

# Maximal Hilbert Functions

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## 1 Introduction

A standard graded algebra is a graded ring  $G = \bigoplus_{n \geq 0} G_n$ , finitely generated over  $G_0$  by its elements of degree 1,  $G = G_0[G_1]$ . Unless stated otherwise, those will be the only kind we shall treat. When  $G_0$  is an Artinian local ring, we denote the Hilbert function of  $G$  by  $H_G(n) = \ell(G_n)$ , and its Hilbert–Poincaré series by

$$P_G(t) = \sum_{n \geq 0} H_G(n)t^n.$$

This is a rational function  $P_G(t) = \frac{h(t)}{(1-t)^d}$ , where  $h(1) = \deg(G)$ ,  $d = \dim G$  are respectively the *degree* or *multiplicity* of  $G$  and its dimension. We note by  $\mathcal{P}_G(t)$  the corresponding Hilbert polynomial

$$\mathcal{P}_G(t) = e_0(G) \binom{t+d-1}{d-1} - e_1(G) \binom{t+d-2}{d-2} + \cdots + (-1)^{d-1} e_{d-1}(G).$$

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In particular  $e_0(G) = \deg(G) = h(1)$ .

There are also iterated versions of these functions and we will make use of  $H_G^1(n) = \sum_{i \leq n} H_G(i)$ , and the corresponding Hilbert series  $P_G^1(t) = \frac{P_G(t)}{1-t}$  and Hilbert polynomial

$$\mathcal{P}_G^1(t) = e_0(G) \binom{t+d}{d} - e_1(G) \binom{t+d-1}{d-1} + \cdots + (-1)^d e_d(G)$$

The integers  $e_0(G), \dots, e_d(G)$  are called the Hilbert coefficients of  $G$ . Where there is no danger of confusion, we will write for short  $e_i$  instead of  $e_i(G)$ .

There is a great deal of interest on the structure of the set  $\mathcal{H}$  of these functions. Our approach to them takes into account the partially ordered structure afforded by the definition

$$P_G(t) \geq P_{G'}(t) \Leftrightarrow H_G(n) \geq H_{G'}(n), \quad \forall n.$$

Thus for a given a condition  $\mathfrak{C}$  on Hilbert functions, we define  $H(\mathfrak{C})$  to be the set of all Hilbert functions satisfying  $\mathfrak{C}$ . Two of the main questions are to search for the extremal members of  $H(\mathfrak{C})$  and to ascertain when it is finite. Among these sets we will consider  $H(d, e_0)$ , defined by all algebras with a given dimension  $d$  and multiplicity  $e_0$ , and its subset  $H(d, e_0, e_1)$ .

One of the most significant classes of these algebras arise as associated graded rings of filtrations of Noetherian local rings, particularly of the following kind. Let  $(R, \mathfrak{m})$  be a Noetherian local and let  $I$  be an  $\mathfrak{m}$ -primary ideal. The Hilbert function of the associated graded ring

$$\text{gr}_I(R) = \bigoplus_{n \geq 0} I^n / I^{n+1}$$

is significant for its role as a control of the blowup process of  $\text{Spec}(R)$  along the subvariety  $V(I)$ . A challenging problem consists in relating  $P_I(t) = P_{\text{gr}_I(R)}(t)$  directly to  $R$  and  $I$  as  $\text{gr}_I(R)$  may fail to inherit some of the arithmetical (e.g. Cohen–Macaulayness) properties of  $R$ .

We will now describe some of our results. Each deals with one of the general aspects mentioned above of the set of Hilbert functions of algebras of a fixed dimension. Section 2 deals with general bounds for the set  $H(d, e_0(I))$  where  $e_0(I) := e_0(\text{gr}_I(R))$ . It is centered on estimates of the following kind:

**Theorem 2.2** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring of dimension  $d \geq 1$  and let  $I$  be an  $\mathfrak{m}$ -primary ideal in  $R$ . If  $J = (x_1, \dots, x_d)$  is a system of parameters in  $I$ , then*

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}.$$

When  $R$  is Cohen–Macaulay,  $\ell(R/J) = e_0(I)$ , which gives the formula above a convenient expression. In this case it shows that  $H(d, e_0(I))$  has a unique maximal element.

The next section considers  $H(d, e_0, e_1)$ . It is generally framed by the fact that the parameters  $e_0$  and  $e_1$  are not independent of one another. In addition, for fine control, we will require Cohen–Macaulay hypotheses to sharpen the results of the previous section.

Our first general bound in  $H(d, e_0, e_1)$  was partially motivated by a calculation in [3].

**Theorem 3.3** *Let  $(R, \mathfrak{m})$  be a Cohen–Macaulay local ring of dimension  $d \geq 1$  and let  $I$  be an  $\mathfrak{m}$ –primary ideal. Then*

$$P_R^1(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^{d+1}}$$

where  $s = e_1(I) - e_0(I) + \ell(R/I)$ .

This gets further refined when we consider the tangent cones of Cohen–Macaulay local rings. The maximal Hilbert function in  $H(d, e_0, e_1)$  (Theorem 3.6) is then

$$P_R^1(t) \leq \frac{1 + bt + (e_0 - b - 2)t^2 + t^{\delta+2}}{(1-t)^{d+1}},$$

where  $b$  and  $\delta$  are certain functions of  $e_0$  and  $e_1$ .

It is quite clear how to exploit these bounds in order to derive finiteness results for the number of Hilbert functions. What is required is a mechanism to limit the postulation numbers of the algebras. To that end we employ the notion of *extended* or *cohomological* multiplicity introduced in [2]. In section 4 we give a bound on the number of Hilbert functions that have a given dimension and given extended multiplicity. This notion is however a strong requirement in comparison to the ordinary multiplicity. In counterpoint it provides effective bounds on the corresponding Hilbert coefficients:

**Theorem 4.3** *Let  $\text{Deg}(\cdot)$  be a cohomological degree function on graded algebras. Given two positive integers  $A, d$ , there exist only a finite number of Hilbert functions associated to standard graded algebras  $G$  over Artinian rings such that  $\dim G = d$  and  $\text{Deg}(G) \leq A$ . Furthermore, there are integers  $b_i$  dependent only on  $\dim G$  such that  $|e_i| \leq b_i \text{Deg}(G)^{i+1}$ , where  $e_i$ , for  $0 \leq i \leq \dim G$ , are the coefficients of the Hilbert polynomial  $P_G^1(t)$ .*

## 2 Boundedness of Hilbert Functions

In this section we develop general bounds for the Hilbert functions of algebras of a given dimension  $d$  and a given multiplicity. The algebras considered throughout will be either Noetherian local rings or standard graded algebras  $G = \bigoplus_{n \geq 0} G_n$  where  $G_0$  is an Artinian

local ring. For simplicity only of expression we will drop ‘Noetherian’. It will be harmlessly assumed that the residue fields of these local rings are infinite. This is achieved, without changing the Hilbert functions, in the usual manner: replacing the local ring  $(R, \mathfrak{m})$  by  $R[X]_{\mathfrak{m}R[X]}$ , where  $X$  is an indeterminate over  $R$ . For definitions and basic results we shall use [1], [5] and [15].

Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d$  and let  $I$  be an  $\mathfrak{m}$ -primary ideal in  $R$ . We denote by

$$H_I(n) = \ell(I^n/I^{n+1})$$

the Hilbert function of  $I$ . In the case  $I = \mathfrak{m}$ , we write  $H_R(n)$ . If we let

$$H_I^1(n) = \sum_{j=0}^n H_I(j) = \ell(R/I^{n+1})$$

then  $H_I^1(n) - H_I^1(n-1) = H_I(n)$ .

Let  $P_I(t) = \sum_{n \geq 0} H_I(n)t^n$  be the Hilbert series of  $I$ , then

$$P_I^1(t) = \sum_{n \geq 0} H_I^1(n)t^n = \frac{P_I(t)}{(1-t)}.$$

The following proposition can be considered a consequence of a well known result proved by Singh, of which we provide a short proof.

**Proposition 2.1** *Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d \geq 1$  and let  $I$  be an  $\mathfrak{m}$ -primary ideal in  $R$ . If  $x \in I$  and  $\bar{I} = I/(x)$ , then*

1.  $H_I(n) = H_{\bar{I}}^1(n) - \ell(I^{n+1} : x/I^n)$  for every  $n \geq 0$ .
2.  $P_I(t) \leq \frac{P_{\bar{I}}(t)}{(1-t)}$  and if the equality holds, then  $\text{depth gr}_I(R) > 0$ .

*Proof.* We let  $\bar{R} = R/xR$ , then from the exact sequence

$$0 \rightarrow (I^{n+1} : x)/I^n \rightarrow R/I^n \rightarrow R/I^{n+1} \rightarrow \bar{R}/\bar{I}^{n+1} \rightarrow 0,$$

induced by multiplication by  $x$ , we get the equality 1.

For the second assertion, Since

$$\frac{P_{\bar{I}}(t)}{(1-t)} = P_{\bar{I}}^1(t) = \sum_{n \geq 0} H_{\bar{I}}^1(n)t^n,$$

the assertion 2. follows by 1. In particular if the equality holds, then  $I^{n+1} : x = I^n$  for every  $n$ , hence  $x^* \in I/I^2$  is regular in  $\text{gr}_I(R)$  and so  $\text{depth gr}_I(R) > 0$ .  $\square$

**Theorem 2.2** *Let  $(R, \mathfrak{m})$  be a local ring of dimension  $d \geq 1$  and let  $I$  be an  $\mathfrak{m}$ -primary ideal in  $R$ . If  $J = (x_1, \dots, x_d)$  is an ideal generated by a system of parameters in  $I$ , then*

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}.$$

*If the equality holds, then  $\text{depth gr}_I(R) \geq d - 1$ .*

*Proof.* We induct on  $d$ . Let  $d = 1$  and  $J = (x)$  where  $x$  is a parameter in  $I$ . Since

$$\begin{aligned} \frac{\ell(R/I) + \ell(I/J)t}{(1-t)} &= \frac{\ell(R/I) + (\ell(R/J) - \ell(R/I))t}{(1-t)} = \\ &\ell(R/I) + \ell(R/J)t + \ell(R/J)t^2 + \dots + \ell(R/J)t^n + \dots \end{aligned}$$

and  $H_I(0) = \ell(R/I)$ , we need only to prove that for every  $n \geq 1$   $H_I(n) \leq \ell(R/xR)$ . We remark that

$$\begin{aligned} R \supseteq I^n \supseteq I^{n+1} \supseteq xI^n \\ R \supseteq xR \supseteq xI^n, \end{aligned}$$

so that

$$\ell(R/xR) + \ell(xR/xI^n) = \ell(R/I^n) + \ell(I^n/xI^n)$$

Since  $\ell(R/I^n) \geq \ell(xR/xI^n)$ , we get

$$\ell(R/xR) \geq \ell(I^n/xI^n) = H_I(n) + \ell(I^{n+1}/xI^n) \geq H_I(n).$$

Suppose  $d \geq 2$ , and let  $x_1$  be a parameter in  $I$ . We denote by  $\bar{R} = R/x_1R$ ,  $\bar{I} = I/x_1R$ ,  $\bar{J} = J/x_1R$ . Then  $\bar{I}$  is a primary ideal in the local ring  $\bar{R}$  which has  $\dim \bar{R} = d - 1$ . By induction we have

$$P_{\bar{I}}(t) \leq \frac{\ell(\bar{R}/\bar{I}) + \ell(\bar{I}/\bar{J})t}{(1-t)^{d-1}} = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^{d-1}}.$$

Since  $\frac{1}{(1-t)} \geq 0$ , we get

$$\frac{P_I(t)}{(1-t)} \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}.$$

The first assertion follows since by Proposition 2.1,  $P_I(t) \leq \frac{P_I(t)}{(1-t)}$ .

We assume now  $d \geq 2$  and

$$P_I(t) = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d}$$

and we prove  $\text{depth gr}_I(R) \geq d - 1$ . If  $J' = (x_1, \dots, x_{d-2})$ , then  $R' = R/J'$  is a local ring of dimension 2 and we remark that  $\ell(R/I)$  and  $\ell(I/J)$  are invariants by going modulo  $J'$ . By [11, Lemma 2.2.], we may assume  $d = 2$ ,  $P_I(t) = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^2}$  and we have to prove  $\text{depth gr}_I(R) > 0$ . With the above notations, if the equality holds, then

$$P_I(t) = \frac{P_{I'}(t)}{(1-t)},$$

and, by Proposition 2.1, we have  $\text{depth gr}_I(R) > 0$ .  $\square$

**Remark 2.3** If the equality holds in Theorem 2.2, we proved  $\text{depth gr}_I(R) \geq d - 1$ . It follows in particular  $\text{depth } R \geq d - 1$ . The following example shows that  $R$  is not necessarily Cohen-Macaulay. Let  $R = k[[X, Y, Z]]/(X^2, XY, XZ, Y^3) = k[[x, y, z]]$  and  $I = (x, y, z)$ .  $R$  is a 1-dimensional ring which is not Cohen-Macaulay and

$$P_I(t) = \frac{\ell(R/I) + \ell(I/J)t}{(1-t)} = \frac{1 + 2t}{(1-t)}.$$

Nearly the same treatment applies to standard graded algebras, which we state for later reference.

**Proposition 2.4** *Let  $G = \bigoplus_{n \geq 0} G_n$  be a standard graded algebra over the Artinian local ring  $G_0$  of dimension  $d \geq 1$ . If  $h \in G_1$  and  $\overline{G} = G/(h)$ , one has*

$$P_G(t) \leq \frac{P_{\overline{G}}(t)}{1-t}$$

and if the equality holds, then  $\text{depth } G > 0$ .

**Corollary 2.5** *Let  $(G_0, \mathfrak{m})$  be an Artinian local ring and let  $G = \bigoplus_{n \geq 0} G_n$  be a standard algebra of dimension  $d \geq 1$ . If  $J$  is the ideal generated by an homogeneous system of parameters in  $G$ , then*

$$P_G(t) \leq \frac{\ell(G_0) + (\ell(G/J) - \ell(G_0))t}{(1-t)^d}.$$

If the equality holds, then  $\text{depth } G \geq d - 1$ .

*Proof.* Proposition 2.4 could be used to reduce to the dimension 1 case, but instead we derive the assertion from Theorem 2.2.

Set  $I = G_+ = \bigoplus_{n \geq 1} G_n$ , and note that  $I$  is primary for the irrelevant maximal ideal  $M$  of  $G$ ,  $\text{gr}_I(G) \simeq G$ , and that  $J$  is generated by a system of parameters. Note also that

the associated graded rings and lengths are not changed if  $G$  or the localization  $G_M$  are considered. Theorem 2.2 can now be applied directly.  $\square$

These results show that the Hilbert function of  $I$  is bounded by the rational function

$$\frac{\ell(R/I) + (\ell(R/J) - \ell(R/I))t}{(1-t)^d}$$

for any ideal  $J$  generated by a system of parameters that yields minimal length for  $R/J$ . It is however not clear which number this turns out to be except when  $R$  is Cohen–Macaulay when we have:

**Corollary 2.6** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring of dimension  $d \geq 1$ . If  $I$  is an  $\mathfrak{m}$ -primary ideal in  $R$  and  $J = (x_1, \dots, x_d)$  is the ideal generated by a superficial sequence for  $I$ , then*

$$P_I(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I))t}{(1-t)^d}.$$

*Proof.* Under our assumptions, if  $x_1, \dots, x_d$  is a superficial sequence for  $I$ , then it is a system of parameters in  $I$  and  $\ell(R/J) = e_0(I)$ .  $\square$

This formula, in case  $I = \mathfrak{m}$ , was obtained in [2] through different means. We shall now explain the difference between the two sides of the formula above. Let  $J$  be a minimal reduction of the ideal  $I$ . Since  $R$  has an infinite residue field,  $J$  is generated by a regular sequence. Now consider the construction of the Sally module of  $I$  relative to  $J$ : it is simply the  $R[Jt]$ -module  $S_J(I)$  defined by the natural exact sequence

$$0 \rightarrow I \cdot R[Jt] \rightarrow I \cdot R[It] \rightarrow S_J(I) \rightarrow 0.$$

$S_J(I) = 0$  exactly when  $I^2 = JI$ , that is when  $I$  has so-called *minimal multiplicity*. In all the other cases,  $S_J(I)$  has no embedded primes and  $\dim S_J(I) = \dim R = d$ .

A calculation in [20] (see also [21]) shows that

$$P_I(t) = \frac{\ell(R/I) + (e_0(I) - \ell(R/I))t}{(1-t)^d} - (1-t)P_{S_J(I)}(t). \quad (1)$$

In particular this formula gives the following equivalence of the inequality (1):

**Corollary 2.7** *The Hilbert function of  $S_J(I)$  is non-decreasing.*

This answers a question raised in [19, p. 385].

**Remark 2.8** One application of this formula is to employ the technique of [2] to obtain estimates for the reduction number of the ideal  $I$ . In other words, to find a minimal reduction  $J$  of  $I$  and an integer  $r$  for which an equality  $I^{r+1} = JI^r$  holds.

For simplicity of notation, set  $a := \ell(R/I)$  and  $b := \ell(I/J)$ . From the inequality of Hilbert functions

$$P_I(t) \leq \frac{\ell(R/I) + \ell(I/J)t}{(1-t)^d},$$

we have that for each positive integer  $n$ ,

$$\ell(I^n/\mathfrak{m}I^n) \leq \ell(I^n/I^{n+1}) \leq a \binom{n+d-1}{d-1} + b \binom{n+d-2}{d-1}.$$

According to [4], if for some integer  $n$  we bound the right hand side of this inequality by  $\binom{n+d}{d}$ , we have found a reduction  $J$  of  $I$  with reduction number  $< n$ . This is easy to work out since the inequality is quadratic:

$$(n+d)(n+d-1) > ad(n+d-1) + bdn,$$

will be satisfied for (set  $c = a + b = \ell(R/J)$ ):

$$r \leq dc - 2d + 1 + \sqrt{(a-1)(d-1)d}.$$

The inequality  $r \leq dc - 2d + 1$  is the bound in [2] for the Cohen–Macaulay case, so that  $\sqrt{(a-1)(d-1)d}$  is a penalty for the lack of that condition.

### 3 Maximal Functions of $H(d, e_0(I), e_1(I))$

Throughout this section  $(R, \mathfrak{m})$  will be a Cohen–Macaulay local ring of dimension  $d > 0$ . Let  $I$  be an  $\mathfrak{m}$ –primary ideal and we denote by  $e_0(I)$  and  $e_1(I)$  the first two coefficients of the Hilbert polynomial of  $I$ . We will now consider the set  $H(d, e_0(I), e_1(I))$  of the Hilbert functions defined by these parameters.

A first difficulty presents itself by the fact that  $e_0(I)$  and  $e_1(I)$  are loosely related. In case  $I = \mathfrak{m}$  (see [10]):

$$e_0(\mathfrak{m}) - 1 \leq e_1(\mathfrak{m}) \leq \binom{e_0(\mathfrak{m}) - 1}{2}.$$

Actually there are more strict relations when Hilbert functions of primary ideals are considered. Here is an instance:

**Proposition 3.1** *Let  $(R, \mathfrak{m})$  be a local Cohen–Macaulay ring of dimension one, let  $I$  be an  $\mathfrak{m}$ –primary ideal. If  $e_0(I) \neq e_0(\mathfrak{m})$  then*

$$e_1(I) \leq \binom{e_0(I) - 2}{2}.$$

*Proof.* The condition  $e_0(I) \neq e_0(\mathfrak{m})$  means, by the theorem of Rees (see [12]), that  $\mathfrak{m}$  is not the integral closure of  $I$ . This implies that for each positive integer  $n$ ,  $I^{n+1} \neq \mathfrak{m}I^n$ , and therefore  $\ell(I^n/I^{n+1}) < \ell(I^n/\mathfrak{m}I^n)$ .

If  $(x)$  is a minimal reduction of  $I$ , the Hilbert function of  $I$  can be written

$$H_I(n) = \ell(I^n/I^{n+1}) = e_0(I) - \ell(I^{n+1}/xI^n),$$

so it only reaches its stable value of  $e_0(I)$  when  $n = r$ , the reduction number of  $I$ , that is the smallest  $r$  for which  $I^{r+1} = xI^r$ . We claim that for all  $n \leq r$ ,  $\ell(I^n/I^{n+1}) \geq n + 2$ . Indeed otherwise we would have  $\ell(I^n/\mathfrak{m}I^n) \leq n$ , which by the main theorem of [4] would lead to an equality  $I^n = xI^{n-1}$ , contradicting the definition of  $r$ . This means that we have

$$\begin{aligned} e_1(I) &= \sum_{n=0}^r (e_0(I) - \ell(I^n/I^{n+1})) \\ &\leq e_0(I) - \ell(R/I) + \sum_{n=1}^r (e_0(I) - (n+2)) \\ &= e_0(I) - \ell(R/I) + r(e_0(I) - 2) - \binom{r+1}{2}. \end{aligned}$$

Since  $r \leq e_0(I) - 1$ , we may assume that  $e_0(I) \geq 3$  (otherwise  $I = (x)$  and the result follows). Substituting we have the desired inequality.  $\square$

There is another relationship between  $e_0(I)$  and  $e_1(I)$  that improves the basic inequality of Corollary 2.6. Let  $R$  be a Cohen-Macaulay local ring of dimension one. If  $x$  is a superficial element for  $I$ , then for every  $j$

$$H_I(j) = e_0(I) - v_j$$

where  $v_j = \ell(I^{j+1}/xI^j)$ . In particular  $v_0 = e_0(I) - \ell(R/I)$ ,  $v_1 = e_0(I) - \ell(I/I^2)$  and if  $v_n = 0$  for some  $n$ , then  $v_j = 0$  for every  $j \geq n$ . It is known that **[need a reference]**

$$e_1(I) = \sum_{j \geq 0} v_j$$

We define

$$\delta := e_1(I) - 2e_0(I) + \ell(R/I) + \ell(I/I^2)$$

and we remark that

$$\delta = e_1(I) - v_0 - v_1 = \sum_{j \geq 2} v_j \geq 0$$

The following result on the Hilbert series of a Cohen-Macaulay local ring of dimension one will be crucial in different applications.

**Proposition 3.2** *Let  $(R, \mathfrak{m})$  be a local Cohen–Macaulay ring of dimension one and let  $I$  be an  $\mathfrak{m}$ –primary ideal. We denote by  $h = \ell(I/I^2) - \ell(R/I)$  and by  $\delta$  the integer defined above. Then*

$$P_R^1(t) \leq \frac{\ell(R/I) + ht + (e_0(I) - h - \ell(R/I) - 1)t^2 + t^{\delta+2}}{(1-t)^2}.$$

*Proof.* We prove that

$$\begin{aligned} P_R^1(t) &\leq \frac{\ell(R/I) + ht + (e_0(I) - h - \ell(R/I) - 1)t^2 + t^{\delta+2}}{(1-t)^2} \\ &= \ell(R/I) + \sum_{n \geq 1} [(n-1)e_0(I) + h + 2\ell(R/I) - \min\{n-1, \delta\}]t^n. \end{aligned}$$

We have for every  $n \geq 0$

$$H_I^1(n) = (n+1)e_0(I) - \sum_{j=0}^n v_j = (n-1)e_0(I) + h + 2\ell(R/I) - \sum_{j=2}^n v_j$$

We may assume  $n \geq 2$  and we have to prove

$$\sum_{j=2}^n v_j \geq \min\{n-1, \delta\}.$$

The inequality follows since if  $v_n \neq 0$ , then  $\sum_{j=2}^n v_j \geq n-1$ , otherwise  $\sum_{j=2}^n v_j = \sum_{j \geq 2} v_j = \delta$ .  $\square$

**Theorem 3.3** *Let  $(R, \mathfrak{m})$  be a Cohen–Macaulay local ring of dimension  $d \geq 1$  and let  $I$  be an  $\mathfrak{m}$ –primary ideal. Then*

$$P_R^1(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^{d+1}}$$

where  $s = e_1(I) - e_0(I) + \ell(R/I)$ .

*Proof.* We induct on  $d \geq 1$ . Suppose  $d = 1$  and we have to prove

$$\begin{aligned} P_R^1(t) &\leq \frac{\ell(R/I) + ht + (e_0(I) - \ell(R/I) - 1)t^2 + t^{s+1}}{(1-t)^2} \\ &= \ell(R/I) + \sum_{n \geq 1} [ne_0(I) + \ell(R/I) - \min\{n, s\}]t^n. \end{aligned}$$

Let  $\delta$  and  $v_i$  be the already defined integers, then  $s = e_1(I) - e_0(I) + \ell(R/I) = \sum_{j \geq 1} v_j$  and  $\delta = \sum_{j \geq 2} v_j$ . By Proposition 3.2 it is enough to show for every  $n \geq 1$

$$(n-1)e_0(I) + \ell(I/I^2) + \ell(R/I) - \min\{n-1, \sum_{j \geq 2} v_j\} \leq ne_0(I) + \ell(R/I) - \min\{n, \sum_{j \geq 1} v_j\}$$

or equivalently

$$e_0(I) \geq \ell(I/I^2) + \min\{n, \sum_{j \geq 1} v_j\} - \min\{n-1, \sum_{j \geq 2} v_j\}.$$

Now  $e_0(I) = H_I(1) + v_1 = \ell(I/I^2) + v_1$  and the conclusion follows since

$$v_1 + \min\{n-1, \sum_{j \geq 2} v_j\} \geq \min\{n, \sum_{j \geq 1} v_j\}.$$

Suppose now  $d \geq 2$  and let  $x$  be a superficial element for  $I$ . Then  $\bar{R} = R/(x)$  is a local Cohen–Macaulay ring of dimension  $d-1$ . In particular if  $\bar{I} = I/(x)$ , then

$$e_0(I) = e_0(\bar{I}), \quad e_1(I) = e_1(\bar{I}), \quad \ell(R/I) = \ell(\bar{R}/\bar{I})$$

By the inductive assumption it follows

$$P_{\bar{R}}^1(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d}.$$

By Proposition 2.1, we have  $P_R(t) \leq \frac{P_{\bar{R}}(t)}{1-t}$ , hence  $P_R^1(t) \leq \frac{P_{\bar{R}}^1(t)}{1-t}$  and the result follows.  $\square$

**Corollary 3.4** *Let  $(R, \mathfrak{m})$  be a Cohen–Macaulay local ring of dimension  $d \geq 2$  and let  $I$  be an  $\mathfrak{m}$ –primary ideal. Then*

$$P_R(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d}$$

where  $s = e_1(I) - e_0(I) + \ell(R/I)$ . If the equality holds, then  $\text{depth gr}_I(R) \geq d-1$ .

*Proof.* Let  $x$  be a superficial element for  $I$ . Then  $\bar{R} = R/(x)$  is a local Cohen–Macaulay ring of dimension  $d-1$ . In particular if  $\bar{I} = I/(x)$ , then

$$e_0(I) = e_0(\bar{I}), \quad e_1(I) = e_1(\bar{I}), \quad \ell(R/I) = \ell(\bar{R}/\bar{I})$$

By Theorem 3.3 it follows

$$P_{\bar{R}}^1(t) \leq \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d}$$

and the result follows now by Proposition 2.1 since  $P_R(t) \leq P_{\bar{R}}^1(t)$ .

We assume now  $d \geq 2$  and

$$P_I(t) = \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d}$$

and we prove  $\text{depth gr}_I(R) \geq d - 1$ . If  $x_1, \dots, x_{d-2}$  is a superficial sequence and  $J' = (x_1, \dots, x_{d-2})$ , then  $R' = R/J'$  is a local Cohen–Macaulay ring of dimension 2. Moreover  $\ell(R/I)$  and  $e_0(I)$  are invariants by going modulo  $J'$ .

By [11, Lemma 2.2.], we may assume  $d = 2$ ,

$$P_I(t) = \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^2}$$

and we have to prove  $\text{depth gr}_I(R) > 0$ .

With the above notations, we have in particular  $P_R(t) = P_{\bar{R}}^1(t)$  and the assertion follows again by Proposition 2.1.  $\square$

**Remark 3.5** If the equality holds in Corollary 3.4, that is

$$P_R(t) = \frac{\ell(R/I) + (e_0(I) - \ell(R/I) - 1)t + t^{s+1}}{(1-t)^d}$$

then either  $e_0(I) = \ell(I/I^2) + (1-d)\ell(R/I)$  if  $s = 0$  or  $e_0(I) = \ell(I/I^2) + (1-d)\ell(R/I) + 1$  otherwise.

Let  $x_1, \dots, x_{d-1}$  a superficial sequence in  $I$  and we set  $\bar{R} = R/(x_1, \dots, x_{d-1})$ ,  $\bar{I} = I/(x_1, \dots, x_{d-1})$ . We have  $e_0(I) = e_0(\bar{I})$ ,  $e_1(I) = e_1(\bar{I})$ . By Corollary 3.4 we have  $\text{depth gr}_I(R) \geq d - 1$  and this implies in particular  $I^2 \cap (x_1, \dots, x_{d-1}) = (x_1, \dots, x_{d-1})I$ . Hence  $\ell(I/I^2) = \ell(\bar{I}/\bar{I}^2) + \ell((x_1, \dots, x_{d-1})/(x_1, \dots, x_{d-1})I) = \ell(\bar{I}/\bar{I}^2) + (d-1)\ell(R/I)$ .

We may deduce now the assertion from the one dimensional case. In fact by the proof of Corollary 3.4 and Theorem 3.3, if the Hilbert series of  $I$  is maximal, then  $P_{\bar{I}}^1$  is maximal and this is possible if either  $v_1 = 0$  or  $v_1 = 1$ . Equivalently  $e_0(\bar{I}) = \ell(\bar{I}/\bar{I}^2) = \ell(I/I^2) + (1-d)\ell(R/I)$  if  $s = 0$ ,  $e_0(\bar{I}) = \ell(\bar{I}/\bar{I}^2) + 1 = \ell(I/I^2) + (1-d)\ell(R/I) + 1$  otherwise.

We may improve the previous bound in case of the maximal ideal. Let  $R$  be a local Cohen–Macaulay ring of dimension  $d$ , we will write  $H_R(n)$ , and  $P_R(t)$  instead of  $H_{\mathfrak{m}}(n)$  and  $P_{\mathfrak{m}}(t)$ . If  $h$  is the embedding codimension of  $R$ , that is  $h = H_R(1) - d$ , the following inequalities hold (see [8] and [6])

$$2e_0(\mathfrak{m}) - h - 2 \leq e_1(\mathfrak{m}) \leq \binom{e_0(\mathfrak{m})}{2} - \binom{h}{2}$$

We define

$$\begin{aligned} b &= \max\{n : \binom{n}{2} \leq \binom{e_0(\mathfrak{m})}{2} - e_1(\mathfrak{m})\} \\ \alpha &:= e_1(\mathfrak{m}) - 2e_0(\mathfrak{m}) + b + 2. \end{aligned}$$

Since  $\binom{h}{2} \leq \binom{e_0(\mathfrak{m})}{2} - e_1(\mathfrak{m})$ , we have  $b \geq h$ . In particular  $\alpha = e_1(\mathfrak{m}) - 2e_0(\mathfrak{m}) + b + 2 \geq e_1(\mathfrak{m}) - 2e_0(\mathfrak{m}) + h + 2 \geq 0$ .

**Theorem 3.6** *Let  $(R, \mathfrak{m})$  be a Cohen–Macaulay local ring of dimension  $d \geq 1$ . We denote by  $b$  and  $\alpha$  the integers defined above. Then*

$$P_R^1(t) \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^{d+1}}.$$

*Proof.* We induct on  $d \geq 1$ . Suppose  $d = 1$  and we prove that

$$\begin{aligned} P_R^1(t) &\leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^2} \\ &= 1 + \sum_{n \geq 1} [(n-1)e_0(\mathfrak{m}) + b + 2 - \min\{n-1, \alpha\}]t^n. \end{aligned}$$

If we apply Proposition 3.2 in the case  $I = \mathfrak{m}$  we obtain

$$\begin{aligned} P_R^1(t) &\leq \frac{1 + ht + (e_0(\mathfrak{m}) - h - 2)t^2 + t^{\delta+2}}{(1-t)^2} \\ &= 1 + \sum_{n \geq 1} [(n-1)e_0(\mathfrak{m}) + h + 2 - \min\{n-1, \delta\}]t^n, \end{aligned}$$

where  $\delta = e_1(\mathfrak{m}) - 2e_0(\mathfrak{m}) + h + 2$ . Then for every  $n \geq 1$  we have to prove

$$h - \min\{n-1, \delta\} \leq b - \min\{n-1, \alpha\}$$

Since  $\alpha = b - h + \delta$ , we get

$$b - h + \delta \geq \min\{n-1, b - h + \delta\}$$

and the assertion follows.

Suppose now  $d \geq 2$  and let  $x$  be a superficial element in  $R$ , then  $\overline{R} = R/xR$  is a local Cohen–Macaulay ring of dimension  $d-1$  and maximal ideal  $\overline{\mathfrak{m}} = \mathfrak{m}/(x)$ . In particular

$$e_0(\mathfrak{m}) = e_0(\overline{\mathfrak{m}}), \quad e_1(\mathfrak{m}) = e_1(\overline{\mathfrak{m}}), \quad h(R) = h(\overline{R}).$$

Then

$$P_R^1(t) \leq \frac{P_{\overline{R}}^1(t)}{(1-t)} \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^{d+1}}$$

and the assertion follows.  $\square$

**Corollary 3.7** *Let  $(R, \mathfrak{m})$  be a Cohen-Macaulay local ring of dimension  $d \geq 2$ . We denote by  $b$  and  $\alpha$  the integers defined above. Then*

$$P_R(t) \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^d}.$$

*If the equality holds, then  $\text{depth gr}_{\mathfrak{m}}(R) \geq d - 1$ .*

*Proof.* Let  $x$  be a superficial element for  $R$ , then  $R/xR$  is a local Cohen-Macaulay ring of dimension  $d - 1 \geq 1$ , in particular  $e_0(\mathfrak{m}) = e_0(\mathfrak{m}/(x))$ ,  $e_1(\mathfrak{m}) = e_1(\mathfrak{m}/(x))$ ,  $h(R) = h(R/xR)$  and, by Proposition 2.1, we have

$$P_R(t) \leq P_{R/xR}^1(t)$$

The assertion follows by Theorem 3.6. □

**Remark 3.8** If we apply Corollary 2.6 with  $I = \mathfrak{m}$ , we obtain

$$P_R(t) \leq \frac{1 + (e_0(\mathfrak{m}) - 1)t}{(1-t)^d}.$$

We remark that, if  $d \geq 2$ , Corollary 3.7 improves this bound. To prove this, note that  $\frac{1}{(1-t)^{d-2}} \geq 0$ , so that we only need to prove that

$$\frac{1 + bt + (e_0(\mathfrak{m}) - b - 2)t^2 + t^{\alpha+2}}{(1-t)^2} \leq \frac{1 + (e_0(\mathfrak{m}) - 1)t}{(1-t)^2}$$

that is

$$(n-1)e_0(\mathfrak{m}) + b + 2 - \min\{n-1, \alpha\} \leq ne_0(\mathfrak{m}) + 1$$

or equivalently

$$e_0(\mathfrak{m}) - b - 1 + \min\{n-1, \alpha\} \geq 0$$

Since  $\binom{b}{2} \leq \binom{e_0(\mathfrak{m})}{2} - e_1(\mathfrak{m})$ , we have  $e_0(\mathfrak{m}) \geq b + 1$  and the assertion follows.

**Remark 3.9** Corollary 3.7 does not hold if  $(R, \mathfrak{m})$  is a Cohen-Macaulay local ring of dimension one.

We recall that if  $e_1(\mathfrak{m}) = e_0(\mathfrak{m}) + 1$ , then by [7, Proposition 2.4] there are only two possible Hilbert series

$$\frac{1 + (e_0(\mathfrak{m}) - 3)t + 2t^2}{(1-t)^d} \quad \text{or} \quad \frac{1 + (e_0(\mathfrak{m}) - 2)t + t^3}{(1-t)^d}.$$

Note that  $e_0(\mathfrak{m}) \geq 4$ , hence  $b = e_0(\mathfrak{m}) - 2$  and  $\alpha = 1$ . If  $d = 1$ , these series are not comparable, while

$$P_R^1(t) \leq \frac{1 + (e_0(\mathfrak{m}) - 2)t + t^3}{(1 - t)^2}.$$

With the usual notations, for every Cohen–Macaulay local ring  $R$  of dimension  $d \geq 1$  it is possible to prove that

$$P_R(t) \leq \frac{1 + bt + (e_0(\mathfrak{m}) - b - 1)t^2}{(1 - t)^d}.$$

If  $e_1(\mathfrak{m}) = e_0(\mathfrak{m}) + 1$ , we get

$$P_R(t) \leq \frac{1 + (e_0(\mathfrak{m}) - 2)t + t^2}{(1 - t)^d}$$

but this is not satisfactory since coefficient  $e_1$  for the above series is not necessarily the given  $e_1(\mathfrak{m})$ .

## 4 Finiteness of Hilbert Functions

Let  $(R, \mathfrak{m})$  be a Cohen–Macaulay local of dimension  $d$  and let  $I$  be an  $\mathfrak{m}$ –primary ideal of multiplicity  $e_0 = e_0(I)$ . Denote by  $H(d, e_0)$  the set of all Hilbert functions of the algebras  $\text{gr}_I(R)$ . In [13] and [14] it is proved that  $H(d, e_0)$  is a finite set. Its difficult proof is accomplished by providing very large bounds on the coefficients of the Hilbert polynomials and on the Castelnuovo–Mumford regularity of the algebra  $\text{gr}_I(R)$  in terms of  $e_0$ . As their authors point out, the assertion fails if  $R$  is not Cohen–Macaulay.

Our result in this section shows that using a different notion of multiplicity one obtains a weaker finiteness theorem which applies to arbitrary graded algebras.

We recall the notion of *extended multiplicity*, or *cohomological degree*, introduced in [2]. Let  $S$  be either a graded algebra generated by its elements of degree 1 or a local ring. An *extended degree* is a function  $\text{Deg}(\cdot)$  on finitely generated  $S$ –modules (graded in the case of the former ring) satisfying the following conditions:

- (i) If  $L = \Gamma_{\mathfrak{m}}(M)$  is the submodule of elements of  $M$  which are annihilated by a power of the maximal ideal (maximal irrelevant ideal in the graded case) and  $\overline{M} = M/L$ , then

$$\text{Deg}(M) = \text{Deg}(\overline{M}) + \ell(L),$$

where  $\ell(\cdot)$  is the ordinary length function.

(ii) (Bertini's rule) If  $S$  has positive depth and  $h \in S$  is a generic hyperplane section on  $M$ , then

$$\text{Deg}(M) \geq \text{Deg}(M/hM).$$

(iii) (The calibration rule) If  $M$  is a Cohen–Macaulay module, then

$$\text{Deg}(M) = \text{deg}(M),$$

where  $\text{deg}(M)$  is the ordinary multiplicity of the module  $M$ .

In [18] an instance of such functions was constructed:

**Definition 4.1** Let  $M$  be a finitely generated graded module over the graded algebra  $A$  and let  $S$  be a Gorenstein graded algebra mapping onto  $A$ , with maximal graded ideal  $\mathfrak{m}$ . Assume that  $\dim S = r$ ,  $\dim M = d$ . The *homological degree* of  $M$  is the integer

$$\text{hdeg}(M) = \text{deg}(M) + \sum_{i=r-d+1}^r \binom{d-1}{i-r+d-1} \cdot \text{hdeg}(\text{Ext}_S^i(M, S))$$

This expression becomes more compact when  $\dim M = \dim S = d > 0$ :

$$\text{hdeg}(M) = \text{deg}(M) + \sum_{i=1}^d \binom{d-1}{i-1} \cdot \text{hdeg}(\text{Ext}_S^i(M, S)).$$

It is important to note that  $\text{hdeg}(\cdot)$  is defined recursively on the dimension of the module; we refer to [18] and [2] for more technical aspects of these definitions.

Given any cohomological degree, [9] proposed a method to construct another extended degree function where equality holds in the Bertini's condition (2). We now restate Corollary 2.5 in the language of these functions.

**Corollary 4.2** Let  $G = \bigoplus_{n \geq 0} G_n$  be a standard graded algebra over an Artinian ring and let  $\text{Deg}(\cdot)$  be any extended degree function defined on  $G$ . If  $\dim G = d \geq 1$  then

$$P_G(t) \leq \frac{\ell(G_0) + (\text{Deg}(G) - \ell(G_0)) \cdot t}{(1-t)^d}.$$

In other words, for all  $n \geq 0$

$$H_G(n) \leq \text{Deg}(G) \binom{d+n-2}{d-1} + \ell(G_0) \binom{d+n-2}{d-2}.$$

*Proof.* Let  $J$  be an ideal that is generated by a system of parameters of degree 1 that is generic for the function  $\text{Deg}(\cdot)$  chosen; according to [2, Proposition 2.3],  $\ell(G/J) \leq \text{Deg}(G)$ . Now replace  $\ell(G/J)$  by  $\text{Deg}(G)$  in the estimate of Corollary 2.5.  $\square$

**Theorem 4.3** *Let  $\text{Deg}(\cdot)$  be an extended degree function on graded algebras. Given two positive integers  $A, d$ , there exist only a finite number of Hilbert functions associated to standard graded algebras  $G$  over Artinian rings such that  $\dim G = d$  and  $\text{Deg}(G) \leq A$ . Furthermore, there are integers  $b_i$  dependent only on  $\dim G$  such that  $|e_i| \leq b_i \text{Deg}(G)^{i+1}$ , where  $e_i$ , for  $0 \leq i \leq \dim G$ , are the coefficients of the Hilbert polynomial  $P_G^1(t)$ .*

*Proof.* We are first going to show that the number of Hilbert polynomials of these algebras

$$P_G^1(n) = e_0 \binom{n+d}{d} - e_1 \binom{n+d-1}{d-1} + \cdots + (-1)^d e_d, \quad n \gg 0, \quad (2)$$

is finite by finding bounds for the  $|e_i|$  in terms of  $\text{Deg}(G)$ .

We induct on  $d$ . The assertion is clear if  $d = 0$ . For  $d \geq 1$ , we set  $\tilde{G} = G/H_{G_+}^0(G)$ . Note that  $G$  and  $\tilde{G}$  have the same Hilbert polynomial and  $\text{Deg}(\tilde{G}) \leq \text{Deg}(G)$ . Let  $h \in \tilde{G}_1$  be a generic hyperplane section for the function  $\text{Deg}(\cdot)$  and set  $\bar{G} = \tilde{G}/(h)$ . Since

$$H_{\bar{G}}^1(n)/\bar{G}_{n+1} = e_0 \binom{n+d-1}{d-1} - e_1 \binom{n+d-2}{d-2} + \cdots + (-1)^{d-1} e_{d-1}, \quad n \gg 0,$$

and  $\text{Deg}(\bar{G}) \leq \text{Deg}(\tilde{G}) \leq \text{Deg}(G)$ , by the induction hypothesis the  $|e_i|$ , for  $i < d$ , are all bounded as functions of  $\text{Deg}(G)$ .

We may return to the algebra  $G$ . We recall that the difference between the Hilbert function of  $G$  and its Hilbert polynomial is given by ([1, Theorem 4.3.5(b)])

$$H_G(n) - \mathcal{P}_G(n) = \sum_{i=0}^d (-1)^i \ell(H_{G_+}^i(G)_n). \quad (3)$$

We now make a key point on the vanishing of  $H_{G_+}^i(G)_n$ , for  $n \geq 0$ . If we denote by  $a_i(G)$  the largest  $n$  for which this group does not vanish, we have the well-known description of the Castelnuovo–Mumford regularity of the algebra  $G$ ,

$$\text{reg}(G) = \sup\{a_i(G) + i \mid i \geq 0\}.$$

However, according to [2]  $\text{Deg}(G) > \text{reg}(G)$  so that  $H_G(n) = \mathcal{P}_G(n)$  for  $n \geq \text{Deg}(G)$ .

From this it is clear how to bound the coefficient  $e_d$  of the Hilbert polynomial of  $G$ . One way to proceed is to add the terms in (3) up to  $n = r = \text{Deg}(G)$  to get

$$H_G^1(r) - \sum_{i=0}^{d-1} (-1)^i e_i \binom{d+r-i}{d-i} = K,$$

where we set

$$K = \sum_{i=0}^d (-1)^i \ell(H_{G_+}^i(G)_{\geq 0}).$$

Note that  $(-1)^d e_d = K$ . A crude estimate for  $K$  is obtained from the inequality

$$|K| \leq |H_G^1(r)| + \left| \sum_{i=0}^{d-1} (-1)^i e_i \binom{d+r-i}{d-i} \right|,$$

where now we replace  $\ell(G/G_{r+1})$  by the estimate given in Corollary 4.2,

$$H_G^1(r) \leq \text{Deg}(G) \binom{d+r-1}{d} + \ell(G_0) \binom{d+r-1}{d-1}.$$

It follows that  $|e_d|$  is bounded by a polynomial (of degree  $d+1$  when the degrees of the  $e_i$  are tracked carefully) in  $\text{Deg}(G)$ .

The finiteness of the number of Hilbert functions now follows from the finiteness of the possible Hilbert polynomials, the bound on the postulation numbers and of another application of Corollary 4.2.  $\square$

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