ is integral over $A$ and $B_{\mathfrak{p}} = A_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ of $A$ then $e(A,B) = 0$. The converse holds in case $B$ is equidimensional and universally catenary.*

(b) *If $B$ is integral over $A$ then $e_\infty(A,B) = 0$. The converse holds in case $B$ is equidimensional and universally catenary.*

**Remark 3.6** In the last corollary the "birationality" assumption to the effect that $B_{\mathfrak{p}} = A_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ of $A$ could be weakened again by looking only at those primes $\mathfrak{p} \in V(\mathfrak{m}A)$ with $\dim A/\mathfrak{p} = d$. It may be of interest to point out that neither condition is used in either [10, 4.5 and 4.11] or [13, 5.10] where rather the stronger assumption that $\dim A/\mathfrak{m}A < d$ is imposed to prove that $e(A,B) = 0$ if $B$ is integral over $A$.

The next result "explains" the different behavior of $e(A,B)$ and $e_\infty(A,B)$ in Corollary 3.5, while also giving estimates for $e(A,B)$ in terms of the multiplicity of a single ring.

**Proposition 3.7** *With the assumptions of* (3.1) *one has:*

(a) $e(A,B) = e_\infty(A,B)$ *if and only if* $B_\mathfrak{p} = A_\mathfrak{p}$ *for every prime* $\mathfrak{p} \in V(\mathfrak{m}A)$ *with* $\dim A/\mathfrak{p} = d$.

(b) *If* $e(A,B) \neq 0$ *and if* $B_\mathfrak{p} = A_\mathfrak{p}$ *for every prime* $\mathfrak{p} \in V(\mathfrak{m}A)$ *with* $\dim A/\mathfrak{p} = d$ *(e.g., for every minimal prime* $\mathfrak{p}$ *of* $A$*) then* $e(A,B) = e(G/0:_G B_1G)$.

(c) *If* $e(A,B) \neq 0$ *and if for every prime* $\mathfrak{p} \in V(\mathfrak{m}A)$ *with* $\dim A/\mathfrak{p} = d$ *either* $B_\mathfrak{p} = A_\mathfrak{p}$ *or* $A_\mathfrak{p}$ *is reduced, then* $e(A,B) \geq e(G/0:_G B_1G)$.

**Proof.** By Proposition 3.2(c) and the associativity formula for multiplicities, $e(A,B) = e_\infty(A,B)$ if and only if for every prime $\mathfrak{q} \in V(0:_G B_1G)$ with $\dim G/\mathfrak{q} = d$ one has $(B_1G)_\mathfrak{q} = (B_tG)_\mathfrak{q}$ for $t \gg 0$, or equivalently, $(B_1G)_\mathfrak{q} = G_\mathfrak{q}$. But this means that $\dim G/(B_1G + 0:_G B_1G) < d$. By Theorem 2.3(a), the latter inequality holds if and only if $B_\mathfrak{p} = A_\mathfrak{p}$ for every prime $\mathfrak{p}$ of $A$ with $\dim A/\mathfrak{p} = d$. Here it suffices to consider primes $\mathfrak{p} \in V(\mathfrak{m}A)$ because $\lambda(B_1/A_1) < \infty$.

(b) By part (a), Proposition 3.2(c) and Proposition 3.4, $e(A,B) = e(G/0:_G (B_1G)^\infty) \leq e(G/0:_G B_1G)$. Thus, the assertion will follow from part (c).

(c) Again by Proposition 3.2(c) and the associativity formula for multiplicities, it remains to prove that $\lambda((B_1G)_\mathfrak{q}) \geq \lambda((G/0:_G B_1G)_\mathfrak{q})$ for every prime $\mathfrak{q} \in V(0:_G B_1G)$ with $\dim G/\mathfrak{q} = d$. We may clearly assume that $B_1G \subset \mathfrak{q}$ as otherwise the assertion is obvious. By the identification $G/B_1G = A$ there exists a prime $\mathfrak{p}$ of $A$ with $\mathfrak{q}/B_1G = \mathfrak{p}$. Notice that $\dim A/\mathfrak{p} = d$. Furthermore by Theorem 2.3(a), $A_\mathfrak{p} \neq B_\mathfrak{p}$. In particular $\mathfrak{m}A \subset \mathfrak{p}$ since $\lambda(B_1/A_1) < \infty$. Now our assumption forces $A_\mathfrak{p}$ to be reduced, hence $(G/B_1G)_\mathfrak{q} = A_\mathfrak{p}$ is a field, which shows that $(B_1G)_\mathfrak{q}$ is the maximal ideal of the Artinian local ring $G_\mathfrak{q}$. Thus obviously, $\lambda((B_1G)_\mathfrak{q}) = \lambda(G_\mathfrak{q}) - 1 \geq \lambda((G/0:_G B_1G)_\mathfrak{q})$. □

# 4 Integrality in codimension one

In this section we give a general criterion for integrality in terms of local behaviour at primes of "small" height.

**Theorem 4.1** *Let* $A \subset B$ *be an extension of rings with* $A$ *Noetherian, locally equidimensional and universally catenary. Assume that:*

- *The extension* $A_\mathfrak{p} \subset B_\mathfrak{p}$ *is integral for every prime* $\mathfrak{p}$ *of* $A$ *with* $\dim A_\mathfrak{p} \leq 1$;
- *Every minimal prime of* $B$ *contracts to a minimal prime of* $A$.

*Then the extension is integral.*

**Proof.** We effect various reduction steps in order to assume that $A$ is a complete local ring and that $B$ is a Noetherian domain.

1. We may assume that $A$ is local: This is easy since if $C$ is the integral closure of $A$ in $B$ then $C_\mathfrak{m}$ is the integral closure of $A_\mathfrak{m}$ in $B_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of $A$.

2. We may assume that $A$ is complete: In other words, we wish to replace $A \subset B$ by $\widehat{A} \subset \widehat{A} \otimes_A B$, where $\widehat{A}$ denotes the completion of $A$ with respect to its maximal ideal.

By faithful flatness, integrality will descend from the latter extension to the former. We now check that our assumptions are preserved while passing to the new extension. The properties of being equidimensional and universally catenary are well known to be preserved by completion (cf. [17, 31.7]). To see that the contraction hypothesis is preserved as well, let $\mathfrak{q}$ be a minimal prime of $\widehat{A} \otimes_A B$. Since the map from $B$ to $\widehat{A} \otimes_A B$ satisfies the going-down property (because of flatness), $\mathfrak{q} \cap B$ is a minimal prime of $B$. Therefore, by one of the standing assumptions, $\mathfrak{q} \cap A$ is a minimal prime of $A$. Since $A_{\mathfrak{q}\cap A}$ is an Artinian local ring, the dimension of $\widehat{A}_{\mathfrak{q}\cap \widehat{A}}$ is not changed if $A_{\mathfrak{q}\cap A}$ is replaced by its residue field. But then the extension $\widehat{A}_{\mathfrak{q}\cap \widehat{A}} \subset \widehat{A}_{\mathfrak{q}\cap \widehat{A}} \otimes_{A_{\mathfrak{q}\cap A}} B_{\mathfrak{q}\cap A}$ is flat and it follows that the map from $\widehat{A}_{\mathfrak{q}\cap \widehat{A}}$ to $(\widehat{A} \otimes_A B)_{\mathfrak{q}}$ is flat as well. Therefore, $\dim \widehat{A}_{\mathfrak{q}\cap \widehat{A}} = 0$, by going down, thus showing that $\mathfrak{q} \cap \widehat{A}$ is a minimal prime of $\widehat{A}$.

Finally, let $\mathfrak{p}$ be a prime of $\widehat{A}$ of height at most one. By going down, $\dim A_{\mathfrak{p}\cap A} \leq 1$, hence the extension $A_{\mathfrak{p}\cap A} \subset B_{\mathfrak{p}\cap A}$ is integral and, by base change, so is the extension $\widehat{A}_{\mathfrak{p}\cap A} = \widehat{A} \otimes_A A_{\mathfrak{p}\cap A} \subset \widehat{A} \otimes_A B_{\mathfrak{p}\cap A} = (\widehat{A} \otimes_A B)_{\mathfrak{p}\cap A}$. From this, by localizing at $\mathfrak{p}$, it follows that the extension $\widehat{A}_{\mathfrak{p}} \subset (\widehat{A} \otimes_A B)_{\mathfrak{p}}$ is also integral.

3. We may assume that $B$ is reduced: This is again easy as integrality lifts from $A/\sqrt{0} \subset B/\sqrt{0}$ to $A \subset B$.

4. We may assume that $B$ is a Noetherian ring: We may replace $B$ by any intermediate Noetherian subring $A \subset C \subset B$ (e.g., a finitely generated $A$-algebra contained in $B$). The contraction property is preserved in the new extension $A \subset C$ because every minimal prime of $C$ is contracted from a minimal prime of $B$.

5. We may assume that $B$ is a domain: Let $\mathfrak{q}_1, \dots \mathfrak{q}_n$ be the minimal primes of the reduced Noetherian ring $B$. Set $A_i = A/\mathfrak{q}_i \cap A \subset B_i = B/\mathfrak{q}_i$. Since $B \subset B_1 \times \cdots \times B_n$ in a natural way, the integrality of $A \subset B$ is implied by that of all the extensions $A_i \subset B_i$. Also, for every $i$ and every prime $\tilde{\mathfrak{p}}$ of $A_i$ of height at most one, let $\mathfrak{p}$ be the preimage of $\tilde{\mathfrak{p}}$ in $A$. Since $\mathfrak{q}_i \cap A$ is a minimal prime of $A$ and $A$ is an equidimensional catenary local ring, it follows that $\dim A_{\mathfrak{p}} \leq 1$. Then by assumption, $A_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ is integral, hence so is the extension $(A_i)_{\tilde{\mathfrak{p}}} \subset (B_i)_{\tilde{\mathfrak{p}}}$.

This concludes the sequence of reduction steps, so we now assume that $A$ is a complete local ring and that $B$ is a Noetherian domain. We may cleraly go one slight step further and assume that $A$ is not a field and that $B$ is a finitely generated $A$-algebra. Let $K$ and $L$ denote the fields of fractions of $A$ and $B$, respectively, let $C$ denote the integral closure of $A$ in $L$ and let $\mathfrak{Q}$ denote the set of all height one primes of $C$. Since the field extension $K \subset L$ is finite and $A$ is a complete local ring, $C$ is a finite $A$-module ([16, p. 234]). Since $A$ is universally catenary, the dimension formula for finitely generated ring extensions then shows that every maximal ideal of $C$ has height $d = \dim A$ ([17, 15.6]). In addition, $C$ is a catenary domain. Therefore, if $\mathfrak{q} \in \mathfrak{Q}$ then $\dim C/\mathfrak{q} = d - 1$, hence $\dim A/\mathfrak{q} \cap A = d - 1$, which gives $\dim A_{\mathfrak{q}\cap A} = 1$. Thus, by our assumption, $B_{\mathfrak{q}\cap A} \subset C_{\mathfrak{q}\cap A}$ and hence $B \subset C_{\mathfrak{q}}$. Finally, $C = \bigcap_{\mathfrak{q}\in\mathfrak{Q}} C_{\mathfrak{q}}$ because $C$ is a normal domain. Therefore $B \subset C$, as required. □

**Corollary 4.2** *In addition to the assumptions of* (2.1)*, suppose that* $R = A_0 = B_0$ *is local with maximal ideal* $\mathfrak{m}$ *and* $B$ *is equidimensional and universally catenary. Then:*

(a) $A \subset B$ *is an integral extension if and only if every minimal prime of* $B$ *contracts to a minimal prime of* $A$ *and the extension* $A_p \subset B_p$ *is integral for every prime* $p$ *of* $R$ *such that* height $pA_p \leq 1$.

(b) *Assume that* height $\mathfrak{m}B > 0$ *and that* $A \subset B$ *is an integral extension locally on the punctured spectrum of* $R$*. If* $A \subset B$ *is not integral then* $\dim A/\mathfrak{m}A = \dim B - 1$ .

**Proof.** (a) To prove that $A \subset B$ is an integral extension in the presence of the remaining conditions, notice that $A$ is equidimensional by Lemma 2.2(b.ii). Now the assertion follows from Theorem 4.1. Conversely, it is clear that if $A \subset B$ is an integral extension, with $B$ equidimensional, then minimal primes of $B$ contract to minimal primes of $A$.

(b) By Lemma 2.2(b.ii), every minimal prime of $B$ contracts to a minimal prime of $A$ and $A$ is equidimensional with $\dim A = \dim B$. Now part (a) implies that height $\mathfrak{m}A = 1$, hence $\dim A/\mathfrak{m}A = \dim B - 1$. □

Part (b) of the preceding corollary was suggested to us by S. Kleiman because of its similarity to an earlier result of his with A. Thorup ([14, 1.1.1]).

Once more, using Theorem 4.1, we can give a short proof of another result by these authors that has to do with the equidimensionality of a suitable fiber ring ([14, 1.1.2]). For this we need first the following observation.

**Lemma 4.3** *In addition to the assumptions of* (2.1)*, suppose that the ideal* $B_1B$ *contains a regular element and let* $I$ *be an ideal of* $A_0 = B_0$ *such that* $A_1 \subset IB_1$*. If* $\mathfrak{p}$ *is a prime ideal of* $A$ *containing* $IA$ *then the extension* $A_\mathfrak{p} \subset B_\mathfrak{p}$ *is not integral.*

**Proof.** Suppose to the contrary. Then for fixed $r >> 0$ there exists an element $a \in A \setminus \mathfrak{p}$ such that $aB_r \subset \sum_{i=0}^{r-1} AB_i$. Notice that $a \notin \sqrt{IA}$. Now

$$aB_r \subset AA_1B_{r-1} \subset AIB_1B_{r-1} = IAB_r.$$

Thus, $a\,(B_rA) \subset (IA)(B_rA)$ and, since $B_rA$ is a finite faithful $A$-module, $a$ is integral over the ideal $IA$. Hence $a \in \sqrt{IA}$; a contradiction. □

**Theorem 4.4** ([14]) *In addition to the assumptions of* (2.1)*, suppose that* $R = A_0 = B_0$ *is local, equidimensional and universally catenary. Set* $\mathfrak{a} = \operatorname{ann}_R B_1/A_1$ *and assume that* height $\mathfrak{a} > 0$*. If* $B$ *is a polynomial ring over* $R$ *then every minimal prime of* $\mathfrak{a}A$ *has height one.*

**Proof.** Arguing by way of contradiction, let $\mathfrak{p}$ be a minimal prime of $\mathfrak{a}A$ such that $\dim A_\mathfrak{p} > 1$. After localizing $R$ at the contraction of $\mathfrak{p}$ we may assume that $\mathfrak{p} \cap R = \mathfrak{m}$, the maximal ideal of $R$. Since $B_1$ is a free $R$-module by assumption, one has a decomposition of $R$-modules $A_1 = F \oplus E$, $B_1 = F \oplus G$ with $E \subset \mathfrak{m}G$. Set $A' = R[E] \subset B' = S_R(G)$. Now, $\operatorname{ann}_R B'_1/A'_1 = \operatorname{ann}_R G/E = \mathfrak{a} \neq R$, $B'$ is a polynomial ring over $R$ with $B'_1 \neq 0$ and $A$ is a polynomial ring over $A'$. Then $\mathfrak{p} \cap A'$ is a minimal prime of the ideal $\mathfrak{a}A'$ with $\dim A'_{\mathfrak{p}\cap A'} > 1$. Thus, replacing $A \subset B$ by $A' \subset B'$, we may assume that $A_1 \subset \mathfrak{m}B_1$, $B_1 \neq 0$ at the outset. Applying Lemma 4.3 with $I = \mathfrak{m}$ we conclude that $A_\mathfrak{p} \subset B_\mathfrak{p}$ cannot be integral.

On the other hand, since height $\mathfrak{a}B > 0$, Lemma 2.2(b.ii) shows that minimal primes of $B$ contract to minimal primes of $A$ and $A$ is equidimensional. Furthermore, $A_\mathfrak{p} \subset B_\mathfrak{p}$ is integral locally on the punctured spectrum of $A_\mathfrak{p}$ and $\dim A_\mathfrak{p} > 1$. But then Theorem 4.1 implies that $A_\mathfrak{p} \subset B_\mathfrak{p}$ is integral; a contradiction. □

## 5 Rees algebras of modules

We now deduce various consequences for Rees algebras of modules. Most results will be straightforward applications of the previous sections, which we assemble for convenience.

Recall first a few basic notions. Let $R$ be a Noetherian ring and $E$ an $R$–module with a fixed embedding $E \subset R^e$. The *Rees algebra* of $E$, denoted by $\mathcal{R}(E)$, is the image of the induced map of symmetric algebras $S_R(E) \longrightarrow S_R(R^e)$, where $S_R(R^e) \simeq R[T_1, \ldots, T_e]$ is a polynomial ring. As in the case of ideals one sees that the minimal primes of $\mathcal{R}(E)$ are exactly the contractions of $pR[T_1, \ldots, T_e]$, $p$ a minimal prime of $R$. Thus $\dim \mathcal{R}(E) \leq \dim R + e$ (see, e.g., [25, 1.2.2]). If height ann $R^e/E > 0$ then $\dim \mathcal{R}(E) = \dim R + e$, and if in addition $R$ is equidimensional then so is $\mathcal{R}(E)$ (loc.cit.). Notice that any finitely generated torsionfree $R$–module $E$ having rank $e$ admits an embedding $E \subset R^e$ with height ann $R^e/E > 0$.

The *analytic spread* of $E$, denoted by $\ell(E)$, is the Krull dimension of $k \otimes_R \mathcal{R}(E)$ in case $R$ is a local ring with residue field $k$. Now consider a pair of modules $F \subset E \subset R^e$, giving rise to inclusions $FE^n \subset E^{n+1} := \mathcal{R}(E)_{n+1}$. One says that $F$ is a *reduction* of $E$ if $E^{n+1} = FE^n$ for $n \gg 0$ or, equivalently, if $\mathcal{R}(E)$ is integral over $\mathcal{R}(F)$. The minimal $n$ such that $E^{n+1} = FE^n$ is called the *reduction number* of $E$ relative to $F$ and is denoted by $r_F(E)$. If $R$ is a local ring with infinite residue field then the smallest number of generators of $F$, among all reductions $F$ of $E$, equals $\ell(E)$.

We specialize the setup of the previous sections to the homogeneous inclusion of $R$–algebras $A = \mathcal{R}(F) \subset B = \mathcal{R}(E)$. Notice that in this case the Rees algebra $\mathcal{R} = \mathcal{R}_B(A_1 B) = \mathcal{R}_B(FB)$ is a homomorphic image of $\mathcal{R}(E \oplus F)$ and coincides with it when $F$ and $E$ are ideals. As before we also consider the associated graded ring $G = \mathrm{gr}_{FB}(B)$. Finally, observe that if height ann $E/F > 0$ then $B_{\mathfrak{p}} = A_{\mathfrak{p}}$ for every minimal prime $\mathfrak{p}$ of $A$.

**Theorem 5.1** ([8], [9], [10], [13]) *Let $R$ be a Noetherian local ring of dimension $d$, let $F \subset E \subset R^e$ be $R$–modules, and assume that $\lambda(E/F) < \infty$.*

(a) *For $n \gg 0$, $\lambda(E^n/F^n)$ is a polynomial function $f(n)$ of degree at most $d + e - 1$.*

(b) *If* height ann $R^e/E > 0$ *then*

$$f(n) = \frac{a}{(d+e-1)!} n^{d+e-1} + \textit{lower terms},$$

*where $a = e(\mathcal{R}(F), \mathcal{R}(E))$. If* height ann $R^e/F > 0$ *and $a \neq 0$, then $a = e(G/0:_G FG)$.*

(c) *If $F$ is a reduction of $E$ then the degree of $f(n)$ is $< d+e-1$, and the converse holds in case $R$ is equidimensional and universally catenary and* height ann $R^e/E > 0$.

**Proof.** The assertions follow immediately from Propositions 3.2 and 3.7(b) and Corollary 3.5(a). It only remains to check that the degree of $f(n)$ is $< e-1$ if $(R, \mathfrak{m})$ is Artinian and $F$ is a reduction of $E$. But then either $E = R^e$ and hence $F = E$ because a free module does not admit a proper reduction, or else we may assume that $E \subset R^{e-1} \oplus \mathfrak{m} \subset R^e$ and thus $f(n)$ has degree $< e - 1$. $\square$

**Corollary 5.2** ([1], [22]) *Let $R$ be an equidimensional universally catenary Noetherian local ring of dimension $d$ and let $J \subset I$ be ideals such that $\lambda(I/J) < \infty$. Then for $n \gg 0$, $\lambda(I^n/J^n)$ is a polynomial function $f(n)$ of degree at most $d$. Furthermore, $J$ is a reduction of $I$ if and only if the degree of $f(n)$ is $< d$.*

**Proof.** We replace the local ring $(R, \mathfrak{m})$ by $\widetilde{R} = R[X]_{(\mathfrak{m},X)}$ where $X$ is a variable, and the $R$–ideals $J \subset I$ by the $\widetilde{R}$–ideals $\widetilde{J} = (J, X) \subset \widetilde{I} = (I, X)$. This does not change our assumptions and conclusions because

$$\lambda(I^n/J^n) = \lambda(\widetilde{I}^n/\widetilde{J}^n) - \lambda(\widetilde{I}^{n-1}/\widetilde{J}^{n-1})$$

and $J$ is a reduction of $I$ if and only if $\widetilde{J}$ is a reduction of $\widetilde{I}$. Thus we may assume that height $I > 0$, and then the result follows from Theorem 5.1. □

Recall that the Buchsbaum–Rim multiplicity ([5, pp. 213-214]) arises in the context of an embedding $F \subset E = R^e$, where $\lambda(R^e/F) < \infty$.

**Corollary 5.3** ([5]) *Let $R$ be a Noetherian local ring of dimension $d$ and let $F \subsetneq R^e$ be $R$–modules with $\lambda(R^e/F) < \infty$. Then for $n \gg 0$, $\lambda(S_n(R^e)/F^n)$ is a polynomial in $n$ of degree $d + e - 1$.*

**Proof.** By Theorem 5.1(a), for $n \gg 0$, $\lambda(S_n(R^e)/F^n)$ is a polynomial of degree at most $d+e-1$. To show that the degree cannot be less, we may complete $R$ and factor out a minimal prime of maximal dimension to assume that $R$ is a complete domain. Clearly, this does not perturb the hypothesis that $F \neq R^e$. Now the assertion follows from Theorem 5.1(c) since the free module $R^e$ does not admit a proper reduction. □

**Remark 5.4** The positive integer $a$ that occurs in the polynomial

$$\lambda(S_n(R^e)/F^n) = \frac{a}{(d+e-1)!} n^{d+e-1} + \text{lower terms}, \qquad n \gg 0$$

is the Buchsbaum–Rim multiplicity $\mathrm{br}(F)$ of the embedding $F \subset R^e$. This number is also expressed as an Euler characteristic of the Buchsbaum–Rim complex ([5, 4.3]). If $d > 0$ then $\mathrm{br}(F) = e(G/0:_G FG)$ by Theorem 5.1(b).

**Corollary 5.5** *Let $R$ be an equidimensional universally catenary Noetherian local ring of dimension $d > 0$ and let $F \subset E \subsetneq R^e$ be $R$–modules with $\lambda(R^e/F) < \infty$. Then $F$ is a reduction of $E$ if and only if $\mathrm{br}(F) = \mathrm{br}(E)$.*

**Proof.** By Remark 5.4, $\mathrm{br}(F) = \mathrm{br}(E)$ if and only if for $n \gg 0$,

$$\lambda(E^n/F^n) = \lambda(S_n(R^e)/F^n) - \lambda(S_n(R^e)/E^n)$$

is a polynomial of degree $< d + e - 1$. According to Theorem 5.1(c), the latter condition is equivalent to $F$ being a reduction of $E$. □

The special case $e = 1$ is a classical result of Rees ([21, 3.2]) and the first reduction criterion based on multiplicities: Let $(R, \mathfrak{m})$ be an equidimensional universally catenary

Noetherian local ring and let $J \subset I$ be $\mathfrak{m}$–primary $R$–ideals; then $J$ is a reduction of $I$ if and only if $e(J) = e(I)$.

The next reduction criterion has been proved by McAdam for ideals ([18, 4.1]) and later by Rees for the case of modules ([23, 2.5]). We obtain it as a direct consequence of Corollary 4.2.

**Proposition 5.6** ([8], [13], [23]) *Let $R$ be an equidimensional universally catenary Noetherian local ring, let $F \subset E \subset R^e$ be $R$–modules with* height ann $R^e/F > 0$, *and write $\ell = \ell(F)$. Then $F$ is a reduction of $E$ if and only if $F_p$ is a reduction of $E_p$ for every prime $p$ of $R$ with* ann $E/F \subset p$ *and* $\dim R_p = \ell(F_p) - e + 1 \leq \ell - e + 1$.

The next result, which has also been shown by Kleiman and Thorup ([13, 10.9]), is a generalization of Corollary 5.5. It follows immediately from Proposition 5.6 and Theorem 5.1(c).

**Corollary 5.7** ([8], [13]) *Let $R$ be an equidimensional universally catenary Noetherian local ring and let $F \subset E \subset R^e$ be $R$–modules with* height ann $R^e/F > 0$. *Set $\ell = \ell(F)$ and assume that* height ann $E/F \geq \ell - e + 1$. *Then $F$ is a reduction of $E$ if and only if the degree of the polynomial $\lambda((E^n/F^n)_p)$ $(n \gg 0)$ is $< \ell$ for every minimal prime $p$ of* ann $E/F$ *with* $\dim R_p = \ell - e + 1 = \ell(F_p) - e + 1$.

This result has been proved earlier by Böger in the case of equimultiple ideals ([2, Satz 1]), where it takes a slightly different form. For modules, it has also been considered by Katz who introduced the notion of equimultiple module ([8, 2.5]). Recall that if $R$ is a Noetherian local ring and $F \subset R^e$ is a submodule of a free $R$-module, we say that $F$ is equimultiple if height $\operatorname{ann} R^e/F \geq \ell - e + 1$, where $\ell = \ell(F)$.

**Proposition 5.8** ([8]) *Let $R$ be an equidimensional universally catenary Noetherian local ring and let $F \subset E \subset R^e$ be $R$–modules with $F \subset R^e$ equimultiple and* height ann $R^e/F > 0$. *Then $F$ is a reduction of $E$ if and only if $\sqrt{\text{ann } R^e/F} = \sqrt{\text{ann } R^e/E}$ and $\operatorname{br}(F_p) = \operatorname{br}(E_p)$ for every minimal prime $p$ of the latter ideal.*

**Proof.** If $F$ is a reduction of $E$ then $\sqrt{\text{ann } R^e/F} = \sqrt{\text{ann } R^e/E}$ because a free module does not admit any proper reduction. Furthermore since $F$ is equimultiple, the prime ideals $p$ with ann $E/F \subset p$ and $\dim R_p \leq \ell - e + 1$ are necessarily minimal primes of ann $R^e/F$. Now the asserted equivalence follows from Proposition 5.6 and Corollary 5.5. □

If $G$ is the associated graded ring of a proper ideal in an equidimensional universally catenary Noetherian local ring of dimension $d$ then $G$ is again equidimensional of dimension $d$ and one can describe the contractions of the minimal primes of $G$. As a generalization to modules, we have:

**Corollary 5.9** *Let $R$ be an equidimensional universally catenary Noetherian local ring of dimension $d$, let $F \subsetneq R^e$ be $R$–modules and set $\mathfrak{a} = $ ann $R^e/F$. Assume that* height $\mathfrak{a} > 0$.

(a) *$\mathcal{R}(F)/\mathfrak{a}\mathcal{R}(F)$ is equidimensional of dimension $d + e - 1$.*

(b) *A prime $p$ of $R$ is a contraction of a minimal prime of $\mathcal{R}(F)/\mathfrak{a}\mathcal{R}(F)$ if and only if $\mathfrak{a} \subset p$ and $\ell(F_p) = \dim R_p + e - 1$.*

**Proof.** Part (a) is a reformulation of Theorem 4.4, while (b) follows from applying (a) in the ring $R_p$. $\square$

### Reduction numbers

Narrowing down still more our setup to the case of ideals, we now show how certain multiplicities attached to the module $B_1G$ may also be used to estimate reduction numbers of ideals. This will be made possible by the way the elements $x \in B_1$ act as nilpotent endomorphisms on the module $B_1G$.

For that, we recall the following concepts from [27, pp. 237–238]. Let $S$ be a Noetherian ring, let $M$ be a finitely generated $S$–module and let $\mathcal{A} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ be the set of associated primes of $M$. For each $\mathfrak{p} \in \mathcal{A}$, denote

$$\mathrm{mult}_M(\mathfrak{p}) = \lambda_{S_\mathfrak{p}}(H^0_\mathfrak{p}(M_\mathfrak{p})).$$

For each chain $C$ of primes $\mathfrak{p}_{i_1} \subset \mathfrak{p}_{i_2} \subset \cdots \subset \mathfrak{p}_{i_s}$ in $\mathcal{A}$, set

$$\mathrm{mult}_M(C) = \sum_{j=1}^{s} \mathrm{mult}_M(\mathfrak{p}_{i_j}).$$

Finally define $\mathrm{mult}(M) = \max_C\{\mathrm{mult}_M(C)\}$, where the maximum is taken over all chains of primes in $\mathcal{A}$. We have the following fact, akin in spirit to [27, Proposition 9.2.2].

**Proposition 5.10** *Let $S$ be a Noetherian ring, let $M$ be a finitely generated $S$–module, and let $\varphi_1, \ldots, \varphi_n$ be a collection of commuting nilpotent endomorphisms of $M$. If $n \geq \mathrm{mult}(M)$ then $\varphi_n \cdots \varphi_1 = 0$. In particular, if $\mathfrak{a} \subset \sqrt{\mathrm{ann}\, M}$ then $\mathfrak{a}^{\mathrm{mult}(M)} \subset \mathrm{ann}\, M$.*

**Proof.** Write $L = \varphi_n \cdots \varphi_1(M)$. To show that $L = 0$, suppose otherwise and let $\mathfrak{m}$ be one of its associated prime ideals. Clearly $\mathfrak{m} \in \mathcal{A}$. Localizing at this ideal we may assume that $R$ is a local ring and its maximal ideal is an associated prime of $M$. Note that the definition of $\mathrm{mult}_M(\mathfrak{p})$ for prime ideals $\mathfrak{p} \subset \mathfrak{m}$ is unaffected by the localization.

We make use of the observation that if $\varphi$ and $\psi$ are commuting nilpotent endomorphisms of a module and $\varphi \neq 0$, then $\ker(\varphi) \neq \ker(\psi\varphi)$. When we apply this to a module of length $n$ it yields that the product $\varphi_n \cdots \varphi_1$ of such morphisms must vanish.

We now induct on the maximal length of chains in $\mathcal{A}$, taking special account of the fact that $\mathrm{mult}_M(\mathfrak{m})$ occurs in each of the terms $\mathrm{mult}_M(C)$. Let $s = \mathrm{mult}_M(\mathfrak{m})$; for each other associated prime $\mathfrak{p} \subset \mathfrak{m}$ of $M$, we have that $\mathrm{mult}(M_\mathfrak{p}) \leq \mathrm{mult}(M) - s$. This implies, by the induction hypothesis, that $\mathfrak{m}$ is the only possible associated prime of the module $N = \varphi_{n-s} \cdots \varphi_1(M)$. Therefore $N \subset H^0_\mathfrak{m}(M)$. Since the latter is a module of length $s$ and the $\varphi_i$ are commuting nilpotent endomorphisms of $H^0_\mathfrak{m}(M)$, it follows that $\varphi_n \cdots \varphi_{n-s+1}(H^0_\mathfrak{m}(M)) = 0$, which implies the assertion. $\square$

To use this result notice that some of the numbers $\mathrm{mult}_M(\mathfrak{p})$ occur in the associativity formula for the multiplicity of $M$. Therefore, if $M$ is a finitely generated graded module over a standard graded Noetherian ring $S$ with $S_0$ Artinian and if all associated primes of $M$ have the same dimension, it follows that $\operatorname{mult}(M) \leq e(M)$. We will use this remark in the following situation. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$ and let $J \subset I$ be $\mathfrak{m}$–primary ideals with $J$ a complete intersection and a reduction of $I$. Consider the homogeneous inclusion of $R$-algebras $A = \mathcal{R}(J) \subset B = \mathcal{R}(I)$. Let as before $G = \operatorname{gr}_{A_1 B}(B)$. By Proposition 3.2(a), $\lambda([B_1 G]_n) = \lambda(I^n/J^n) = \lambda(R/J^n) - \lambda(R/I^n)$. Hence, for $n \gg 0$,

$$\begin{aligned} \lambda([B_1 G]_n) &= e_0(I)\binom{n+d-1}{d} - [e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots] \\ &= e_1(I)\binom{n+d-2}{d-1} + \cdots . \end{aligned} \qquad (1)$$

**Corollary 5.11** *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$ and let $J \subset I$ be $\mathfrak{m}$–primary ideals with $J$ is a complete intersection and a reduction of $I$. Assume that* $\operatorname{depth} R[IT, JU]_{\mathfrak{q}} \geq 2$ *for every prime $\mathfrak{q}$ of $R[IT, JU]$ such that* $\dim R[IT, JU]_{\mathfrak{q}} \geq 3$. *Then* $\mathrm{r}_J(I) \leq e_1(I)$.

**Proof.** Notice that $R[IT, JU] = \mathcal{R}_B(A_1 B)$. Our assumption on $\mathcal{R}_B(A_1 B)$ implies that every associated prime of $G = \operatorname{gr}_{A_1 B}(B)$ has dimension at least $d$, hence the same holds for the $G$-module $B_1 G \subset G$. Since $\dim B_1 G \leq d$ by (1), it follows that every associated prime of the $G$-module $B_1 G$ has dimension $d$. By the remark following Proposition 5.10, $\operatorname{mult}(B_1 G) \leq e(B_1 G)$, whereas $e(B_1 G) = e_1(I)$ by (1). The last part of Proposition 5.10 then shows that $(B_1 G)^{e_1(I)}\, B_1 G = 0$ or equivalently, $(B_1 B)^{e_1(I)+1} \subset A_1 B$. This implies the asserted inequality $\mathrm{r}_J(I) \leq e_1(I)$. $\square$

# 6 Homogeneous algebras over Artinian rings

Consider a homogeneous inclusion $A \subset B$ of standard graded Noetherian rings with $A_0 = B_0$ Artinian local. In this section we wish to compare the multiplicities of $A$ and $B$.

**Proposition 6.1** *Let $A \subset B$ be a homogeneous inclusion of standard graded Noetherian rings of the same dimension with $A_0 = B_0$ Artinian local. Let $\mathfrak{P}$ denote the set of primes of $A$ with* $\dim A/\mathfrak{p} = \dim A$. *Let $r \geq 1$ be an integer such that $B_{\mathfrak{p}}$ contains a free $A_{\mathfrak{p}}$-module of rank $r$ for every prime $\mathfrak{p} \in \mathfrak{P}$. Then:*

(a) *$r\, e(A) + e_\infty(A, B) \leq e(B)$ and equality holds if and only if $B_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank $r$ for every prime $\mathfrak{p} \in \mathfrak{P}$.*

(b) *If $B$ is integral over $A$ and $B_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank $r$ for every prime $\mathfrak{p} \in \mathfrak{P}$, then $r\, e(A) = e(B)$. The converse holds if $B$ is equidimensional.*

**Proof.** (a) Write $M_t = \sum_{i=0}^{t-1} B_i A \subset B$ for $t \gg 0$. Since $\dim M_t = \dim A = \dim B$ and $\lambda([M_t]_n) + \lambda([B/M_t]_n) = \lambda(B_n)$, Proposition 3.2 shows that $e(M_t) + e_\infty(A, B) = e(B)$. On the other hand, $(M_t)_{\mathfrak{p}}$ contains a free $A_{\mathfrak{p}}$-module of rank $r$ for every prime $\mathfrak{p} \in \mathfrak{P}$. Thus, $e(M_t) \geq r\, e(A)$ and equality holds if and only if $(M_t)_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank $r$ for every prime $\mathfrak{p} \in \mathfrak{P}$ and $t \gg 0$. But the last condition means that $B_{\mathfrak{p}}$ is free of rank $r$ over $A_{\mathfrak{p}}$ because in either case $(M_t)_{\mathfrak{p}} = B_{\mathfrak{p}}$ for $t \gg 0$.

(b) This part is an immediate consequence of (a) and Corollary 3.5(b). $\square$

It may be useful to consider also the case where $\dim A \neq \dim B$. For this we introduce the following notion. Let $k$ be a field and $R$ a finitely generated $k$–algebra which is a domain; we write $\deg_k(R) = \min\{\mathrm{rank}_A R\}$, where $A$ ranges over all Noether normalizations of $R$. For convenience, we single out some typical behaviour of this numerical invariant.

- For every maximal ideal $\mathfrak{n}$ of $R$, $\deg_k(R) \geq e(R_{\mathfrak{n}})$.

Indeed, if $A$ is any Noether normalization of $R$, $\mathfrak{m} = \mathfrak{n} \cap A$, and $(\widehat{A}, \widehat{\mathfrak{m}})$ is the $\mathfrak{m}$–adic completion of $A$, then $\widehat{R_{\mathfrak{n}}}$ is a direct summand of the $\widehat{A}$–module $\widehat{A} \otimes_A R$, and therefore

$$\mathrm{rank}_A R = \mathrm{rank}_{\widehat{A}} \widehat{A} \otimes_A R \geq \mathrm{rank}_{\widehat{A}} \widehat{R_{\mathfrak{n}}} = e_{\widehat{\mathfrak{m}}}(\widehat{R_{\mathfrak{n}}}) \geq e_{\widehat{\mathfrak{m}}\widehat{R_{\mathfrak{n}}}}(\widehat{R_{\mathfrak{n}}}) \geq e(\widehat{R_{\mathfrak{n}}}) = e(R_{\mathfrak{n}}).$$

- If $R$ is standard graded with $R_0 = k$ infinite then $\deg_k(R) = e(R)$.

This follows from the previous inequality $\deg_k(R) \geq e(R_{\mathfrak{n}})$, $\mathfrak{n} \in \text{m–Spec}(R)$, and from Proposition 6.1(b).

- Let $k$ be infinite and, say, $R = k[y_1, \ldots, y_n]$ with $\dim R = d$. Let $\{z_{ij} \mid 1 \leq i \leq d, 1 \leq j \leq n\}$ be variables over $k$. Set $k' = k(\{z_{ij}\})$, $x_i = \sum_{j=1}^n z_{ij} y_j$ and $A' = k'[x_1, \ldots, x_d] \subset R' = k' \otimes_k R$. Then

$$\deg_k(R) \leq \mathrm{rank}_{A'} R'. \tag{2}$$

To see this write $k'' = k[\{z_{ij}\}] \subset k'$ and $A'' = k''[x_1, \ldots, x_d] \subset R'' = k'' \otimes_k R$ and let $S$ denote the integral closure of $A''$ in $R''$. Since $R'' \subset k'S$ (see, e.g., [6, 7.3]) and $R''$ is a finitely generated $A''$-algebra, it follows that $R'' \subset S_b$, for some $0 \neq b \in k''$. Now $R''_b$ is a finite $A''_b$-module. Say, $b = g(z_{ij})$ as an element of the polynomial ring $k''$. On the other hand, by the so called generic freeness lemma, $R''_a$ is a free module over $A''_a$ (whose rank is necessarily $\mathrm{rank}_{A'} R'$) for some $0 \neq a \in A''$. Write $a = f(z_{ij}) \in R[\{z_{ij}\}] \subset K[\{z_{ij}\}]$, with $K$ standing for the field of fractions of $R$. Let $(\alpha_{ij}) \in k^{dn}$ be so chosen that $f(\alpha_{ij}) g(\alpha_{ij}) \neq 0$ ($k$ is infinite). Let $\overline{\phantom{x}}$ denote the evaluation map $z_{ij} \mapsto \alpha_{ij}$ from $R''$ to $R$. Then $A = \overline{A''}$ is a Noether normalization of $R = \overline{R''}$ because $0 \neq \overline{b} \in k$, and $\mathrm{rank}_A R \leq \mathrm{rank}_{A''} R''$ because $0 \neq \overline{a} \in A$. This shows that $\deg_k(R) \leq \mathrm{rank}_{A''} R'' = \mathrm{rank}_{A'} R'$, as required.

**Proposition 6.2** *Let $A \subset B$ be a homogeneous inclusion of standard graded Noetherian domains with $A_0 = B_0$ an infinite field and let $K$ stand for the field of fractions of $A$. Then $\deg_K(K \otimes_A B)\, e(A) \leq e(B)$.*

**Proof.** Write $t := \mathrm{trdeg}_A B = \dim B - \dim A$ and let $x_1, \ldots, x_t$ be linear forms in $B$ such that $K[x_1, \ldots, x_t]$ is a Noether normalization of $K \otimes_A B$. Replacing $A$ by the polynomial ring $A[x_1, \ldots, x_t]$ does not change $e(A)$ while $\deg_K(K \otimes_A B)$ can only increase. Thus we may assume that $\dim A = \dim B$ and $\deg_K(K \otimes_A B) = \mathrm{rank}_A B$, in which case the assertion follows from Proposition 6.1(a). $\square$

**Proposition 6.3** *Let $S$ be a Noetherian domain which is a standard bigraded ring with $S_{(0,0)}$ a field. Set $R = \bigoplus_{i \geq 0} S_{(i,0)}$, $\mathfrak{m} = \bigoplus_{i > 0} S_{(i,0)}$, and let $K$ be the field of fractions of $R$. Then, considering $S$ with the standard grading by total degree, one has $e(K \otimes_R S) e(R) \leq e(S)$, and equality holds if and only if $\dim R + \dim S/\mathfrak{m}S \leq \dim S$.*

**Proof.** We may assume that $S_{(0,0)}$ is infinite. Write $t := \operatorname{trdeg}_R S = \dim S - \dim R$ and let $x_1, \ldots, x_t$ be general elements in $S_{(0,1)}$. Set $A = R[x_1, \ldots, x_t]$. As $K \otimes_R S$ is standard graded with $[K \otimes_R S]_0 = K$, one has $e(K \otimes_R S) = \operatorname{rank}_A S$. Furthermore, since $A$ is a polynomial ring over $R$, $\dim A = \dim S$ and $e(A) = e(R)$. Now by Proposition 6.1, $e(K \otimes_R S)\, e(R) \leq e(S)$ with equality holding if and only if $S$ is integral over $A = R[x_1, \ldots, x_t]$. But the latter means that $\dim S/\mathfrak{m}S \leq t$. $\square$

We next come to the main result of this section, in which we deal with the correction term $e_\infty(A, B)$ occurring in Proposition 6.1(a).

**Theorem 6.4** *Let $A \subset B$ be a homogeneous inclusion of standard graded Noetherian rings of the same dimension with $A_0 = B_0$ Artinian local and $B$ equidimensional. Write $\mathfrak{m} = A_1 A$, let $\mathfrak{P}$ be the set of all primes $\mathfrak{p}$ of $A$ with $\dim A/\mathfrak{p} = \dim A$ and let $\mathfrak{Q}$ be the set of all primes $\mathfrak{q}$ in $V(\mathfrak{m}B)$ with $\dim B/\mathfrak{q} = \dim B/\mathfrak{m}B$. Let $r \geq 1$ be an integer. Assume that:*

- *$A_\mathfrak{p}$ is reduced and $\operatorname{rank}_{A_\mathfrak{p}} B_\mathfrak{p} \geq r$ for every prime $\mathfrak{p} \in \mathfrak{P}$;*
- *$\dim B/\mathfrak{m}B > 0$ (equivalently, $B$ is not integral over $A$).*

(a) *Then*

$$r\, e(A) + \sum_{\mathfrak{q} \in \mathfrak{Q}} e_{\mathfrak{m}B_\mathfrak{q}}(B_\mathfrak{q})\, e(B/\mathfrak{q}) \leq e(B).$$

*This inequality is an equality if* (i) *$\operatorname{rank}_{A_\mathfrak{p}} B_\mathfrak{p} = r$ for every $\mathfrak{p} \in \mathfrak{P}$ and* (ii) *$\dim B/\mathfrak{m}B = 1$. Conversely, equality implies* (i) *and, in case every minimal prime of $B$ contracts to a prime in $\mathfrak{P}$, also* (ii).

(b) *If for every $\mathfrak{q} \in \mathfrak{Q}$, either $\mathfrak{m}B_\mathfrak{q}$ is prime or $B_\mathfrak{q}$ is Cohen–Macaulay (for instance, if $\operatorname{Proj}(B/\mathfrak{m}B)$ is reduced or $\operatorname{Proj}(B)$ is Cohen–Macaulay), then*

$$r\, e(A) + e(B/\mathfrak{m}B) \leq e(B).$$

*This inequality is an equality if* (i) *$\operatorname{rank}_{A_\mathfrak{p}} B_\mathfrak{p} = r$ for every $\mathfrak{p} \in \mathfrak{P}$,* (ii) *$\dim B/\mathfrak{m}B = 1$, and* (iii) *$\mathfrak{m}B_\mathfrak{q}$ is a complete intersection for every $\mathfrak{q} \in \mathfrak{Q}$. Conversely, equality implies* (i)*, in case every minimal prime of $B$ contracts to a prime in $\mathfrak{P}$, also* (ii)*, and, in case $B_\mathfrak{q}$ is unmixed for every $\mathfrak{q} \in \mathfrak{Q}$, also* (iii).

**Proof.** (a) First recall that $G = \operatorname{gr}_{A_1 B}(B)$ is equidimensional with $\dim G = \dim B$ by Lemma 2.2(a,b.i).

Clearly, $\operatorname{height}(B_1 G + 0 :_G (B_1 G)^\infty) \geq 1$. We claim that equality holds here if every minimal prime of $B$ contracts to a prime in $\mathfrak{P}$. To see this, let $\mathfrak{q}$ be a minimal prime of $B$ with $\dim B/(\mathfrak{m}B + \mathfrak{q}) > 0$, and write $\bar{B} = B/\mathfrak{q}$, $\bar{A} = A/\mathfrak{q} \cap A$, $\bar{G} = \operatorname{gr}_{\bar{A}_1 \bar{B}}(\bar{B})$. By our assumption, $\dim \bar{A} = \dim A = \dim B = \dim \bar{B}$. As $\bar{G}$ is equidimensional with $\dim \bar{G} = \dim G$, the height of $B_1 G + 0 :_G (B_1 G)^\infty$ can only increase as we pass to $\bar{B}$. But the latter ring is analytically irreducible, hence by [3, 2.5], $\bar{G}$ is connected in codimension one. Now $\bar{B}_1 \bar{G}$ is not nilpotent since $\bar{B}$ is not integral over $\bar{A}$ by the choice of $\mathfrak{q}$. On the other hand, $\operatorname{height} \bar{B}_1 \bar{G} = 0$, since $\dim \bar{G}/\bar{B}_1 \bar{G} = \dim \bar{A}$ (cf. (2.1)) and $\dim \bar{A} = \dim \bar{B} = \dim \bar{G}$ by the above. It follows that the $\bar{G}$-ideal $\bar{B}_1 \bar{G} + 0 :_{\bar{G}} (\bar{B}_1 \bar{G})^\infty$ has height at most one. Thus, indeed $\operatorname{height}(B_1 G + 0 :_G (B_1 G)^\infty) = 1$.

By Proposition 6.1(a), $r\,e(A) + e_{\infty}(A,B) \leq e(B)$ and equality holds if and only if condition (i) obtains. Furthermore, since the extension $A \subset B$ is not integral, Corollary 2.7 and Proposition 3.4 show that $e_{\infty}(A,B) = e(G/0{:}_G\,(B_1G)^{\infty})$. Finally, by the above, height $(B_1G + 0{:}_G\,(B_1G)^{\infty}) \geq 1$ and this inequality is an equality if the minimal primes of $B$ contract to primes belonging to $\mathfrak{P}$. Therefore, part (a) will follow once we have shown that

$$e(G/0{:}_G\,(B_1G)^{\infty}) \geq \sum_{\mathfrak{q}\in\mathfrak{Q}} e_{\mathfrak{m}B_{\mathfrak{q}}}(B_{\mathfrak{q}})\, e(B/\mathfrak{q}), \tag{3}$$

and that equality holds in (3) if and only if

$$\text{height}\,(B_1G + Q) \geq \dim B/\mathfrak{m}B \tag{4}$$

for every $Q \in \mathfrak{Q}'$, where $\mathfrak{Q}'$ denotes the set of all minimal primes of $G$ that contain $0{:}_G\,(B_1G)^{\infty}$.

For each $\mathfrak{q}_i \in \mathfrak{Q}$ let $\mathfrak{Q}_i = \{Q_{ij}\}$ be the set of all minimal primes of $G$ contracting to $\mathfrak{q}_i$. Notice that $\{(Q_{ij})_{\mathfrak{q}_i} \mid Q_{ij} \in \mathfrak{Q}_i\}$ is the set of minimal primes of $G_{\mathfrak{q}_i} = B_{\mathfrak{q}_i} \otimes_B G$. By the equidimensionality of $B$ and $B_{\mathfrak{q}_i}$ one has $\dim G/Q_{ij} = \dim G$ and $\dim G_{\mathfrak{q}_i}/(Q_{ij})_{\mathfrak{q}_i} = \dim G_{\mathfrak{q}_i}$. Furthermore $B_1 \not\subset \mathfrak{q}_i$ as $\dim B/\mathfrak{q}_i = \dim B/\mathfrak{m}B > 0$. Therefore $(0{:}_G\,(B_1G)^{\infty})_{Q_{ij}} = 0$. It follows that $Q_{ij} \in \mathfrak{Q}'$ and that $(G/0{:}_G\,(B_1G)^{\infty})_{Q_{ij}} = G_{Q_{ij}}$. Thus by the associativity formula for multiplicities,

$$e(G/0{:}_G\,(B_1G)^{\infty}) \geq \sum_{i,j} \lambda(G_{Q_{ij}})\, e(G/Q_{ij}), \tag{5}$$

and equality holds if and only if $\mathfrak{Q}' = \bigcup_i \mathfrak{Q}_i$, which means that every $Q$ in $\mathfrak{Q}'$ has a contraction $\mathfrak{q}$ that belongs to $\mathfrak{Q}$. The last condition is triggered by (4). Indeed, if $x_1, \ldots, x_t$ are forms in $B$ whose images are a system of parameters of $B/\mathfrak{q}$, then

$$\dim B/\mathfrak{m}B \leq \text{height}\,(B_1G + Q) = \text{height}\,((x_1,\ldots,x_t)G + Q) \leq t + \text{height}\ Q = \dim B/\mathfrak{q},$$

which shows that $\mathfrak{q} \in \mathfrak{Q}$.

By (2.1), $G/Q_{ij}$ is a standard bigraded domain over a field, $R = \bigoplus_{l\geq 0}[G/Q_{ij}]_{(l,0)} = B/\mathfrak{q}_i$ and $\mathrm{Quot}(R) \otimes_R G/Q_{ij} = G_{\mathfrak{q}_i}/(Q_{ij})_{\mathfrak{q}_i}$. Thus by Proposition 6.3,

$$e(G/Q_{ij}) \geq e(G_{\mathfrak{q}_i}/(Q_{ij})_{\mathfrak{q}_i})\, e(B/\mathfrak{q}_i), \tag{6}$$

with equality holding if and only if $\dim B/\mathfrak{q}_i + \dim G/(B_1G + Q_{ij}) \leq \dim G/Q_{ij}$, where $\dim B/\mathfrak{q}_i = \dim B/\mathfrak{m}B$ and $\dim G/Q_{ij} = \dim G$. Thus equality in (6) for every $i,j$ means that height $(B_1G + Q) \geq \dim B/\mathfrak{m}B$ whenever $Q \in \bigcup_i \mathfrak{Q}_i$.

From (5) and (6) and the remarks following them we conclude that

$$e(G/0{:}_G\,(B_1G)^{\infty}) \geq \sum_i (\sum_j \lambda(G_{Q_{ij}})\, e(G_{\mathfrak{q}_i}/(Q_{ij})_{\mathfrak{q}_i}))\, e(B/\mathfrak{q}_i), \tag{7}$$

and that this inequality is an equality if and only if (4) holds. To arrive at (3) notice that again by the associativity formula,

$$\sum_j \lambda(G_{Q_{ij}})\, e(G_{\mathfrak{q}_i}/(Q_{ij})_{\mathfrak{q}_i}) = e(G_{\mathfrak{q}_i}). \tag{8}$$

As $G_{\mathfrak{q}_i} = \mathrm{gr}_{\mathfrak{m}B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$,

$$e(G_{\mathfrak{q}_i}) = e_{\mathfrak{m}B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i}). \tag{9}$$

Now (8) and (9) show that (7) is equivalent to (3).

(b) Since for every $\mathfrak{q} \in \mathfrak{Q}$, $\mathfrak{m}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$ or $B_{\mathfrak{q}}$ is Cohen–Macaulay, one has $\lambda((B/\mathfrak{m}B)_{\mathfrak{q}}) \leq e_{\mathfrak{m}B_{\mathfrak{q}}}(B_{\mathfrak{q}})$. This inequality is an equality if $\mathfrak{m}B_{\mathfrak{q}}$ is a complete intersection, and the converse holds in case $B_{\mathfrak{q}}$ is unmixed. Now (b) follows from (a) and the associativity formula. $\square$

The next result covers cases in which $A$ and $B$ may have different dimensions.

**Corollary 6.5** *Let $A \subset B$ be a homogeneous inclusion of standard graded Noetherian domains with $A_0 = B_0$ an infinite field, let $K$ denote the field of fractions of $A$ and set $\mathfrak{m} = A_1A$. Assume that $\dim A + \dim B/\mathfrak{m}B \neq \dim B$ and that for every $\mathfrak{q}$ in $V(\mathfrak{m}B)$ with $\dim B/\mathfrak{q} = \dim B/\mathfrak{m}B$ either $\mathfrak{m}B_{\mathfrak{q}}$ is prime or $B_{\mathfrak{q}}$ is Cohen–Macaulay. Then*

$$\deg_K(K \otimes_A B)\, e(A) + e(B/\mathfrak{m}B) \leq e(B).$$

**Proof.** Write $t := \mathrm{trdeg}_A B = \dim B - \dim A$. As in (2), we take a suitable field extension $k \subset k'$ of $k = A_0 = B_0$ and generic linear forms $x_1, \ldots, x_t$ in $B' = k' \otimes_k B$. Write $A' = (k' \otimes_k A)[x_1, \ldots, x_t]$, $\mathfrak{m}' = A'_1 A'$. Then $\deg_K(K \otimes_A B) \leq \mathrm{rank}_{A'} B'$. Since the elements $x_1, \ldots, x_t$ are algebraically independent over $k' \otimes_k A$, one has $\dim A' = \dim B'$ and $e(A) = e(A')$. Furthermore, $B'/\mathfrak{m}'B' \simeq (k' \otimes_k B/\mathfrak{m}B)/(x_1, \ldots, x_t)$ with $x_1, \ldots, x_t$ general linear forms of $k' \otimes_k B/\mathfrak{m}B$. Thus, as $\dim B/\mathfrak{m}B > t$, one has $\dim B'/\mathfrak{m}'B' > 0$ and $e(B/\mathfrak{m}B) = e(B'/\mathfrak{m}'B')$. For every prime $\mathfrak{q}'$ in $V(\mathfrak{m}'B')$ with $\dim B'/\mathfrak{q}' = \dim B'/\mathfrak{m}'B' > 0$, $B_1 \not\subset \mathfrak{q}'$ and therefore $(B'/\mathfrak{m}'B')_{\mathfrak{q}'}$ is $B$–isomorphic to a polynomial ring over $B/\mathfrak{m}B$, localized at the extension of $\mathfrak{q} = \mathfrak{q}' \cap B$. As $x_1, \ldots, x_t$ are contained in $\mathfrak{q}'$ and are general linear forms in $k' \otimes_k B/\mathfrak{q}$, it follows that $0 < \dim B'/\mathfrak{q}' \leq \max\{0, \dim B/\mathfrak{q} - t\}$. Thus $\dim B'/\mathfrak{q}' \leq \dim B/\mathfrak{q} - t$, and hence $\dim B/\mathfrak{q} \geq \dim B'/\mathfrak{q}' + t = \dim B'/\mathfrak{m}'B' + t \geq \dim B/\mathfrak{m}B$. As $\mathfrak{q}$ is in $V(\mathfrak{m}B)$, our assumptions yield that $(B/\mathfrak{m}B)_{\mathfrak{q}}$ is a domain or $B_{\mathfrak{q}}$ is Cohen–Macaulay, and then by the above, the same holds true for $(B'/\mathfrak{m}'B')_{\mathfrak{q}'}$ or $B'_{\mathfrak{q}'}$. Now we apply Theorem 6.4(b) to conclude that $(\mathrm{rank}_{A'} B')\, e(A') + e(B'/\mathfrak{m}'B') \leq e(B')$, which yields $\deg_K(K \otimes_A B)\, e(A) + e(B/\mathfrak{m}B) \leq e(B)$. $\square$

**Theorem 6.6** *Let $R$ be a reduced equidimensional standard graded Noetherian ring of dimension $d > 0$ with $R_0 = k$ a field and let $I$ be an $R$–ideal of height $g$ generated by forms of degree $s > 0$. Let $\mathfrak{P}$ denote the set of all primes $\mathfrak{p}$ of $k[I_s] \subset k[R_s] \subset R$ with $\dim k[I_s]/\mathfrak{p} = \dim k[I_s]$, and let $\mathfrak{R}$ denote the set of all primes $p$ in $V(I) \cap \mathrm{Proj}(R)$ with $\dim R/p = \dim R/I$. Let $r \geq 1$ be an integer such that $\mathrm{rank}_{\,k[I_s]_{\mathfrak{p}}}\, k[R_s]_{\mathfrak{p}} \geq r$ for every $\mathfrak{p} \in \mathfrak{P}$.*

(a) *Assume that $\ell(I) = d$. Then*

$$e(k[I_s]) \leq \left( e(R)s^g - \sum_{p \in \mathfrak{R}} e_{I_p}(R_p)\, e(R/p) \right) \frac{s^{d-g-1}}{r}.$$

*This inequality is an equality if $\mathrm{rank}_{\,k[I_s]}\, k[R_s] = r$ and $d \leq g+1$. The converse holds in case $R$ is a domain.*

(b) *Assume that $\ell(I) = d \neq g$ and that for every $p \in \mathfrak{R}$, either $I_p$ is prime or $R_p$ is Cohen–Macaulay. Then*

$$e(k[I_s]) \leq (e(R)s^g - e(R/I))\ \frac{s^{d-g-1}}{r}.$$

*This inequality is an equality if* $\operatorname{rank}_{k[I_s]} k[R_s] = r$, $d = g+1$ *and $I_p$ is a complete intersection for every $p \in \mathfrak{R}$. The converse holds in case $R$ is a domain.*

(c) *Assume that $\ell(I) \neq g$ (i.e., $I$ is not equimultiple), that $R$ is a domain with $k$ infinite and that, again, for every $p \in \mathfrak{R}$, either $I_p$ is prime or $R_p$ is Cohen–Macaulay. Then the same inequality as in* (b) *holds with $r$ replaced by* $\deg_K(K \otimes_{k[I_s]} k[R_s])$, *where $K$ is the field of fractions of $k[I_s]$.*

**Proof.** Notice that $k[R_s] = R^{(s)}$ is the $s$th Veronese subring of $R$. There are natural isomorphisms from $\operatorname{Proj}(R)$ to $\operatorname{Proj}(R^{(s)})$ and from $\operatorname{Proj}(R/I)$ to $\operatorname{Proj}((R/I)^{(s)}) = \operatorname{Proj}(R^{(s)}/I^{(s)})$. They induce a one-to-one correspondence between the set $\mathfrak{R}$ and the set of all primes $\mathfrak{q}$ in $V(I^{(s)}) \cap \operatorname{Proj}(R^{(s)})$ with $\dim R^{(s)}/\mathfrak{q} = \dim R^{(s)}/I^{(s)}$. Now let $p \in \mathfrak{R}$. By the above isomorphisms, if $(R/I)_p$ is a domain then so is $(R^{(s)}/I^{(s)})_{p^{(s)}}$, and if $R_p$ is Cohen–Macaulay then so is $R^{(s)}_{p^{(s)}}$. Furthermore $R_p$ is an étale local extension of $R^{(s)}_{p^{(s)}}$, and $I = I^{(s)}R$. Therefore, $I_p$ is a complete intersection if and only if $I^{(s)}_{p^{(s)}}$ is, and $e_{I^{(s)}_{p^{(s)}}}(R^{(s)}_{p^{(s)}}) = e_{I_p}(R_p)$. Finally recall that $e(R^{(s)}) = e(R)\, s^{d-1}$ and, if $d \neq g$, $e(R^{(s)}/I^{(s)}) = e(R/I)\, s^{d-g-1}$.

We rescale the grading of both $k[I_s]$ and $R^{(s)}$ so that $k[I_s] \subset R^{(s)}$ is a homogeneous inclusion of standard graded $k$–algebras. Set $\mathfrak{m} = I_s k[I_s]$. We have $R^{(s)}/\mathfrak{m}R^{(s)} = R^{(s)}/I^{(s)}$ and $k[I_s] \simeq k \otimes_R \mathcal{R}(I)$. Therefore $\dim k[I_s] = \ell(I)$, $\dim R^{(s)} = d$, $\dim R^{(s)}/\mathfrak{m}R^{(s)} = d - g$, and $R^{(s)}$ is reduced and equidimensional. We may clearly assume that $d \neq g$ as the assertion of (a) is otherwise trivial. Thus, by the above remarks, Theorem 6.4 and Corollary 6.5 apply to the homogeneous inclusion $k[I_s] \subset R^{(s)}$ and yield parts (a), (b) and part (c), respectively. $\square$

**Remark 6.7** For the subsequent applications, we emphasize again that, after rescaling $k[I_s]$ to a standard $k$-algebra, $k[I_s]$ is isomorphic to the special fiber $k \otimes_R \mathcal{R}(I)$ as standard $k$-algebras (see (2.1)).

## Applications

**Application 6.8** If $\operatorname{char} k = 0$ and $R$ is a domain, then Theorem 6.6 yields upper bounds for the reduction number $\mathrm{r}_J(I)$ of $I$ relative to any reduction $J$, because $\mathrm{r}_J(I) < e(k \otimes_R \mathcal{R}(I))$ by [26, Theorem 7 and Proposition 9].

**Application 6.9** In addition to the assumptions of Theorem 6.6(b) suppose that $k$ is infinite and that either (i) $R_p$ is Gorenstein or (ii) $\nu(I_p) = g$ for every $p \in \mathfrak{R}$. There exist forms $\alpha_1, \ldots, \alpha_g$ of degree $s$ in $I$ so that $J = (\alpha_1, \ldots, \alpha_g)$ has height $g$ and, in case (ii), $I_p = J_p$ for every $p \in \mathfrak{R}$. One has $e(R)s^g - e(R/I) \leq e(R/J) - e(R/I) = e(R/(J\colon I))$, and hence by the theorem, $e(k \otimes_R \mathcal{R}(I)) \leq e(R/(J\colon I))\, \frac{s^{d-g-1}}{r}$.

**Application 6.10** Let $R$ be a polynomial ring over a field $k$ with $\dim R = 3$ and $I$ a linearly presented perfect $R$–ideal of grade 2 with $\ell(I) = 3$ and $\nu(I) = n$. In this case the estimate of Theorem 6.6(b) with $r = 1$ yields $e(k \otimes_R \mathcal{R}(I)) \leq \binom{n-1}{2}$. By Theorem 6.6(b) and [19, 1.3], equality holds if and only if $I$ is generically a complete intersection. The theorem also implies that for such ideals the extension $k[I_{n-1}] \subset R^{(n-1)}$ is birational.

**Application 6.11** Let $R$ be a polynomial ring over a field $k$ with $\dim R = 4$ and $I$ a linearly presented Gorenstein $R$-ideal of grade 3 with $\ell(I) = 4$ and $\nu(I) = 5$. In this case the formula of Theorem 6.6(b) reads $e(k \otimes_R \mathcal{R}(I)) \leq 3$. On the other hand, since $I$ is syzygetic ([7, 4.11]) but not of linear type and $k \otimes_R \mathcal{R}(I)$ is a hypersurface ring, one has $e(k \otimes_R \mathcal{R}(I)) \geq 3$. Thus, $e(k \otimes_R \mathcal{R}(I)) = 3$. Now Theorem 6.6(b) shows that $I$ is generically a complete intersection and that the extension $k[I_2] \subset R^{(2)}$ is birational.

**Application 6.12** Let $\phi : X \to Y$ be a dominant rational map of projective varieties (with given embeddings in projective spaces), where $X$ is assumed to be reduced and irreducible. Suppose that $\phi$ is given by forms of fixed degree $s$ and let $\mathcal{B} \subset X$ stand for the corresponding zero locus. Let $d = \dim X$, $g = \operatorname{codim}(\mathcal{B})$, and assume that $g \leq \dim Y$. If $X$ is Cohen–Macaulay locally at the irreducible components of $\mathcal{B}$ or if $\mathcal{B}$ is generically reduced, then by Theorem 6.6(c)

$$\deg(Y) \leq \deg(X)\, s^d - \deg(\mathcal{B})\, s^{d-g}.$$

The application often yields estimates for the degree of the Gauss image of a projective variety. For hypersurfaces with isolated singularities the next application gives a better result.

**Application 6.13** (Teissier–Plücker formula) Let $k$ be an algebraically closed field and let $X \subset \mathbb{P}^n_k$ be a reduced and irreducible hypersurface of degree $e > 1$. Assume that the dual variety $X' \subset \mathbb{P}^{n'}_k$ is again a hypersurface (i.e., $X'$ is non-deficient). Notice that $X'$ is also the image of the Gauss map of $X \subset \mathbb{P}^n_k$. Let $r$ be the degree of this map and let $\mathfrak{R}$ denote the (possibly empty) set of irreducible components of $\operatorname{Sing}(X)$ of maximal dimension. Let $f$ be the irreducible homogeneous polynomial defining $X$. Set $R = k[X_0, \ldots, X_n]/(f)$, $I = (\partial f/\partial X_0, \ldots, \partial f/\partial X_n)R$ (the jacobian ideal of $R$), and write $\mathcal{I}$ for the ideal sheaf corresponding to the homogeneous ideal $I$. Finally, let $g = \operatorname{height} I$ (the codimension of $\operatorname{Sing}(X)$ in case $X$ is singular). Now, since the special fiber $k \otimes_R \mathcal{R}(I)$ is the homogeneous coordinate ring of $X' \subset \mathbb{P}^{n'}_k$, Theorem 6.6(a) yields the following inequality

$$\deg(X') \leq \frac{1}{r}\left( e(e-1)^{n-1} - \sum_{x \in \mathfrak{R}} e_{\mathcal{I}_x}(\mathcal{O}_{X,x})\, \deg(x)\, (e-1)^{n-g-1} \right),$$

with equality holding if and only if $X$ has at most isolated singularities.

We next allow $X$ to be reducible. It is known that in characteristic zero the Gauss map of a hypersurface with non-deficient dual has degree one ([11, Theorem 4], cf. also [20, Proposition 3.3]). If in addition $X$ has at most isolated singularities then the formula of Theorem 6.6(a) specializes to the formula proved by Teissier ([12], [15], [20], [24]):

$$\deg(X') = e(e-1)^{n-1} - \sum_{x \in \operatorname{Sing}(X)} e_{\mathcal{I}_x}(\mathcal{O}_{X,x}).$$

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