CHAPTER I

PROJECTIVE MODULES AND VECTOR BUNDLES

The basic objects studied in algebraic K-theory are projective modules over a ring, and vector bundles over schemes. In this first chapter we introduce the cast of characters. Much of this information is standard, but collected here for ease of reference in later chapters.

Here are a few running conventions we will use. The word *ring* will always mean an associative ring with 1 ($1 \neq 0$). If R is a ring, the word *R-module* will mean right R-module unless explicitly stated otherwise.

§1. Free modules, GL_n and stably free modules

If R is a field, or a division ring, then R-modules are called *vector spaces*. Classical results in linear algebra state that every vector space has a basis, and that the rank (or dimension) of a vector space is independent of the choice of basis. However, much of this fails for arbitrary rings.

As with vector spaces, a *basis* of an R-module M is a subset $\{e_i\}_{i\in I}$ such that every element of M can be expressed in a unique way as a finite sum $\sum e_i r_i$ with $r_i \in R$. We say that a module M is *free* if it has a basis. If M has a fixed ordered basis we call M a *based free module*, and define the *rank* of the based free module M to be the cardinality of its given basis. Homomorphisms between based free modules are naturally identified with matrices over R.

The canonical example of a based free module is R^n with the usual basis; it consists of n-tuples of elements of R, or "column vectors" of length n.

Unfortunately, there are rings for which $R^n \cong R^{n+t}$, $t \neq 0$. We make the following definition to avoid this pathology, referring the curious reader to the exercises for more details. (If κ is an infinite cardinal number, let $R^{(\kappa)}$ denote a free module on a basis of cardinality κ; every basis of $R^{(\kappa)}$ has cardinality κ. In particular $R^{(\kappa)}$ cannot be isomorphic to R^n for finite n. See ch.2, 5.5 of [Cohn65].)

Definition 1.1 (IBP). We say that a ring R satisfies the (right) *invariant basis property (or IBP)* if R^m and R^n are not isomorphic for $m \neq n$. In this case, the rank of a free R-module M is an invariant, independent of the choice of basis of M.

Most of the rings we will consider satisfy the invariant basis property. For example, commutative rings satisfy the invariant basis property, and so do group rings $\mathbb{Z}[G]$. This is because a ring R must satisfy the IBP if there exists a ring map $f\colon R \to F$ from R to a field or division ring F. (If R is commutative we may take $F = R/\mathfrak{m}$, where $\mathfrak{m}$ is any maximal ideal of R.) To see this, note that any basis of M maps to a basis of the vector space $V = M \otimes_R F$; since $\dim V$ is independent of the choice of basis, any two bases of M must have the same cardinality.

Our choice to use right modules dictates that we write R-module homomorphisms on the left. In particular, homomorphisms $R^n \to R^m$ may be thought of as $m \times n$ matrices with entries in R, acting on the column vectors in R^n by matrix multiplication. We write $M_n(R)$ for the ring of $n \times n$ matrices, and write $GL_n(R)$ for the group of invertible $n \times n$ matrices, *i.e.*, the automorphisms of R^n. We will usually write $R^\times$ for the group $GL_1(R)$ of *units* in R.

EXAMPLE 1.1.1. Any finite-dimensional algebra R over a field (or division ring) F must satisfy the IBP, because the rank of a free R-module M is an invariant:

$$\operatorname{rank}(M) = \dim_F(M)/\dim_F(R).$$

For a simple artinian ring R we can say even more. Classical Artin-Wedderburn theory states that $R = M_n(F)$ for some n and F, and that every right R-module M is a direct sum of copies of the (projective) R-module V consisting of row vectors over F of length n. Moreover, the number of copies of V is an invariant of M, called its *length*; the length is also $\dim_F(M)/n$ since $\dim_F(V) = n$. In this case we also have $\operatorname{rank}(M) = \operatorname{length}(M)/n = \dim_F(M)/n^2$.

There are noncommutative rings which do not satisfy the IBP, *i.e.*, which have $R^m \cong R^n$ for some $m \neq n$. Rank is not an invariant of a free module over these rings. One example is the infinite matrix ring $\operatorname{End}_F(F^\infty)$ of endomorphisms of an infinite-dimensional vector space over a field F. Another is the cone ring $C(R)$ associated to a ring R. (See the exercises.)

Unimodular rows and stably free modules

DEFINITION 1.2. An R-module P is called *stably free* (of rank $n-m$) if $P \oplus R^m \cong R^n$ for some m and n. (If R satisfies the IBP then the rank of a stably free module is easily seen to be independent of the choice of m and n.) Conversely, the kernel of any surjective $m \times n$ matrix $\sigma\colon R^n \to R^m$ is a stably free module, because a lift of a basis for R^m yields a decomposition $P \oplus R^m \cong R^n$.

This raises a question: when are stably free modules free? Over some rings every stably free module is free (fields, $\mathbb{Z}$ and the matrix rings $M_n(F)$ of Example 1.1.1 are classical cases), but in general this is not so even if R is commutative; see example 1.2.2 below.

1.2.1. The most important special case, at least for inductive purposes, is when $m = 1$, *i.e.*, $P \oplus R \cong R^n$. In this case σ is a row vector, and we call σ a *unimodular row*. It is not hard to see that the following conditions on a sequence $\sigma = (r_1, ..., r_n)$ of elements in R are equivalent for each n:

- σ is a unimodular row;
- $R^n \cong P \oplus R$, where $P = \ker(\sigma)$ and the projection $R^n \to R$ is σ;
- $R = r_1R + \cdots + r_nR$;
- $1 = r_1s_1 + \cdots + r_ns_n$ for some $s_i \in R$.

If $R^n \cong P \oplus R$ with P free, then a basis of P would yield a new basis for R^n and hence an invertible matrix g whose first row is the unimodular row $\sigma\colon R^n \to R$ corresponding to P. This gives us a general criterion: P is a free module if and only if the corresponding unimodular row may be completed to an invertible matrix. (The invertible matrix is in $GL_n(R)$ if R satisfies the IBP).

When R is commutative, every unimodular row of length 2 may be completed. Indeed, if $r_1 s_1 + r_2 s_2 = 1$, then the desired matrix is:

$$\begin{pmatrix} r_1 & r_2 \\ -s_2 & s_1 \end{pmatrix}$$

Hence $R^2 \cong R \oplus P$ implies that $P \cong R$. In §3 we will obtain a stronger result: every stably free module of rank 1 is free. The fact that R is commutative is crucial; in Ex. 1.6 we give an example of a unimodular row of length 2 which cannot be completed over $D[x, y]$, D a division ring.

Example 1.2.2. Here is an example of a unimodular row σ of length 3 which cannot be completed to an element of $GL_3(R)$. Hence $P = \ker(\sigma)$ is a rank 2 stably free module P which is not free, yet $P \oplus R \cong R^3$. Let σ be the unimodular row $\sigma = (x, y, z)$ over the commutative ring $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 = 1)$. Every element (f, g, h) of R^3 yields a vector field in 3-space ($\mathbb{R}^3$), and σ is the vector field pointing radially outward. Therefore an element in P yields a vector field in 3-space tangent to the 2-sphere S^2. If P were free, a basis of P would yield two tangent vector fields on S^2 which are linearly independent at every point of S^2 (because together with σ they span the tangent space of 3-space at every point). It is well known that this is impossible: you can't comb the hair on a coconut. Hence P cannot be free.

The following theorem describes a "stable range" in which stably free modules are free (see 2.3 for a stronger version). A proof may be found in [Bass, V.3.5], using the "stable range" condition (S_n) of Ex. 1.5 below. Example 1.2.2 shows that this range is sharp.

Bass Cancellation Theorem for Stably Free Modules 1.3. *Let R be a commutative noetherian ring of Krull dimension d. Then every stably free R-module of rank $> d$ is a free module. Equivalently, every unimodular row of length $n \geq d + 2$ may be completed to an invertible matrix.*

The study of stably free modules has a rich history, and we cannot do it justice here. An excellent source for further information is the book [Lam].

EXERCISES

1.1 *Semisimple rings.* A nonzero R-module M is called *simple* if it has no submodules other than 0 and M, and *semisimple* if it is the direct sum of simple modules. A ring R is called *semisimple* if R is a semisimple R-module. If R is semisimple, show that R is a direct sum of a finite (say n) number of simple modules. Then use the Jordan-Hölder Theorem, part of which states that the length of a semisimple module is an invariant, to show that every stably free module is free. In particular, this shows that semisimple rings satisfy the IBP. *Hint:* Observe that length$= n \cdot$ rank is an invariant of free R-modules.

1.2 (P.M. Cohn) Consider the following conditions on a ring R:
(I) R satisfies the invariant basis property (IBP);
(II) For all m and n, if $R^m \cong R^n \oplus P$ then $m \geq n$;
(III) For all n, if $R^n \cong R^n \oplus P$ then $P = 0$.
If $R \neq 0$, show that (III) $\Rightarrow$ (II) $\Rightarrow$ (I). For examples of rings satisfying (I) but not (II), resp. (II) but not (III), see [Cohn66].

1.3 Show that (III) and the following matrix conditions are equivalent:
(a) For all n, every surjection $R^n \to R^n$ is an isomorphism;
(b) For all n, and $f, g \in M_n(R)$, if $fg = 1_n$, then $gf = 1_n$ and $g \in GL_n(R)$.
Then show that commutative rings satisfy (b), hence (III).

1.4 Show that right noetherian rings satisfy condition (b) of the previous exercise. Hence they satisfy (III), and have the right invariant basis property.

1.5 *Stable Range Conditions.* We say that a ring R satisfies condition (S_n) if for every unimodular row $(r_0, r_1, ..., r_n)$ in R^{n+1} there is a unimodular row $(r'_1, ..., r'_n)$ in R^n with $r'_i = r_i - r_0 t_i$ for some $t_1, ..., t_n$ in R. The *stable range of* R, $sr(R)$, is defined to be the smallest n such that R satisfies condition (S_n). (Warning: our (S_n) is the stable range condition SR_{n+1} of [Bass].)
(a) (Vaserstein) Show that (S_n) holds for all $n \geq sr(R)$.
(b) If $sr(R) = n$, show that all stably free projective modules of rank $\geq n$ are free. Bass' Cancellation Theorem [Bass, V.3.5], which is used to prove 1.3 and 2.3 below, actually states that $sr(R) \leq d+1$ if R is a d-dimensional commutative noetherian ring, or more generally if $\mathrm{Max}(R)$ is a finite union of spaces of dimension $\leq d$.
(c) Show that $sr(R) = 1$ for every artinian ring R. Conclude that all stably free projective R-modules are free over artinian rings.
(d) Show that if I is an ideal of R then $sr(R) \geq sr(R/I)$.
(e) (Veldkamp) If $sr(R) = n$ for some n, show that R satisfies the invariant basis property (IBP). *Hint:* Consider an isomorphism $B \colon R^N \cong R^{N+n}$, and apply (S_n) to convert B into a matrix of the form $\binom{C}{0}$.

1.6 (Ojanguren-Sridharan) Let D be a division ring which is not a field. Choose $\alpha, \beta \in D$ such that $\alpha\beta - \beta\alpha \neq 0$, and show that $\sigma = (x + \alpha, y + \beta)$ is a unimodular row over $R = D[x, y]$. Let $P = \ker(\sigma)$ be the associated rank 1 stably free module; $P \oplus R \cong R^2$. Prove that P is not a free $D[x, y]$-module, using these steps:
(i) If $P \cong R^n$, show that $n = 1$. Thus we may suppose that $P \cong R$ with $1 \in R$ corresponding to a vector $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]$ with $r, s \in R$.
(ii) Show that P contains a vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $f = c_1 x + c_2 y + c_3 xy + c_4 y^2$ and $g = d_1 x + d_2 y + d_3 xy + d_4 x^2$, $(c_i, d_i \in D)$.
(iii) Show that P cannot contain any vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with f and g linear polynomials in x and y. Conclude that the vector in (i) must be quadratic, and may be taken to be of the form given in (ii).
(iv) Show that P contains a vector $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $f = \gamma_0 + \gamma_1 y + y^2$, $g = \delta_0 + \delta_1 x - \alpha y - xy$ and $\gamma_0 = \beta u^{-1} \beta u \neq 0$. This contradicts (iii), so we cannot have $P \cong R$.

1.7 *Direct sum rings.* A ring R (with unit) is called a *direct sum ring* if there is an R-module isomorphism $R \cong R^2$. This implies that $R \cong R^n$ for every finite n. Any homomorphism $R \to S$ makes S into a direct sum ring, so many direct sum rings exist. In this exercise and the next, we give some examples of direct sum rings.

For any ring R, let $R^\infty = R^{(\aleph_0)}$ be a fixed free R-module on a countably infinite basis. Then R^∞ is naturally a left module over the endomorphism ring $E = \mathrm{End}_R(R^\infty)$, and we identify E with the ring of infinite column-finite matrices.

If $R^\infty = V_1 \oplus V_2$ as a left R-module, show that $E = I_1 \oplus I_2$ for the right ideals $I_i = \{f \in E : f(R^\infty) \subseteq V_i\}$. Conversely, if $E = I_1 \oplus I_2$ as a right module, show that $R^\infty = V_1 \oplus V_2$, where $V_i = I_i \cdot R^\infty$. Conclude that E is a direct sum ring, and that $I \oplus J = E$ implies that $I \oplus E \cong E$ for every right ideal I of E.

1.8 *Cone Ring.* For any ring R, the endomorphism ring $\mathrm{End}_R(R^\infty)$ of the previous exercise contains a smaller ring, namely the subring $C(R)$ consisting of row-and-column finite matrices. The ring $C(R)$ is called the *cone ring* of R. Show that $C(R)$ is a direct sum ring.

1.9 To see why our notion of stably free module involves only finitely generated free modules, let R^∞ be the infinitely generated free module of exercise 1.7. Prove that if $P \oplus R^m \cong R^\infty$ then $P \cong R^\infty$. *Hint:* The image of R^m is contained in some $R^n \subseteq R^\infty$. Writing $R^\infty \cong R^n \oplus F$ and $Q = P \cap R^n$, show that $P \cong Q \oplus F$ and $F \cong F \oplus R^m$. This trick is a version of the Eilenberg Swindle 2.8 below.

1.10 *Excision for GL_n.* If I is a ring without unit, let $\mathbb{Z} \oplus I$ be the canonical augmented ring with underlying abelian group $\mathbb{Z} \oplus I$. Let $GL_n(I)$ denote the kernel of the map $GL_n(\mathbb{Z} \oplus I) \to GL_n(\mathbb{Z})$, and let $M_n(I)$ denote the matrices with entries in I. If $g \in GL_n(I)$ then clearly $g - 1_n \in M_n(I)$.

(i) Characterize the set of all $x \in M_n(I)$ such that $1_n + x \in GL_n(I)$.

(ii) If I is an ideal in a ring R, show that $GL_n(I)$ is the kernel of $GL_n(R) \to GL_n(R/I)$, and so is independent of the choice of R.

(iii) If $x = (x_{ij})$ is any nilpotent matrix in $M_n(I)$, such as a strictly upper triangular matrix, show that $1_n + x \in GL_n(I)$.

1.11 (Whitehead) If $g \in GL_n(R)$, verify the following identity in $GL_{2n}(R)$:

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Conclude that if $S \to R$ is a ring surjection then there is a matrix $h \in GL_{2n}(S)$ mapping to the block diagonal matrix with entries g, g^{-1} displayed above.

1.12 *Radical Ideals.* A 2-sided ideal I in R is called a *radical ideal* if $1+x$ is a unit of R for every $x \in I$, *i.e.*, if $(\forall x \in I)(\exists y \in I)(x + y + xy = 0)$. Every ring has a largest radical ideal, called the *Jacobson radical* of R; it is the intersection of the maximal left ideals of R.

(i) Show that every nil ideal is a radical ideal. (A *nil ideal* is an ideal in which every element is nilpotent.)

(ii) A ring R is *local* if it has a unique maximal 2-sided ideal $\mathfrak{m}$, and every element of $R - \mathfrak{m}$ is a unit. If R is local, show that $R/\mathfrak{m}$ is a field or division ring.

(iii) If I is a radical ideal of R, show that $M_n(I)$ is a radical ideal of $M_n(R)$ for every n. *Hint:* Use elementary row operations to diagonalize any matrix which is congruent to 1_n modulo I.

(iv) If I is a radical ideal, show that $GL_n(R) \to GL_n(R/I)$ is surjective for each n. That is, there is a short exact sequence of groups:

$$1 \to GL_n(I) \to GL_n(R) \to GL_n(R/I) \to 1.$$

(v) If I is a radical ideal, show that $sr(R) = sr(R/I)$, where sr is the stable range of Exercise 1.5. Conclude that $sr(R) = 1$ for every local ring R.

1.13 A *von Neumann regular ring* is a ring R such that for every $r \in R$ there is an $x \in R$ such that $r = rxr$. It is called *unit-regular* if for every $r \in R$ there is a *unit* $x \in R$ such that $r = rxr$. If R is von Neumann regular, show that:

(a) for every $r \in R$, $R = rR \oplus (1 - rx)R$. *Hint:* $(rx)^2 = rx$.

(b) R is unit-regular $\iff$ R has stable range 1 (in the sense of Exercise 1.5);

(c) If R is unit-regular then R satisfies condition (III) of Exercise 1.2. (The converse does not hold; see Example 5.10 of [Gdearl].)

A *rank function* on R is a set map $\rho: R \to [0,1]$ such that: (i) $\rho(0) = 0$ and $\rho(1) = 1$; (ii) $\rho(x) > 0$ if $x \neq 0$; (iii) $\rho(xy) \leq \rho(x), \rho(y)$; and (iv) $\rho(e + f) = \rho(e) + \rho(f)$ if e, f are orthogonal idempotents in A. Goodearl and Handelman proved (18.4 of [Gdearl]) that if R is a simple von Neumann ring then:

$$\text{(III) holds} \iff R \text{ has a rank function.}$$

(d) Let F be a field or division ring. Show that the matrix ring $M_n(F)$ is unit-regular, and that $\rho_n(g) = \operatorname{rank}(g)/n$ is a rank function on $M_n(F)$. Then show that the ring $\operatorname{End}_F(F^\infty)$ is von Neumann regular but not unit-regular.

(e) Consider the union R of the matrix rings $M_{n!}(F)$, where we embed $M_{n!}(F)$ in $M_{(n+1)!}(F) \cong M_{n!}(F) \otimes M_{n+1}(F)$ as $M_{n!} \otimes 1$. Show that R is a simple von Neumann regular ring, and that the union of the ρ_n of (c) gives a rank function $\rho: R \to [0,1]$ with image $\mathbb{Q} \cap [0,1]$.

(f) Show that a commutative ring R is von Neumann regular if and only if it is reduced and has Krull dimension 0. These rings are called *absolutely flat rings* by Bourbaki, since every R-module is flat. Use Exercise 1.12 to conclude that every commutative 0-dimensional ring has stable range 1 (and is unit-regular).

§2. Projective modules

DEFINITION 2.1. An R-module P is called *projective* if there exists a module Q so that the direct sum $P \oplus Q$ is free. This is equivalent to saying that P satisfies the *projective lifting property:* For every surjection $s: M \to N$ of R-modules and every map $g: P \to N$ there exists a map $f: P \to M$ so that $g = sf$.

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle\exists f}\swarrow & \downarrow g & & \\ M & \xrightarrow{s} & N & \to & 0 \end{array}$$

To see that these are equivalent, first observe that free modules satisfy this lifting property; in this case f is determined by lifting the image of a basis. To see that all projective modules satisfy the lifting property, extend g to a map from a free module $P \oplus Q$ to N and lift that. Conversely, suppose that P satisfies the projective lifting property. Choose a surjection $\pi: F \to P$ with F a free module; the lifting property splits π, yielding $F \cong P \oplus \ker(\pi)$.

If P is a projective module, then P is generated by n elements if and only if there is a decomposition $P \oplus Q \cong R^n$. Indeed, the generators give a surjection $\pi: R^n \to P$, and the lifting property yields the decomposition.

We will focus most of our attention on the category $\mathbf{P}(R)$ of finitely generated projective R-modules; the morphisms are the R-module maps. Since the direct sum of projectives is projective, $\mathbf{P}(R)$ is an additive category. We may regard $\mathbf{P}$ as a covariant functor on rings, since if $R \to S$ is a ring map then up to coherence there is an additive functor $\mathbf{P}(R) \to \mathbf{P}(S)$ sending P to $P \otimes_R S$. (Formally, there is an additive functor $\mathbf{P}'(R) \to \mathbf{P}(S)$ and an equivalence $\mathbf{P}'(R) \to \mathbf{P}(R)$; see Ex. 2.16.)

Hom and $\otimes$. If P is a projective R-module, then it is well-known that $P \otimes_R -$ is an exact functor on the category of (left) R-modules, and that $\mathrm{Hom}_R(P, -)$ is an exact functor on the category of (right) R-modules. (See [WHomo], for example.) That is, any exact sequence $0 \to L \to M \to N \to 0$ of R-modules yields exact sequences

$$0 \to P \otimes L \to P \otimes M \to P \otimes N \to 0$$

and

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0.$$

Examples 2.1.1. Of course free modules and stably free modules are projective.

(1) If F is a field (or a division ring) then every F-module (vector space) is free, but this is not so for all rings.
(2) Consider the matrix ring $R = M_n(F)$, $n > 1$. The R-module V of Example 1.1.1 is projective but not free, because $\mathrm{length}(V) = 1 < n = \mathrm{length}(R)$.
(3) *Componentwise free modules.* Another type of projective module arises for rings of the form $R = R_1 \times R_2$; both $P = R_1 \times 0$ and $Q = 0 \times R_2$ are projective but cannot be free because the element $e = (0, 1) \in R$ satisfies $Pe = 0$ yet $R^n e \neq 0$. We say that a module M is *componentwise free* if there is a decomposition $R = R_1 \times \cdots \times R_c$ and integers n_i such that $M \cong R_1^{n_1} \times \cdots \times R_c^{n_c}$. It is easy to see that all componentwise free modules are projective.
(4) *Topological Examples.* Other examples of nonfree projective modules come from topology, and will be discussed more in section 4 below. Consider the ring $R = C^0(X)$ of continuous functions $X \to \mathbb{R}$ on a compact topological space X. If $\eta\colon E \to X$ is a vector bundle then by Ex. 4.8 the set $\Gamma(E) = \{s\colon X \to E : \eta s = 1_X\}$ of continuous sections of η forms a projective R-module. For example, if T^n is the trivial bundle $\mathbb{R}^n \times X \to X$ then $\Gamma(T^n) = R^n$. I claim that if E is a nontrivial vector bundle then $\Gamma(E)$ cannot be a free R-module. To see this, observe that if $\Gamma(E)$ were free then the sections $\{s_1, ..., s_n\}$ in a basis would define a bundle map $f\colon T^n \to E$ such that $\Gamma(T^n) = \Gamma(E)$. Since the kernel and cokernel bundles of f have no nonzero sections they must vanish, and f is an isomorphism.

   When X is compact, the category $\mathbf{P}(R)$ of finitely generated projective $C^0(X)$-modules is actually equivalent to the category of vector bundles over X; this result is called *Swan's Theorem.* (See Ex. 4.9 for a proof.)

Idempotents 2.1.2. An element e of a ring R is called *idempotent* if $e^2 = e$. If $e \in R$ is idempotent then $P = eR$ is projective because $R = eR \oplus (1 - e)R$. Conversely, given any decomposition $R = P \oplus Q$, there are unique elements $e \in P$, $f \in Q$ such that $1 = e + f$ in R. By inspection, e and $f = 1 - e$ are idempotent, and $ef = fe = 0$. Thus idempotent elements of R are in 1-1 correspondence with decompositions $R \cong P \oplus Q$.

If $e \neq 0, 1$ and R is commutative then $P = eR$ cannot be free, because $P(1-e) = 0$ but $R(1-e) \neq 0$. The same is true for noetherian rings by Ex.1.4, but obviously cannot be true for rings such that $R \cong R \oplus R$; see Ex.1.2 (III).

Every finitely generated projective R-module arises from an idempotent element in a matrix ring $M_n(R)$. To see this, note that if $P \oplus Q = R^n$ then the projection-inclusion $R^n \to P \to R^n$ is an idempotent element e of $M_n(R)$. By inspection, the image $e(R^n)$ of e is P. The study of projective modules via idempotent elements can be useful, especially for rings of operators on a Banach space. (See [Ro96].)

If R is a Principal Ideal Domain (PID), such as $\mathbb{Z}$ or $F[x]$, F a field, then all projective R-modules are free. This follows from the Structure Theorem for modules over a PID (even for infinitely generated projectives).

Not many other classes of rings have all (finitely generated) projective modules free. A famous theorem of Quillen and Suslin states that if R is a polynomial ring (or a Laurent polynomial ring) over a field or a PID then all projective R-modules are free; a good reference for this is Lam's book [Lam]. In particular, if G is a free abelian group then the group ring $\mathbb{Z}[G]$ is the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}, ..., z, z^{-1}]$, and has all projectives free. In contrast, if G is a nonabelian torsion-free nilpotent group, Artamanov proved in [Art] that there are always projective $\mathbb{Z}[G]$-modules P which are stably free but not free: $P \oplus \mathbb{Z}[G] \cong (\mathbb{Z}[G])^2$.

It is an open problem to determine whether all projective $\mathbb{Z}[G]$-modules are stably free when G is a finitely presented torsion-free group. Some partial results and other examples are given in [Lam].

For our purposes, local rings form the most important class of rings with all projectives free. A ring R is called a *local ring* if R has a unique maximal (2-sided) ideal $\mathfrak{m}$, and every element of $R - \mathfrak{m}$ is a unit; $R/\mathfrak{m}$ is either a field or a division ring by Ex. 1.12.

LEMMA 2.2. *If R is a local ring, then every finitely generated projective R-module P is free. In fact $P \cong R^p$, where $p = \dim_{R/\mathfrak{m}}(P/\mathfrak{m}P)$.*

PROOF. We first observe that every element $u \in R$ invertible in $R/\mathfrak{m}$ is a unit of R, *i.e.*, $uv = vu = 1$ for some v. Indeed, by multiplying by a representative for the inverse of $\bar{u} \in R/\mathfrak{m}$ we may assume that $u \in 1 + \mathfrak{m}$. Since $\mathfrak{m}$ is the Jacobson radical of R, any element of $1 + \mathfrak{m}$ must be a unit of R.

Suppose that $P \oplus Q \cong R^n$. As vector spaces over $F = R/\mathfrak{m}$, $P/\mathfrak{m}P \cong F^p$ and $Q/\mathfrak{m}Q \cong F^q$ for some p and q. Since $F^p \oplus F^q \cong F^n$, $p + q = n$. Choose elements $\{e_1, ..., e_p\}$ of P and $\{e'_1, ..., e'_q\}$ of Q mapping to bases of $P/\mathfrak{m}P$ and $Q/\mathfrak{m}Q$. The e_i and e'_j determine a homomorphism $R^p \oplus R^q \to P \oplus Q \cong R^n$, which may be represented by a square matrix $(r_{ij}) \in M_n(R)$ whose reduction $(\bar{r}_{ij}) \in M_n(F)$ is invertible. But every such matrix (r_{ij}) is invertible over R by Exercise 1.12. Therefore $\{e_1, ..., e_p, e'_1, ..., e'_q\}$ is a basis for $P \oplus Q$, and from this it follows that P is free on basis $\{e_1, ..., e_p\}$.

REMARK 2.2.1. Even infinitely generated projective R-modules are free when R is local. See [Kap58].

COROLLARY 2.2.2. *If $\mathfrak{p}$ is a prime ideal of a commutative ring R and P is a finitely generated projective R-module, then the localization $P_\mathfrak{p}$ is isomorphic to $(R_\mathfrak{p})^n$ for some $n \geq 0$. Moreover, there is an $s \in R - \mathfrak{p}$ such that the localization of P away from s is free:*

$$(P[\frac{1}{s}]) \cong (R[\frac{1}{s}])^n.$$

In particular, $P_{\mathfrak{p}'} \cong (R_{\mathfrak{p}'})^n$ for every other prime ideal $\mathfrak{p}'$ of R not containing s.

PROOF. If $P \oplus Q = R^m$ then $P_\mathfrak{p} \oplus Q_\mathfrak{p} = (R_\mathfrak{p})^m$, so $P_\mathfrak{p}$ is a finitely generated projective $R_\mathfrak{p}$-module. Since $R_\mathfrak{p}$ is a local ring, $P_\mathfrak{p}$ is free by 2.2. Now every element of $P_\mathfrak{p}$ is of the form p/s for some $p \in P$ and $s \in R - \mathfrak{p}$. By clearing denominators, we may find an R-module homomorphism $f: R^n \to P$ which becomes an isomorphism upon localizing at $\mathfrak{p}$. As coker(f) is a finitely generated R-module which vanishes upon localization, it is annihilated by some $s \in R - \mathfrak{p}$. For this s, the map $f[\frac{1}{s}]: (R[\frac{1}{s}])^n \to P[\frac{1}{s}]$ is onto. Since $P[\frac{1}{s}]$ is projective, $(R[\frac{1}{s}])^n$ is isomorphic to the direct sum of $P[\frac{1}{s}]$ and a finitely generated $R[\frac{1}{s}]$-module M with $M_\mathfrak{p} = 0$. Since M is annihilated by some $t \in R - \mathfrak{p}$ we have

$$f[\frac{1}{st}]: \quad (R[\frac{1}{st}])^n \xrightarrow{\cong} P[\frac{1}{st}].$$

Suppose that there is a ring homomorphism $f: R \to F$ from R to a field or a division ring F. If M is any R-module (projective or not) then the *rank of M at f* is the integer $\dim_F(M \otimes_R F)$. However, the rank depends upon f, as the example $R = F \times F$, $M = F \times 0$ shows. When R is commutative, every such homomorphism factors through the field $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R, so we may consider rank(M) as a function on the set Spec(R) of prime ideals in R.

Recall that the set Spec(R) of prime ideals of R has the natural structure of a topological space in which the basic open sets are

$$D(s) = \{\mathfrak{p} \in \text{Spec}(R) : s \notin \mathfrak{p}\} \cong \text{Spec}(R[\frac{1}{s}]) \quad \text{for } s \in R.$$

DEFINITION 2.2.3 (RANK). Let R be a commutative ring. The *rank* of a finitely generated R-module M at a prime ideal $\mathfrak{p}$ of R is $\text{rank}_\mathfrak{p}(M) = \dim_{k(\mathfrak{p})} M \otimes_R k(\mathfrak{p})$. Since $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong k(\mathfrak{p})^{\text{rank}_\mathfrak{p}(M)}$, $\text{rank}_\mathfrak{p}(M)$ is the minimal number of generators of $M_\mathfrak{p}$.

If P is a finitely generated projective R-module then $\text{rank}(P): \mathfrak{p} \mapsto \text{rank}_\mathfrak{p}(P)$ is a *continuous* function from the topological space Spec(R) to the discrete topological space $\mathbb{N} \subset \mathbb{Z}$, as we see from Corollary 2.2.2. In this way, we shall view rank(P) as an element of the two sets $[\text{Spec}(R), \mathbb{N}]$ and $[\text{Spec}(R), \mathbb{Z}]$ of continuous maps from Spec(R) to $\mathbb{N}$ and to $\mathbb{Z}$, respectively.

We say that P has *constant rank n* if $n = \text{rank}_\mathfrak{p}(P)$ is independent of $\mathfrak{p}$. If Spec(R) is topologically connected, every continuous function $\text{Spec}(R) \to \mathbb{N}$ must be constant, so every finitely generated projective R-module has constant rank. For example, suppose that R is an integral domain with field of fractions F; then

Spec(R) is connected, and every finitely generated projective R-module P has constant rank: $\mathrm{rank}(P) = \dim_F(P \otimes_R F)$. Conversely, if a projective P has constant rank, then it is finitely generated; see Ex. 2.13 and 2.14.

If a module M is not projective, $\mathrm{rank}(M)$ need not be a continuous function on Spec(R), as the example $R = \mathbb{Z}$, $M = \mathbb{Z}/p$ shows.

Componentwise free modules 2.2.4. Every continuous $f\colon \mathrm{Spec}(R) \to \mathbb{N}$ induces a decomposition of Spec(R) into the disjoint union of closed subspaces $f^{-1}(n)$. In fact, f takes only finitely many values (say $n_1, \dots, n_c$), and it is possible to write R as $R_1 \times \cdots \times R_c$ such that $f^{-1}(n_i)$ is homeomorphic to Spec(R_i). (See Ex. 2.4.) Given such a function f, form the componentwise free R-module:

$$R^f = R_1^{n_1} \times \cdots \times R_c^{n_c}.$$

Clearly R^f has constant rank n_i at every prime in Spec(R_i) and $\mathrm{rank}(R^f) = f$. For $n \geq \max\{n_i\}$, $R^f \oplus R^{n-f} = R^n$, so R^f is a finitely generated projective R-module. Hence continuous functions $\mathrm{Spec}(R) \to \mathbb{N}$ are in 1-1 correspondence with componentwise free modules.

The following variation allows us to focus on projective modules of constant rank in many arguments. Suppose that P is a finitely generated projective R-module, so that $\mathrm{rank}(P)$ is a continuous function. Let $R \cong R_1 \times \cdots \times R_c$ be the corresponding decomposition of R. Then each component $P_i = P \otimes_R R_i$ of P is a projective R_i-module of constant rank and there is an R-module isomorphism $P \cong P_1 \times \cdots \times P_c$.

The next theorem allows us to further restrict our attention to projective modules of rank $\leq \dim(R)$. Its proof may be found in [Bass, IV]. We say that two R-modules M, M' are *stably isomorphic* if $M \oplus R^m \cong M' \oplus R^m$ for some $m \geq 0$.

Bass-Serre Cancellation Theorem 2.3. *Let R be a commutative noetherian ring of Krull dimension d, and let P be a projective R-module of constant rank $n > d$.*
(a) (Serre) $P \cong P_0 \oplus R^{n-d}$ for some projective R-module P_0 of constant rank d.
(b) (Bass) If P is stably isomorphic to P' then $P \cong P'$.
(c) (Bass) For all M, M', if $P \oplus M$ is stably isomorphic to M' then $P \oplus M \cong M'$.

Remark 2.3.1. If P is a projective module whose rank is not constant, then $P \cong P_1 \times \cdots \times P_c$ for some decomposition $R \cong R_1 \times \cdots \times R_c$. (See Ex. 2.4.) In this case, we can apply the results in 2.3 to each P_i individually. The reader is invited to phrase 2.3 in this generality.

Locally Free Modules 2.4. Let R be commutative. An R-module M is called *locally free* if for every prime ideal $\mathfrak{p}$ of R there is an $s \in R - \mathfrak{p}$ so that $M[\frac{1}{s}]$ is a free module. We saw in Corollary 2.2.2 that finitely generated projective R-modules are locally free. In fact, the following are equivalent:

(1) M is a finitely generated projective R-module;
(2) M is a locally free R-module of finite rank (*i.e.*, $\mathrm{rank}_{\mathfrak{p}}(M) < \infty$ for all prime ideals $\mathfrak{p}$);
(3) M is a finitely presented R-module, and for every prime ideal $\mathfrak{p}$ of R:

$$M_{\mathfrak{p}} \text{ is a free } R_{\mathfrak{p}}\text{-module.}$$

Proof. The implication (2) ⇒ (3) follows from the theory of faithfully flat descent; a proof is in [[]II§5.2, Thm. 1]B-AC. Nowadays we would say that M is coherent (locally finitely presented), hence finitely presented; cf. [Hart, II], [B-AC,0_I(1.4.3)]. To see that (3) ⇒ (1), note that finite presentation gives an exact sequence

$$R^m \to R^n \xrightarrow{\varepsilon} M \to 0.$$

We claim that the map $\varepsilon^*\colon \mathrm{Hom}_R(M,R^n) \to \mathrm{Hom}_R(M,M)$ is onto. To see this, recall that being onto is a local property; locally $\varepsilon^*_{\mathfrak{p}}$ is $\mathrm{Hom}(M_{\mathfrak{p}},R^n_{\mathfrak{p}}) \to \mathrm{Hom}(M_{\mathfrak{p}},M_{\mathfrak{p}})$. This is a split surjection because $M_{\mathfrak{p}}$ is projective and $\varepsilon_{\mathfrak{p}}\colon R^n_{\mathfrak{p}} \to M_{\mathfrak{p}}$ is a split surjection. If $s\colon M \to R^n$ is such that $\varepsilon^*(s) = \varepsilon s$ is id_M, then s makes M a direct summand of R^n, and M is a finitely generated projective module.

Open Patching Data 2.5. It is sometimes useful to be able to build projective modules by patching free modules. The following data suffices. Suppose that $s_1, ..., s_c \in R$ form a unimodular row, *i.e.*, $s_1R + \cdots + s_cR = R$. Then $\mathrm{Spec}(R)$ is covered by the open sets $D(s_i) \cong \mathrm{Spec}(R[\frac{1}{s_i}])$. Suppose we are given $g_{ij} \in GL_n(R[\frac{1}{s_is_j}])$ with $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ in $GL_n(R[\frac{1}{s_is_js_k}])$ for every i,j,k. Then

$$P = \{(x_1, ..., x_c) \in \prod_{i=1}^{c} (R[\frac{1}{s_i}])^n : g_{ij}(x_j) = x_i \text{ in } R[\frac{1}{s_is_j}]^n \text{ for all } \ i,j\}$$

is a finitely generated projective R-module by 2.4, because each $P[\frac{1}{s_i}]$ is isomorphic to $R[\frac{1}{s_i}]^n$.

Milnor Squares 2.6. Another type of patching arises from an ideal I in R and a ring map $f\colon R \to S$ such that I is mapped isomorphically onto an ideal of S, which we also call I. In this case R is the "pullback" of S and R/I :

$$R = \{(\bar{r}, s) \in (R/I) \times S : \bar{f}(\bar{r}) = s \text{ modulo } I\};$$

the square

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar{f}} & S/I \end{array}$$

is called a *Milnor square*, because their importance for patching was emphasized by J. Milnor in [Milnor].

One special kind of Milnor square is the *conductor square*. This arises when R is commutative and S is a finite extension of R with the same total ring of fractions. (S is often the integral closure of R). The ideal I is chosen to be the *conductor ideal*, *i.e.*, the largest ideal of S contained in R, which is just $I = \{x \in R : xS \subset R\} = \mathrm{ann}_R(S/R)$. If S is reduced then I cannot lie in any minimal prime of R or S, so the rings R/I and S/I have lower Krull dimension.

Given a Milnor square, we can construct an R-module $M = (M_1, g, M_2)$ from the following "descent data": an S-module M_1, an R/I-module M_2 and a S/I-module isomorphism $g\colon M_2 \otimes_{R/I} S/I \cong M_1/IM_1$. In fact M is the kernel of the R-module map

$$M_1 \times M_2 \to M_1/IM_1, \qquad (m_1, m_2) \mapsto \bar{m}_1 - g(\bar{f}(m_2)).$$

We call M the R-module *obtained by patching* M_1 and M_2 together along g.

An important special case is when we patch S^n and $(R/I)^n$ together along a matrix $g \in GL_n(S/I)$. For example, R is obtained by patching S and R/I together along $g = 1$. We will return to this point when we study $K_1(R)$ and $K_0(R)$.

Here is Milnor's result.

Milnor Patching Theorem 2.7. *In a Milnor square,*

(1) *If P is obtained by patching together a finitely generated projective S-module P_1 and a finitely generated projective R/I-module P_2, then P is a finitely generated projective R-module;*
(2) *$P \otimes_R S \cong P_1$ and $P/IP \cong P_2$;*
(3) *Every finitely generated projective R-module arises in this way;*
(4) *If P is obtained by patching free modules along $g \in GL_n(S/I)$, and Q is obtained by patching free modules along g^{-1}, then $P \oplus Q \cong R^{2n}$.*

We shall prove part (3) here; the rest of the proof will be described in Exercise 2.8. If M is any R-module, the Milnor square gives a natural map from M to the R-module M' obtained by patching $M_1 = M \otimes_R S$ and $M_2 = M \otimes_R (R/I) = M/IM$ along the canonical isomorphism

$$(M/IM) \otimes_{R/I} (S/I) \cong M \otimes_R (S/I) \cong (M \otimes_R S)/I(M \otimes_R S).$$

Tensoring M with $0 \to R \to (R/I) \oplus S \to S/I \to 0$ yields an exact sequence

$$\mathrm{Tor}_1^R(M, S/I) \to M \to M' \to 0,$$

so in general M' is just a quotient of M. However, if M is projective, the Tor-term is zero and $M \cong M'$. Thus every projective R-module may be obtained by patching, as (3) asserts.

Remark 2.7.1. Other examples of patching may be found in [Landsbg].

Eilenberg Swindle 2.8. The following "swindle," discovered by Eilenberg, explains why we restrict our attention to finitely generated projective modules. Let R^∞ be an infinitely generated free module. If $P \oplus Q = R^n$, then

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong R^\infty.$$

Moreover $R^\infty \cong R^\infty \oplus R^\infty$, and if $P \oplus R^m \cong R^\infty$ then $P \cong R^\infty$ (see Ex. 1.9). Here are a few more facts about infinitely generated projective modules:

• (Bass) If R is noetherian, every infinitely generated projective module P is free, unless there is an ideal I such that P/IP has fewer generators than P;

• (Kaplansky) Every infinitely generated projective module is the direct sum of countably generated projective modules;

• (Kaplansky) There are infinitely generated projectives P whose rank is finite but $\mathrm{rank}(P)$ is not continuous on $\mathrm{Spec}(R)$. (See Ex. 2.15.)

EXERCISES

2.1 *Radical ideals.* Let I be a radical ideal in R (Exercise 1.12). If P_1, P_2 are finitely generated projective R-modules such that $P_1/IP_1 \cong P_2/IP_2$, show that $P_1 \cong P_2$. *Hint:* Modify the proof of 2.2, observing that $\mathrm{Hom}(P,Q) \to \mathrm{Hom}(P/I, Q/I)$ is onto.

2.2 *Idempotent lifting.* Let I be a nilpotent ideal, or more generally an ideal that is *complete* in the sense that every Cauchy sequence $\sum_{n=1}^{\infty} x_n$ with $x_n \in I^n$ converges to a unique element of I. Show that there is a bijection between the isomorphism classes of finitely generated projective R-modules and the isomorphism classes of finitely generated projective R/I-modules. To do this, use Ex. 2.1 and proceed in two stages:

(i) Show that every idempotent $\bar{e} \in R/I$ is the image of an idempotent $e \in R$, and that any other idempotent lift is ueu^{-1} for some $u \in 1+I$. *Hint:* it suffices to suppose that $I^2 = 0$ (consider the tower of rings R/I^n). If r is a lift of $\bar{e}$, consider elements of the form $e = r + rxr + (1-r)y(1-r)$ and $(1+xe)e(1+xe)^{-1}$.

(ii) By applying (i) to $M_n(R)$, show that every finitely generated projective R/I-module is of the form P/IP for some finitely generated projective R-module P.

2.3 Let e, e_1 be idempotents in $M_n(R)$ defining projective modules P and P_1. If $e_1 = geg^{-1}$ for some $g \in GL_n(R)$, show that $P \cong P_1$. Conversely, if $P \cong P_1$ show that for some $g \in GL_{2n}(R)$:

$$\begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} = g \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} g^{-1}.$$

2.4 *Rank.* If R is a commutative ring and $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ is a continuous function, show that we can write $R = R_1 \times \cdots \times R_c$ in such a way that $\mathrm{Spec}(R)$ is the disjoint union of the $\mathrm{Spec}(R_i)$, and f is constant on each of the components $\mathrm{Spec}(R_i)$ of R. To do this, proceed as follows.

(i) Show that $\mathrm{Spec}(R)$ is quasi-compact and conclude that f takes on only finitely many values, say $n_1, ..., n_c$. Each $V_i = f^{-1}(n_i)$ is a closed and open subset of $\mathrm{Spec}(R)$ because $\mathbb{Z}$ is discrete.

(ii) It suffices to suppose that R is reduced, *i.e.*, has no non-zero nilpotent elements. To see this, let $\mathfrak{N}$ be the ideal of all nilpotent elements in R, so $R/\mathfrak{N}$ is reduced. Since $\mathrm{Spec}(R) \cong \mathrm{Spec}(R/\mathfrak{N})$, we may apply idempotent lifting (Ex. 2.2).

(iii) Let I_i be the ideal defining V_i, i.e., $I_i = \cap\{\mathfrak{p} : \mathfrak{p} \in V_i\}$. If R is reduced, show that $I_1 + \cdots + I_c = R$ and that for every $i \neq j$ $\;I_i \cap I_j = \varnothing$. Conclude using the Chinese Remainder Theorem, which says that $R \cong \prod R_i$.

2.5 Show that the following are equivalent for every commutative ring R:

(1) $\mathrm{Spec}(R)$ is topologically connected
(2) Every finitely generated projective R-module has constant rank
(3) R has no idempotent elements except 0 and 1.

2.6 *Dual Module.* If P is a projective R-module, show that $\check{P} = \mathrm{Hom}_R(P,R)$ (its *dual module*) is a projective R^{op}-module, where R^{op} is R with multiplication reversed.

Now suppose that R is commutative, so that $R = R^{op}$. Show that $\mathrm{rank}(P) = \mathrm{rank}(\check{P})$ as functions from $\mathrm{Spec}(R)$ to $\mathbb{Z}$. The image τ_P of $\check{P} \otimes P \to R$ is called the *trace* of P; show that $\tau_P^2 = \tau_P$, and that for $\mathfrak{p} \in \mathrm{Spec}(R)$, $P_\mathfrak{p} \neq 0$ if and only if $\tau_P \not\subseteq \mathfrak{p}$.

2.7 *Tensor Product.* Let P and Q be projective modules over a commutative ring R. Show that the tensor product $P \otimes_R Q$ is also a projective R-module, and is finitely generated if P and Q are. Finally, show that

$$\mathrm{rank}(P \otimes_R Q) = \mathrm{rank}(P) \cdot \mathrm{rank}(Q).$$

2.8 *Milnor Patching.* In this exercise we prove the Milnor Patching Theorem 2.7, that any R-module obtained by patching finitely generated projective modules over S and R/I in a Milnor square is a finitely generated projective R-module. Prove the following:

(i) If $g \in GL_n(S/I)$ is the image of a matrix in either $GL_n(S)$ or $GL_n(R/I)$, the patched module $P = (S^n, g, (R/I)^n)$ is a free R-module.

(ii) Show that $(P_1, g, P_2) \oplus (Q_1, h, Q_2) \cong (P_1 \oplus Q_1, \left(\begin{smallmatrix} g & 0 \\ 0 & h \end{smallmatrix}\right), P_2 \oplus Q_2)$.

(iii) If $g \in GL_n(S/I)$, let M be the module obtained by patching S^{2n} and $(R/I)^{2n}$ together along the matrix $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \in GL_{2n}(S/I)$. Use Ex. 1.11 to prove that $M \cong R^{2n}$. This establishes Theorem 2.7, part (4).

(iv) Given $P_1 \oplus Q_1 \cong S^n$, $P_2 \oplus Q_2 \cong (R/I)^n$ and isomorphisms $P_1/IP_1 \cong P_2 \otimes S/I$, $Q_1/IQ_1 \cong Q_2 \otimes S/I$, let P and Q be the R-modules obtained by patching the P_i and Q_i together. By (ii), $P \oplus Q$ is obtained by patching S^n and $(R/I)^n$ together along some $g \in GL_n(S/I)$. Use (iii) to show that P and Q are finitely generated projective.

(v) If $P_1 \oplus Q_1 \cong S^m$ and $P_2 \oplus Q_2 \cong (R/I)^n$, and $g\colon P_1/IP_1 \cong P_2 \otimes S/I$, show that $(Q_1 \oplus S^n) \otimes S/I$ is isomorphic to $(R/I^m \oplus Q_2) \otimes S/I$. By (iv), this proves that (P_1, g, P_2) is finitely generated projective, establishing part (1) of Theorem 2.7.

(vi) Prove part (2) of Theorem 2.7 by analyzing the above steps.

2.9 Consider a Milnor square (2.6). Let P_1, Q_1 be finitely generated projective S-modules, and P_2, Q_2 be finitely generated projective R/I-modules such that there are isomorphisms $g\colon P_2 \otimes S/I \cong P_1/IP_1$ and $h\colon Q_2 \otimes S/I \cong Q_1/IQ_1$.

(i) If $f\colon Q_2 \otimes S/I \cong P_1/IP_1$, show that $(P_1, g, P_2) \oplus (Q_1, h, Q_2)$ is isomorphic to $(Q_1, gf^{-1}h, P_2) \oplus (P_1, f, Q_2)$. *Hint:* Use Ex. 2.8 and the decomposition

$$\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} gf^{-1}h & 0 \\ 0 & f \end{pmatrix} \begin{pmatrix} h^{-1}f & 0 \\ 0 & f^{-1}h \end{pmatrix}.$$

(ii) Conclude that $(S^n, g, R/I^n) \oplus (S^n, h, R/I^n) \cong (S^n, gh, R/I^n) \oplus R^n$.

2.10 Suppose P, Q are modules over a commutative ring R such that $P \otimes Q \cong R^n$ for some $n \neq 0$. Show that P and Q are finitely generated projective R-modules. *Hint:* Find a generating set $\{p_i \otimes q_i | i = 1, ..., m\}$ for $P \otimes Q$; the $p_i \otimes q_j \otimes p_k$ generate $P \otimes Q \otimes P$. Show that $\{p_i\}$ define a split surjection $R^m \to P$.

2.11 Let M be a finitely generated module over a commutative ring R. Show that the following are equivalent for every n:

(1) M is a finitely generated projective module of constant rank n

(2) $M_\mathfrak{p} \cong R_\mathfrak{p}^n$ for every prime ideal $\mathfrak{p}$ of R.

Conclude that in Lemma 2.4 we may add:

(4) M is finitely generated, $M_{\mathfrak{p}}$ is free for every prime ideal $\mathfrak{p}$ of R, and $\text{rank}(M)$ is a continuous function on $\text{Spec}(R)$.

2.12 If $f: R \to S$ is a homomorphism of commutative rings, there is a continuous map $f^*: \text{Spec}(S) \to \text{Spec}(R)$ sending $\mathfrak{p}$ to $f^{-1}(\mathfrak{p})$. If P is a finitely generated projective R-module, show that $\text{rank}(P \otimes_R S)$ is the composition of f^* and $\text{rank}(P)$. In particular, show that if P has constant rank n, then so does $P \otimes_R S$.

2.13 If P is a projective module of constant rank 1, show that P is finitely generated. *Hint:* Show that the trace $\tau_P = R$, and write $1 = \sum f_i(x_i)$.

2.14 If P is a projective module of constant rank r, show that P is finitely generated. *Hint:* Use Ex. 2.13 to show that $\wedge^r(P)$ is finitely generated.

2.15 (Kaplansky) Here is an example of an infinitely generated projective module whose rank is not continuous. Let R be the ring of continuous functions $f: [0,1] \to \mathbb{R}$ on the unit interval and I the ideal of all functions f which vanish on some neighborhood $[0, \varepsilon)$ of 0. Show that I is a projective R-module, yet $\text{rank}(I): \text{Spec}(R) \to \{0,1\}$ is not continuous, so I is neither finitely generated nor free. We remark that every finitely generated projective R-module is free; this follows from Swan's Theorem, since every vector bundle on $[0,1]$ is trivial (by 4.6.1 below).

Hint: Show that the functions $f_n = \max\{0, t - \frac{1}{n}\}$ generate I, and construct a splitting to the map $R^\infty \to I$. To see that $\text{rank}(I)$ is not continuous, consider the rank of I at the primes $\mathfrak{m}_t = \{f \in R : f(t) = 0\}$, $0 \le t \le 1$.

2.16 *Kleisli rectification.* Fix a small category of rings $\mathcal{R}$. By a *big* projective R-module we will mean the choice of a finitely generated projective S-module P_S for each morphism $R \to S$ in $\mathcal{R}$, equipped with an isomorphism $P_S \otimes_S T \to P_T$ for every $S \to T$ over R such that: (i) to the identity of each S we associate the identity of P_S, and (ii) to each commutative triangle of algebras we have a commutative triangle of modules. Let $\mathbf{P}'(R)$ denote the category of big projective R-modules. Show that the forgetful functor $\mathbf{P}'(R) \to \mathbf{P}(R)$ is an equivalence, and that $R \mapsto \mathbf{P}'(R)$ is a contravariant functor from $\mathcal{R}$ to exact categories. In particular, $\mathbf{P}'(R) \to \mathbf{P}(S)$ is an additive functor for each $R \to S$.

§3. The Picard Group of a commutative ring

An *algebraic line bundle* L over a commutative ring R is just a finitely generated projective R-module of constant rank 1. The name comes from the fact that if R is the ring of continuous functions on a compact space X, then a topological line bundle (vector bundle which is locally $\mathbb{R} \times X \to X$) corresponds to an algebraic line bundle by Swan's Theorem (see example 2.1.1(4) or Ex. 4.9 below).

The tensor product $L \otimes_R M \cong M \otimes_R L$ of line bundles is again a line bundle (by Ex. 2.7), and $L \otimes_R R \cong L$ for all L. Thus up to isomorphism the tensor product is a commutative associative operation on line bundles, with identity element R.

Lemma 3.1. *If L is a line bundle, then the dual module $\check{L} = \text{Hom}_R(L, R)$ is also a line bundle, and $\check{L} \otimes_R L \cong R$.*

Proof. Since $\text{rank}(\check{L}) = \text{rank}(L) = 1$ by Ex. 2.6, $\check{L}$ is a line bundle. Consider the evaluation map $\check{L} \otimes_R L \to R$ sending $f \otimes x$ to $f(x)$. If $L \cong R$, this map is clearly an isomorphism. Therefore for every prime ideal $\mathfrak{p}$ the localization

$$(\check{L} \otimes_R L)_{\mathfrak{p}} = (L_{\mathfrak{p}})\check{\ } \otimes_{R_{\mathfrak{p}}} L_{\mathfrak{p}} \to R_{\mathfrak{p}}$$

is an isomorphism. Since being an isomorphism is a local property of an R-module homomorphism, the evaluation map must be an isomorphism.

Definition. the *Picard group* $\mathrm{Pic}(R)$ of a commutative ring R is the set of isomorphism classes of line bundles over R. As we have seen, the tensor product $\otimes_R$ endows $\mathrm{Pic}(R)$ with the structure of an abelian group, the identity element being $[R]$ and the inverse being $L^{-1} = \check{L}$.

Proposition 3.2. Pic *is a functor from commutative rings to abelian groups. That is, if* $R \to S$ *is a ring homomorphism then* $\mathrm{Pic}(R) \to \mathrm{Pic}(S)$ *is a homomorphism sending* L *to* $L \otimes_R S$*.*

Proof. If L is a line bundle over R, then $L \otimes_R S$ is a line bundle over S (see Ex. 2.12), so $\otimes_R S$ maps $\mathrm{Pic}(R)$ to $\mathrm{Pic}(S)$. The natural isomorphism $(L \otimes_R M) \otimes_R S \cong (L \otimes_R S) \otimes_S (M \otimes_R S)$, valid for all R-modules L and M, shows that $\otimes_R S$ is a group homomorphism.

Lemma 3.3. *If* L *is a line bundle, then* $\mathrm{End}_R(L) \cong R$*.*

Proof. Multiplication by elements in R yields a map from R to $\mathrm{End}_R(L)$. As it is locally an isomorphism, it must be an isomorphism.

Determinant line bundle of a projective module

If M is any module over a commutative ring R and $k \geq 0$, the k^{th} *exterior power* $\wedge^k M$ is the quotient of the k-fold tensor product $M \otimes \cdots \otimes M$ by the submodule generated by terms $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for some $i \neq j$. By convention, $\wedge^0 M = R$ and $\wedge^1 M = M$. Here are some classical facts; see [B-AC, ch. 2].

(i) $\wedge^k(R^n)$ is the free module of rank $\binom{n}{k}$ generated by terms $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$. In particular, $\wedge^n(R^n) \cong R$ on $e_1 \wedge \cdots \wedge e_n$.

(ii) If $R \to S$ is a ring map, there is a natural isomorphism $(\wedge^k M) \otimes_R S \cong \wedge^k(M \otimes_R S)$, the first $\wedge^k$ being taken over R and the second being taken over S. In particular, $\mathrm{rank}(\wedge^k M) = \binom{\mathrm{rank}\, M}{k}$ as functions from $\mathrm{Spec}(R)$ to $\mathbb{N}$.

(iii) *(Sum Formula)* There is a natural isomorphism

$$\wedge^k(P \oplus Q) \cong \bigoplus_{i=0}^{k} (\wedge^i P) \otimes (\wedge^{k-i} Q).$$

If P is a projective module of constant rank n, then $\wedge^k P$ is a finitely generated projective module of constant rank $\binom{n}{k}$, because $\wedge^k P$ is locally free: if $P[\frac{1}{s}] \cong (R[\frac{1}{s}])^n$ then $(\wedge^k P)[\frac{1}{s}] \cong (R[\frac{1}{s}])^{\binom{n}{k}}$. In particular, $\wedge^n P$ is a line bundle, and $\wedge^k P = 0$ for $k > n$. We write $\det(P)$ for $\wedge^n P$, and call it the *determinant line bundle* of P.

If the rank of a projective module P is not constant, we define the determinant line bundle $\det(P)$ componentwise, using the following recipe. From §2 we find a decomposition $R \cong R_1 \times \cdots \times R_c$ so that $P \cong P_1 \times \cdots \times P_c$ and each P_i has constant rank n_i as an R_i-module. We then define $\det(P)$ to be $(\wedge^{n_1} P_1) \times \cdots \times (\wedge^{n_c} P_c)$; clearly $\det(P)$ is a line bundle on R. If P has constant rank n, this agrees with our above definition: $\det(P) = \wedge^n P$.

As the name suggests, the determinant line bundle is related to the usual determinant of a matrix. An $n \times n$ matrix g is just an endomorphism of R^n, so it induces an endomorphism $\wedge^n g$ of $\wedge^n R^n \cong R$. By inspection, $\wedge^n g$ is multiplication by $\det(g)$.

Using the determinant line bundle, we can also take the determinant of an endomorphism g of a finitely generated projective R-module P. By the naturality of $\wedge^n$, g induces an endomorphism $\det(g)$ of $\det(P)$. By Lemma 3.3, $\det(g)$ is an element of R, acting by multiplication; we call $\det(g)$ the *determinant* of the endomorphism g.

Here is an application of the determinant construction. Let L, L' be stably isomorphic line bundles. That is, $P = L \oplus R^n \cong L' \oplus R^n$ for some n. The Sum Formula (iii) shows that $\det(P) = L$, and $\det(P) = L'$, so $L \cong L'$. Taking $L' = R$, this shows that R is the only stably free line bundle. It also gives the following slight improvement upon the Cancellation Theorem 2.3 for 1-dimensional rings:

PROPOSITION 3.4. *Let R be a commutative noetherian 1-dimensional ring. Then all finitely generated projective R-modules are completely classified by their rank and determinant. In particular, every finitely generated projective R-module P of rank ≥ 1 is isomorphic to $L \oplus R^f$, where $L = \det(P)$ and $f = \operatorname{rank}(P) - 1$.*

Invertible Ideals

When R is a commutative integral domain (=*domain*), we can give a particularly nice interpretation of $\operatorname{Pic}(R)$, using the following concepts. Let F be the field of fractions of R; a *fractional ideal* is a nonzero R-submodule I of F such that $I \subseteq fR$ for some $f \in F$. If I and J are fractional ideals then their product $IJ = \{\sum x_i y_i : x_i \in I, y_i \in J\}$ is also a fractional ideal, and the set $\operatorname{Frac}(R)$ of fractional ideals becomes an abelian monoid with identity element R. A fractional ideal I is called *invertible* if $IJ = R$ for some other fractional ideal J; invertible ideals are sometimes called *Cartier divisors*. The set of invertible ideals is therefore an abelian group, and one writes $\operatorname{Cart}(R)$ or $\operatorname{Pic}(R, F)$ for this group.

If $f \in F^\times$, the fractional ideal $\operatorname{div}(f) = fR$ is invertible because $(fR)(f^{-1}R) = R$; invertible ideals of this form are called *principal divisors*. Since $(fR)(gR) = (fg)R$, the function $\operatorname{div}\colon F^\times \to \operatorname{Cart}(R)$ is a group homomorphism.

This all fits into the following overall picture (see Ex. 3.7 for a generalization).

PROPOSITION 3.5. *If R is a commutative integral domain, every invertible ideal is a line bundle, and every line bundle is isomorphic to an invertible ideal. If I and J are fractional ideals, and I is invertible, then $I \otimes_R J \cong IJ$. Finally, there is an exact sequence of abelian groups:*

$$1 \to R^\times \to F^\times \xrightarrow{\operatorname{div}} \operatorname{Cart}(R) \to \operatorname{Pic}(R) \to 0.$$

PROOF. If I and J are invertible ideals such that $IJ \subseteq R$, then we can interpret elements of J as homomorphisms $I \to R$. If $IJ = R$ then we can find $x_i \in I$ and

$y_i \in J$ so that $x_1y_1 + \cdots + x_ny_n = 1$. The $\{x_i\}$ assemble to give a map $R^n \to I$ and the $\{y_i\}$ assemble to give a map $I \to R^n$. The composite $I \to R^n \to I$ is the identity, because it sends $r \in I$ to $\sum x_iy_ir = r$. Thus I is a summand of R^n, *i.e.*, I is a finitely generated projective module. As R is an integral domain and $I \subseteq F$, $\mathrm{rank}(I)$ is the constant $\dim_F(I \otimes_R F) = \dim_F(F) = 1$. Hence I is a line bundle.

This construction gives a set map $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$; to show that it is a group homomorphism, it suffices to show that $I \otimes_R J \cong IJ$ for invertible ideals. Suppose that I is a submodule of F which is also a line bundle over R. As I is projective, $I \otimes_R -$ is an exact functor. Thus if J is an R-submodule of F then $I \otimes_R J$ is a submodule of $I \otimes_R F$. The map $I \otimes_R F \to F$ given by multiplication in F is an isomorphism because I is locally free and F is a field. Therefore the composite

$$I \otimes_R J \subseteq I \otimes_R F \xrightarrow{\text{multiply}} F$$

sends $\sum x_i \otimes y_i$ to $\sum x_iy_i$. Hence $I \otimes_R J$ is isomorphic to its image $IJ \subseteq F$. This proves the third assertion.

The kernel of $\mathrm{Cart}(R) \to \mathrm{Pic}(R)$ is the set of invertible ideals I having an isomorphism $I \cong R$. If $f \in I$ corresponds to $1 \in R$ under such an isomorphism then $I = fR = \mathrm{div}(f)$. This proves exactness of the sequence at $\mathrm{Cart}(R)$.

Clearly the units $R^\times$ of R inject into $F^\times$. If $f \in F^\times$ then $fR = R$ if and only if $f \in R$ and f is in no proper ideal, *i.e.*, if and only if $f \in R^\times$. This proves exactness at $R^\times$ and $F^\times$.

Finally, we have to show that every line bundle L is isomorphic to an invertible ideal of R. Since $\mathrm{rank}(L) = 1$, there is an isomorphism $L \otimes_R F \cong F$. This gives an injection $L \cong L \otimes_R R \subset L \otimes_R F \cong F$, *i.e.*, an isomorphism of L with an R-submodule I of F. Since L is finitely generated, I is a fractional ideal. Choosing an isomorphism $\check{L} \cong J$, Lemma 3.1 yields

$$R \cong L \otimes_R \check{L} \cong I \otimes_R J \cong IJ.$$

Hence $IJ = fR$ for some $f \in F^\times$, and $I(f^{-1}J) = R$, so I is invertible.

Dedekind domains

Historically, the most important applications of the Picard group have been for Dedekind domains. A *Dedekind domain* is a commutative integral domain which is noetherian, integrally closed and has Krull dimension 1.

There are many equivalent definitions of Dedekind domain in the literature. Here is another: an integral domain R is Dedekind if and only if every fractional ideal of R is invertible. In a Dedekind domain every nonzero ideal (and fractional ideal) can be written uniquely as a product of prime ideals $\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$. Therefore $\mathrm{Cart}(R)$ is the free abelian group on the set of (nonzero) prime ideals of R, and $\mathrm{Pic}(R)$ is the set of isomorphism classes of (actual) ideals of R.

Another property of Dedekind domains is that every finitely generated torsionfree R-module M is projective. To prove this fact we use induction on $\mathrm{rank}_0(M) = \dim_F(M \otimes F)$, the case $\mathrm{rank}_0(M) = 0$ being trivial. Set $\mathrm{rank}_0(M) = n + 1$. As M is torsionfree, it is a submodule of $M \otimes F \cong F^{n+1}$. The image of M under any nonzero coordinate projection $F^{n+1} \to F$ is a fractional ideal I_0. As I_0 is invertible, the projective lifting property for I_0 shows that $M \cong M' \oplus I_0$ with $\mathrm{rank}_0(M') = n$. By induction, $M \cong I_0 \oplus \cdots \oplus I_n$ is a sum of ideals. By Propositions 3.4 and 3.5, $M \cong I \oplus R^n$ for the invertible ideal $I = \det(M) = I_0 \cdots I_n$.

EXAMPLES. Here are some particularly interesting classes of Dedekind domains.

• A *principal ideal domain* (or PID) is a domain R in which every ideal is rR for some $r \in R$. Clearly, these are just the Dedekind domains with $\mathrm{Pic}(R) = 0$. Examples of PID's include $\mathbb{Z}$ and polynomial rings $k[x]$ over a field k.

• A *discrete valuation domain* (or DVR) is a local Dedekind domain. By Lemma 2.2, a DVR is a PID R with a unique maximal ideal $M = \pi R$. Fixing π, it isn't hard to see that every ideal of R is of the form $\pi^i R$ for some $i \geq 0$. Consequently every fractional ideal of R can be written as $\pi^i R$ for a unique $i \in \mathbb{Z}$. By Proposition 3.5, $F^\times \cong R^\times \times \{\pi^i\}$. There is a (discrete) valuation ν on the field of fractions $F : \nu(f)$ is that integer i such that $fR \cong \pi^i R$.

Examples of DVR's include the p-adic integers $\hat{\mathbb{Z}}_p$, the power series ring $k[[x]]$ over a field k, and localizations $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$.

• Let F be a number field, *i.e.*, a finite field extension of $\mathbb{Q}$. An *algebraic integer* of F is an element which is integral over $\mathbb{Z}$, *i.e.*, a root of a monic polynomial $x^n + a_1 x^{n-1} + \cdots + a_n$ with integer coefficients ($a_i \in \mathbb{Z}$). The set $\mathcal{O}_F$ of all algebraic integers of F is a ring—it is the integral closure of $\mathbb{Z}$ in F. A famous result in ring theory asserts that $\mathcal{O}_F$ is a Dedekind domain with field of fractions F. It follows that $\mathcal{O}_F$ is a lattice in F, *i.e.*, a free abelian group of rank $\dim_{\mathbb{Q}}(F)$.

In Number Theory, $\mathrm{Pic}(\mathcal{O}_F)$ is called the *ideal class group* of the number field F. A fundamental theorem states that $\mathrm{Pic}(\mathcal{O}_F)$ is always a finite group, but the precise structure of the ideal class group is only known for special number fields of small dimension. For example, if $\xi_p = e^{2\pi i/p}$ then $\mathbb{Z}[\xi_p]$ is the ring of algebraic integers of $\mathbb{Q}(\xi_p)$, and the class group is zero if and only if $p \leq 19$; $\mathrm{Pic}(\mathbb{Z}[\xi_{23}])$ is $\mathbb{Z}/3$. More details may be found in books on number theory, such as [BSh].

• If C is a smooth affine curve over a field k, then the coordinate ring R of C is a Dedekind domain. One way to construct a smooth affine curve is to start with a smooth projective curve $\bar{C}$. If $\{p_0, \dots, p_n\}$ is any nonempty set of points on $\bar{C}$, the Riemann-Roch theorem implies that $C = \bar{C} - \{p_0, \dots, p_n\}$ is a smooth affine curve.

If k is algebraically closed, $\mathrm{Pic}(R)$ is a divisible abelian group. Indeed, the points of the Jacobian variety $J(\bar{C})$ form a divisible abelian group, and $\mathrm{Pic}(R)$ is the quotient of $J(\bar{C})$ by the subgroup generated by the classes of the prime ideals of R corresponding to $p_1, ..., p_n$.

This is best seen when $k = \mathbb{C}$, because smooth projective curves over $\mathbb{C}$ are the same as compact Riemann surfaces. If $\bar{C}$ is a compact Riemann surface of genus g, then as an abelian group the points of the Jacobian $J(\bar{C})$ form the divisible group $(\mathbb{R}/\mathbb{Z})^{2g}$. In particular, when $C = \bar{C} - \{p_0\}$ then $\mathrm{Pic}(R) \cong J(\bar{C}) \cong (\mathbb{R}/\mathbb{Z})^{2g}$.

For example, $R = \mathbb{C}[x, y]/(y^2 - x(x-1)(x-\beta))$ is a Dedekind domain with $\mathrm{Pic}(R) \cong (\mathbb{R}/\mathbb{Z})^2$ if $\beta \neq 0, 1$. Indeed, R is the coordinate ring of a smooth affine curve C obtained by removing one point from an elliptic curve (= a projective curve of genus $g = 1$).

The Weil Divisor Class group

Let R be an integrally closed domain (= *normal domain*) with field of fractions F. If R is a noetherian normal domain, it is well-known that:

(i) $R_{\mathfrak{p}}$ is a discrete valuation ring (DVR) for all height 1 prime ideals $\mathfrak{p}$;

(ii) $R = \cap R_{\mathfrak{p}}$, the intersection being over all height 1 primes $\mathfrak{p}$ of R, each $R_{\mathfrak{p}}$ being a subring of F;

(iii) Every $r \neq 0$ in R is contained in only finitely many height 1 primes $\mathfrak{p}$.
An integral domain R satisfying (i), (ii) and (iii) is called a *Krull domain.*

Krull domains are integrally closed because every DVR $R_\mathfrak{p}$ is integrally closed. For a Krull domain R, the group $D(R)$ of *Weil divisors* is the free abelian group on the height 1 prime ideals of R. An *effective* Weil divisor is a divisor $D = \sum n_i[\mathfrak{p}_i]$ with all the $n_i \geq 0$.

We remark that effective divisors correspond to "divisorial" ideals of R, D corresponding to the intersection $\cap \mathfrak{p}_i^{(n_i)}$ of the symbolic powers of the $\mathfrak{p}_i$.

If $\mathfrak{p}$ is a height 1 prime of R, the $\mathfrak{p}$-adic valuation $\nu_\mathfrak{p}(I)$ of an invertible ideal I is defined to be that integer ν such that $I_\mathfrak{p} = \mathfrak{p}^\nu R_\mathfrak{p}$. By (iii), $v_\mathfrak{p}(I) \neq 0$ for only finitely many $\mathfrak{p}$, so $\nu(I) = \sum \nu_\mathfrak{p}(I)[\mathfrak{p}]$ is a well-defined element of $D(R)$. By 3.5, this gives a group homomorphism:

$$\nu\colon \quad \mathrm{Cart}(R) \to D(R).$$

If I is invertible, $\nu(I)$ is effective if and only if $I \subseteq R$. To see this, observe that $\nu(I)$ is effective $\iff I_\mathfrak{p} \subseteq R_\mathfrak{p}$ for all $\mathfrak{p} \iff I \subseteq \cap I_\mathfrak{p} \subseteq \cap R_\mathfrak{p} = R$. It follows that ν is an injection, because if both $\nu(I)$ and $\nu(I^{-1})$ are effective then I and I^{-1} are ideals with product R; this can only happen if $I = R$.

The *divisor class group* $Cl(R)$ of R is defined to be the quotient of $D(R)$ by the subgroup of all $\nu(fR)$, $f \in F^\times$. This yields a map $\mathrm{Pic}(R) \to Cl(R)$ which is evidently an injection. Summarizing, we have proven:

PROPOSITION 3.6. *Let R be a Krull domain. Then* $\mathrm{Pic}(R)$ *is a subgroup of the class group $Cl(R)$, and there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 1 \to & R^\times & \to & F^\times & \overset{\mathrm{div}}{\to} & \mathrm{Cart}(R) & \to & \mathrm{Pic}(R) & \to 0 \\ & \downarrow= & & \downarrow= & & \bigcap \nu & & \bigcap & \\ 1 \to & R^\times & \to & F^\times & \overset{\mathrm{div}}{\to} & D(R) & \to & Cl(R) & \to 0. \end{array}$$

REMARK 3.6.1. The Picard group and the divisor class group of a Krull domain R are invariant under polynomial and Laurent polynomial extensions. That is, $\mathrm{Pic}(R) = \mathrm{Pic}(R[t]) = \mathrm{Pic}(R[t,t^{-1}])$ and $Cl(R) = Cl(R[t]) = Cl(R[t,t^{-1}])$. Most of this assertion is proven in [B-AC, ch.7,§1]; the $\mathrm{Pic}[t,t^{-1}]$ part is proven in [BM, 5.10].

Recall that an integral domain R is called *factorial*, or a *Unique Factorization Domain* (UFD) if every nonzero element $r \in R$ is either a unit or a product of prime elements. (This implies that the product is unique up to order and primes differing by a unit). It is not hard to see that UFD's are Krull domains; the following interpretation in terms of the class group is taken from [Matsu, §20].

THEOREM 3.7. *Let R be a Krull domain. Then R is a UFD $\iff Cl(R) = 0$.*

DEFINITION. A noetherian ring R is called *regular* if every R-module M has a finite resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$ with the P_i projective. Every localization $S^{-1}R$ of a regular ring R is also a regular ring, because $S^{-1}R$-modules are also R-modules, and a localization of an R-resolution is an $S^{-1}R$-resolution.

Now suppose that $(R, \mathfrak{m})$ is a regular local ring. It is well-known [Matsu, §14, 19] that R is a noetherian, integrally closed domain (hence Krull), and that if $s \in \mathfrak{m} - \mathfrak{m}^2$ then sR is a prime ideal.

THEOREM 3.8. *Every regular local ring is a UFD.*

PROOF. We proceed by induction on $\dim(R)$. If $\dim(R) = 0$ then R is a field; if $\dim(R) = 1$ then R is a DVR, hence a UFD. Otherwise, choose $s \in \mathfrak{m} - \mathfrak{m}^2$. Since sR is prime, Ex. 3.8(b) yields $Cl(R) \cong Cl(R[\frac{1}{s}])$. Hence it suffices to show that $S = R[\frac{1}{s}]$ is a UFD. Let $\mathfrak{P}$ be a height 1 prime of S; we have to show that $\mathfrak{P}$ is a principal ideal. For every prime ideal $\mathfrak{Q}$ of S, $S_{\mathfrak{Q}}$ is a regular local ring of smaller dimension than R, so by induction $S_{\mathfrak{Q}}$ is a UFD. Hence $\mathfrak{P}_{\mathfrak{Q}}$ is principal: $xS_{\mathfrak{Q}}$ for some $x \in S$. By 2.4, $\mathfrak{P}$ is projective, hence invertible. Let $\mathfrak{p}$ be the prime ideal of R such that $\mathfrak{P} = \mathfrak{p}[\frac{1}{s}]$ and choose an R-resolution $0 \to P_n \to \cdots \to P_0 \to \mathfrak{p} \to 0$ of $\mathfrak{p}$ by finitely generated projective R-modules P_i. Since R is local, the P_i are free. Since $\mathfrak{P}$ is projective, the localized sequence $0 \to P_n[\frac{1}{s}] \to \cdots \to P_0[\frac{1}{s}] \to \mathfrak{P} \to 0$ splits. Letting E (resp. F) denote the direct sum of the odd (resp. even) $P_i[\frac{1}{s}]$, we have $\mathfrak{P} \oplus E \cong F$. Since stably free line bundles are free, $\mathfrak{P}$ is free. That is, $\mathfrak{P} = xS$ for some $x \in \mathfrak{P}$, as desired.

COROLLARY 3.8.1. *If R is a regular domain, then* $\mathrm{Cart}(R) = D(R)$, *and hence*

$$\mathrm{Pic}(R) = Cl(R).$$

PROOF. We have to show that every height 1 prime ideal $\mathfrak{P}$ of R is invertible. For every prime ideal $\mathfrak{p}$ of R we have $\mathfrak{P}_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ in the UFD $R_{\mathfrak{p}}$. By 2.4 and 3.5, $\mathfrak{P}$ is an invertible ideal.

REMARK 3.8.2. A ring is called *locally factorial* if $R_{\mathfrak{p}}$ is factorial for every prime ideal $\mathfrak{p}$ of R. For example, regular rings are locally factorial by 3.8. The proof of Cor. 3.8.1 shows that if R is a locally factorial Krull domain then $\mathrm{Pic}(R) = Cl(R)$.

Non-normal rings

The above discussion should make it clear that the Picard group of a normal domain is a classical object, even if it is hard to compute in practice. If R isn't normal, we can get a handle on $\mathrm{Pic}(R)$ using the techniques of the rest of this section.

For example, the next lemma allows us to restrict attention to reduced noetherian rings with finite normalization, because the quotient R_{red} of any commutative ring R by its *nilradical* (the ideal of nilpotent elements) is a reduced ring, and every commutative ring is the filtered union of its finitely generated subrings—rings having these properties.

If $\mathcal{A}$ is a small indexing category, every functor $R : \mathcal{A} \to \mathbf{Rings}$ has a colimit $\mathrm{colim}_{\alpha \in \mathcal{A}} R_\alpha$. We say that $\mathcal{A}$ is *filtered* if for every α, β there are maps $\alpha \to \gamma \leftarrow \beta$, and if for any two parallel arrows $\alpha \rightrightarrows \beta$ there is a $\beta \to \gamma$ so that the composites $\alpha \to \gamma$ agree; in this case we write $\varinjlim R_\alpha$ for colim R_α and call it the *filtered direct limit.* (See [WHomo, 2.6.13].)

LEMMA 3.9. (1) $\mathrm{Pic}(R) = \mathrm{Pic}(R_{\mathrm{red}})$.

(2) Pic *commutes with filtered direct limits of rings. In particular, if R is the filtered union of subrings R_α, then* $\mathrm{Pic}(R) \cong \varinjlim \mathrm{Pic}(R_\alpha)$.

PROOF. Part (1) is an instance of idempotent lifting (Ex. 2.2). To prove (2), recall from 2.5 that a line bundle L over R may be given by patching data: a unimodular row $(s_1, ..., s_c)$ and units g_{ij} over the $R[\frac{1}{s_i s_j}]$. If R is the filtered direct limit of rings R_α, this finite amount of data defines a line bundle L_α over one of the R_α, and we have $L = L_\alpha \otimes_{R_\alpha} R$. If L_α and L'_α become isomorphic over R, the isomorphism is defined over some R_β, *i.e.*, L and L' become isomorphic over R_β.

If R is reduced noetherian, its normalization S is a finite product of normal domains S_i. We would like to describe $\mathrm{Pic}(R)$ in terms of the more classical group $\mathrm{Pic}(S) = \prod \mathrm{Pic}(S_i)$, using the conductor square of 2.6. For this it is convenient to assume that S is finite over R, an assumption which is always true for rings of finite type over a field.

More generally, suppose that we are given a Milnor square (2.6):

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar f} & S/I. \end{array}$$

Given a unit β of S/I, the Milnor Patching Theorem 2.7 constructs a finitely generated projective R-module $L_\beta = (S, \beta, R/I)$ with $L_\beta \otimes_R S \cong S$ and $L_\beta / I L_\beta \cong R/I$. In fact L_β is a line bundle, because $\mathrm{rank}(L_\beta) = 1$; every map from R to a field factors through either R/I or S (for every prime ideal $\mathfrak{p}$ of R either $I \subseteq \mathfrak{p}$ or $R_\mathfrak{p} \cong S_\mathfrak{p}$). By Ex. 2.9, $L_\alpha \oplus L_\beta \cong L_{\alpha\beta} \oplus R$; applying $\wedge^2$ yields $L_\alpha \otimes_R L_\beta \cong L_{\alpha\beta}$. Hence the formula $\partial(\beta) = [L_\beta]$ yields a group homomorphism

$$\partial\colon (S/I)^\times \to \mathrm{Pic}(R).$$

Theorem 3.10 (Units-Pic sequence). *Given a Milnor square, the following sequence is exact. Here Δ denotes the diagonal map and $\pm$ denotes the difference map sending $(s, \bar r)$ to $\bar s f(\bar r)^{-1}$, resp. (L', L) to $L' \otimes_S S/I \otimes_{R/I} L^{-1}$.*

$$1 \to R^\times \xrightarrow{\Delta} S^\times \times (R/I)^\times \xrightarrow{\pm} (S/I)^\times \xrightarrow{\partial} \mathrm{Pic}(R) \xrightarrow{\Delta} \mathrm{Pic}(S) \times \mathrm{Pic}(R/I) \xrightarrow{\pm} \mathrm{Pic}(S/I)$$

Proof. Since R is the pullback of S and R/I, exactness at the first two places is clear. Milnor Patching 2.7 immediately yields exactness at the last two places, leaving only the question of exactness at $(S/I)^\times$. Given $s \in S^\times$ and $\bar r \in (R/I)^\times$, set $\beta = \pm(s, \bar r) = \bar s f(\bar r)^{-1}$, where $\bar s$ denotes the reduction of s modulo I. By inspection, $\lambda = (s, \bar r) \in L_\beta \subset S \times R/I$, and every element of L_β is a multiple of λ. It follows that $L_\beta \cong R$. Conversely, suppose given $\beta \in (S/I)^\times$ with $L_\beta \cong R$. If $\lambda = (s, \bar r)$ is a generator of L_β we claim that s and $\bar r$ are units, which implies that $\beta = \bar s f(\bar r)^{-1}$ and finishes the proof. If $s' \in S$ maps to $\beta \in S/I$ then $(s', 1) \in L_\beta$; since $(s', 1) = (xs, x\bar r)$ for some $x \in R$ this implies that $\bar r \in (R/I)^\times$. If $t \in S$ maps to $f(\bar r)^{-1}\beta^{-1} \in S/I$ then $st \equiv 1$ modulo I. Now $I \subset sR$ because $I \times 0 \subset L_\beta$, so $st = 1 + sx$ for some $x \in R$. But then $s(t - x) = 1$, so $s \in S^\times$ as claimed.

Example 3.10.1. (Cusp). Let k be a field and let R be $k[x, y]/(x^3 = y^2)$, the coordinate ring of the cusp in the plane. Setting $x = t^2$, $y = t^3$ makes R isomorphic to the subring $k[t^2, t^3]$ of $S = k[t]$. The conductor ideal from S to R is $I = t^2 S$, so we get a conductor square with $R/I = k$ and $S/I = k[t]/(t^2)$. Now $\mathrm{Pic}(k[t]) = 0$ and $(S/I)^\times \cong k^\times \times k$ with $\alpha \in k$ corresponding to $(1 + \alpha t) \in (S/I)^\times$. Hence $\mathrm{Pic}(R) \cong k$. A little algebra shows that a nonzero $\alpha \in k$ corresponds to the invertible prime ideal $\mathfrak{p} = (1 - \alpha^2 x, x - \alpha y)R$ corresponding to the point $(x, y) = (\alpha^{-2}, \alpha^{-3})$ on the cusp.

EXAMPLE 3.10.2. (Node). Let R be $k[x,y]/(y^2 = x^2 + x^3)$, the coordinate ring of the node in the plane over a field k with $\mathrm{char}(k) \neq 2$. Setting $x = t^2 - 1$ and $y = tx$ makes R isomorphic to a subring of $S = k[t]$ with conductor ideal $I = xS$. We get a conductor square with $R/I = k$ and $S/I \cong k \times k$. Since $(S/I)^\times \cong k^\times \times k^\times$ we see that $\mathrm{Pic}(R) \cong k^\times$. A little algebra shows that $\alpha \in k^\times$ corresponds to the invertible prime ideal $\mathfrak{p}$ corresponding to the point $(x,y) = \left(\frac{4\alpha}{(\alpha-1)^2}, \frac{4\alpha(\alpha+1)}{(\alpha-1)^3}\right)$ on the node corresponding to $t = \left(\frac{1+\alpha}{1-\alpha}\right)$.

Seminormal rings

A reduced commutative ring R is called *seminormal* if whenever $x, y \in R$ satisfy $x^3 = y^2$ there is an $s \in R$ with $s^2 = x$, $s^3 = y$. If R is an integral domain, there is an equivalent definition: R is seminormal if every s in the field of fractions satisfying $s^2, s^3 \in R$ belongs to R. Normal domains are clearly seminormal; the node (3.10.2) is not normal ($t^2 = 1 + x$), but it is seminormal (see Ex. 3.13). Arbitrary products of seminormal rings are also seminormal, because s may be found slotwise. The cusp (3.10.1) is the universal example of a reduced ring which is not seminormal.

Our interest in seminormal rings lies in the following theorem, first proven by C. Traverso for geometric rings and extended by several authors. For normal domains, it follows from Remark 3.6.1 above. Our formulation is taken from [Swan80].

THEOREM 3.11. (Traverso) *The following are equivalent for a commutative ring:*

(1) R_{red} *is seminormal;*
(2) $\mathrm{Pic}(R) = \mathrm{Pic}(R[t])$;
(3) $\mathrm{Pic}(R) = \mathrm{Pic}(R[t_1, ..., t_n])$ *for all* n.

REMARK 3.11.1. If R is seminormal, it follows that $R[t]$ is also seminormal. By Ex. 3.11, $R[t, t^{-1}]$ and the local rings $R_{\mathfrak{p}}$ are also seminormal. However, the $\mathrm{Pic}[t, t^{-1}]$ analogue of Theorem 3.11 fails. For example, if R is the node (3.10.2) then $\mathrm{Pic}(R[t, t^{-1}]) \cong \mathrm{Pic}(R) \times \mathbb{Z}$. For more details, see [We91].

To prove Traverso's theorem, we shall need the following standard result about units of polynomial rings.

LEMMA 3.12. *Let R be a commutative ring with nilradical $\mathfrak{N}$. If $r_0 + r_1 t + \cdots + r_n t^n$ is a unit of $R[t]$ then $r_0 \in R^\times$ and $r_1, ..., r_n$ are nilpotent. Consequently, if $NU(R)$ denotes the subgroup $1 + t\mathfrak{N}[t]$ of $R[t]^\times$ then:*

(1) $R[t]^\times = R^\times \times NU(R)$;
(2) *If R is reduced then $R^\times = R[t]^\times$;*
(3) *Suppose that R is an algebra over a field k. If char$(k) = p$, $NU(R)$ is a p-group. If char$(k) = 0$, $NU(R)$ is a uniquely divisible abelian group (= a $\mathbb{Q}$-module).*

PROOF OF TRAVERSO'S THEOREM. We refer the reader to Swan's paper for the proof that (1) implies (2) and (3). By Lemma 3.9, we may suppose that R is reduced but not seminormal. Choose $x, y \in R$ with $x^3 = y^2$ such that no $s \in R$ satisfies $s^2 = x, s^3 = y$. Then the reduced ring $S = R[s]/(s^2 - x, s^3 - y)_{\mathrm{red}}$ is strictly larger than R. Since $I = xS$ is an ideal of both R and S, we have Milnor squares

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar f} & S/I \end{array} \qquad \text{and} \qquad \begin{array}{ccc} R[t] & \xrightarrow{f} & S[t] \\ \downarrow & & \downarrow \\ R/I[t] & \xrightarrow{\bar f} & S/I[t]. \end{array}$$

The Units-Pic sequence 3.10 of the first square is a direct summand of the Units-Pic sequence for the second square. Using Lemma 3.12, we obtain the exact quotient sequence

$$0 \to NU(R/I) \to NU(S/I) \xrightarrow{\partial} \frac{\operatorname{Pic}(R[t])}{\operatorname{Pic}(R)}.$$

By construction, $s \notin R$ and $\bar s \notin R/I$. Hence $\partial(1 + \bar s t)$ is a nonzero element of the quotient $\operatorname{Pic}(R[t])/\operatorname{Pic}(R)$. Therefore if R isn't seminormal we have $\operatorname{Pic}(R) \neq \operatorname{Pic}(R[t])$, which is the (3) $\Rightarrow$ (2) $\Rightarrow$ (1) half of Traverso's theorem.

EXERCISES

In these exercises, R is always a commutative ring.

3.1 Show that the following are equivalent for every R-module L:
(a) There is a R-module M such that $L \otimes M \cong R$.
(b) L is an algebraic line bundle.
(c) L is a finitely generated R-module and $L_{\mathfrak p} \cong R_{\mathfrak p}$ for every prime ideal $\mathfrak p$ of R.
Hint: Use Exercises 2.10 and 2.11.

3.2 Show that the tensor product $P \otimes_R Q$ of two line bundles may be described using "Open Patching" 2.5 as follows. Find $s_1, ..., s_r \in R$ forming a unimodular row, such that P (resp. Q) is obtained by patching the $R[\frac{1}{s_i}]$ by units g_{ij} (resp. h_{ij}) in $R[\frac{1}{s_i s_j}]^\times$. Then $P \otimes_R Q$ is obtained by patching the $R[\frac{1}{s_i}]$ using the units $f_{ij} = g_{ij} h_{ij}$).

3.3 Let P be a locally free R-module, obtained by patching free modules of rank n by $g_{ij} \in GL_n(R[\frac{1}{s_i s_j}])$. Show that $\det(P)$ is the line bundle obtained by patching free modules of rank 1 by the units $\det(g_{ij}) \in (R[\frac{1}{s_i s_j}])^\times$.

3.4 Let P and Q be finitely generated projective modules of constant ranks m and n respectively. Show that there is a natural isomorphism $(\det P)^{\otimes n} \otimes (\det Q)^{\otimes m} \to \det(P \otimes Q)$. *Hint:* Send $(p_{11} \wedge \cdots \otimes \cdots \wedge p_{mn}) \otimes (q_{11} \wedge \cdots \otimes \cdots \wedge q_{mn})$ to $(p_{11} \otimes q_{11}) \wedge \cdots \wedge (p_{mn} \otimes q_{mn})$. Then show that this map is locally an isomorphism.

3.5 If an ideal $I \subseteq R$ is a projective R-module and $J \subseteq R$ is any other ideal, show that $I \otimes_R J$ is isomorphic to the ideal IJ of R.

3.6 *Excision for* Pic. If I is a commutative ring without unit, let $\operatorname{Pic}(I)$ denote the kernel of the canonical map $\operatorname{Pic}(\mathbb{Z} \oplus I) \to \operatorname{Pic}(\mathbb{Z})$. Write $I^\times$ for the group $GL_1(I)$ of Ex. 1.10. Show that if I is an ideal of R then there is an exact sequence:

$$1 \to I^\times \to R^\times \to (R/I)^\times \xrightarrow{\partial} \operatorname{Pic}(I) \to \operatorname{Pic}(R) \to \operatorname{Pic}(R/I).$$

3.7 (Roberts-Singh) This exercise generalizes Proposition 3.5. Let $R \subseteq S$ be an inclusion of commutative rings. An R-submodule I of S is called an *invertible R-ideal of S* if $IJ = R$ for some other R-submodule J of S.

(i) If $I \subseteq S$ is an invertible R-ideal of S, show that I is finitely generated over R, and that $IS = S$.
(ii) Show that the invertible R-ideals of S form an abelian group $\mathrm{Pic}(R, S)$ under multiplication.
(iii) Show that every invertible R-ideal of S is a line bundle over R. *Hint:* use Ex. 3.5 to compute its rank. Conversely, if I is a line bundle over R contained in S and $IS = S$, then I is an R-ideal.
(iv) Show that there is a natural exact sequence:

$$1 \to R^\times \to S^\times \xrightarrow{\text{div}} \mathrm{Pic}(R, S) \to \mathrm{Pic}(R) \to \mathrm{Pic}(S).$$

3.8 *Relative Class groups.* Suppose that R is a Krull domain and that $R_S = S^{-1}R$ for some multiplicatively closed set S in R. Let $D(R, R_S)$ denote the free abelian group on the height 1 primes $\mathfrak{p}$ of R such that $\mathfrak{p} \cap S \neq \phi$. Since $D(R_S)$ is free on the remaining height 1 primes of R, $D(R) = D(R, R_S) \oplus D(R_S)$.
(a) Show that the group $\mathrm{Pic}(R, R_S)$ of Ex. 3.7 is a subgroup of $D(R, R_S)$, and that there is an exact sequence compatible with Ex. 3.7

$$1 \to R^\times \to R_S^\times \to D(R, R_S) \to Cl(R) \to Cl(R_S) \to 0.$$

(b) Suppose that sR is a prime ideal of R. Prove that $(R[\frac{1}{s}])^\times \cong R^\times \times \mathbb{Z}^n$ and that $Cl(R) \cong Cl(R[\frac{1}{s}])$.
(c) Suppose that every height 1 prime $\mathfrak{p}$ of R with $\mathfrak{p} \cap S \neq \phi$ is an invertible ideal. Show that $\mathrm{Pic}(R, R_S) = D(R, R_S)$ and that $\mathrm{Pic}(R) \to \mathrm{Pic}(R_S)$ is onto. (This always happens if R is a regular ring, or if the local rings R_M are unique factorization domains for every maximal ideal M of R with $M \cap S \neq \emptyset$.)

3.9 Suppose that we are given a Milnor square with $R \subseteq S$. If $\bar{s} \in (S/I)^\times$ is the image of a nonzerodivisor $s \in S$, show that $-\partial(\bar{s}) \in \mathrm{Pic}(R)$ is the class of the ideal $(sS) \cap R$.

3.10 Let R be a 1-dimensional noetherian ring with finite normalization S, and let I be the conductor ideal from S to R. Show that for every maximal ideal $\mathfrak{p}$ of R, $\mathfrak{p}$ is a line bundle $\iff I \not\subseteq \mathfrak{p}$. Using Ex. 3.9, show that these $\mathfrak{p}$ generate $\mathrm{Pic}(R)$.

3.11 If R is seminormal, show that every localization $S^{-1}R$ is seminormal.

3.12 *Seminormality is a local property.* Show that the following are equivalent:
(a) R is seminormal;
(b) $R_\mathfrak{m}$ is seminormal for every maximal ideal $\mathfrak{m}$ of R;
(c) $R_\mathfrak{p}$ is seminormal for every prime ideal $\mathfrak{p}$ of R.

3.13 If R is a pullback of a diagram of seminormal rings, show that R is seminormal. This shows that the node (3.10.2) is seminormal.

3.14 *Normal rings.* A ring R is called *normal* if each local ring $R_\mathfrak{p}$ is an integrally closed domain. If R and R' are normal rings, so is the product $R \times R'$. Show that normal domains are normal rings, and that every reduced 0-dimensional ring is normal. Then show that every normal ring is seminormal.

3.15 *Seminormalization.* Show that every reduced commutative ring R has an extension $R \subseteq {}^+R$ with ${}^+R$ seminormal, satisfying the following universal property:

if S is seminormal, then every ring map $R \to S$ has a unique extension ${}^+R \to S$. The extension ${}^+R$ is unique up to isomorphism, and is called the *seminormalization* of R. *Hint:* First show that it suffices to construct the seminormalization of a noetherian ring R whose normalization S is finite. In that case, construct the seminormalization as a subring of S, using the observation that if $x^3 = y^2$ for $x, y \in R$, there is an $s \in S$ with $s^2 = x$, $s^3 = y$.

3.16 An extension $R \subset R'$ is called *subintegral* if $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a bijection, and the residue fields $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ are isomorphic. Show that the seminormalization $R \subset {}^+R$ of the previous exercise is a subintegral extension.

3.17 Let R be a commutative ring with nilradical $\mathfrak{N}$.
 (a) Show that the subgroup $1 + \mathfrak{N}[t, t^{-1}]$ of $R[t, t^{-1}]^\times$ is the product of the three groups $1 + \mathfrak{N}$, $N_tU(R) = 1 + t\mathfrak{N}[t]$, and $N_{t^{-1}}U(R) = 1 + t^{-1}\mathfrak{N}[t^{-1}]$.
 (b) Show that there is a homomorphism $t\colon [\mathrm{Spec}(R), \mathbb{Z}] \to R[t, t^{-1}]^\times$ sending f to the unit t^f of $R[t, t^{-1}]$ which is t^n on the factor R_i of R where $f = n$. Here R_i is given by 2.2.4 and Ex. 2.4.
 (c) Show that there is a natural decomposition

$$R[t, t^{-1}]^\times \cong R^\times \times N_tU(R) \times N_{t^{-1}}U(R) \times [\mathrm{Spec}(R), \mathbb{Z}],$$

or equivalently, that there is a split exact sequence:

$$1 \to R^\times \to R[t]^\times \times R[t^{-1}]^\times \to R[t, t^{-1}]^\times \to [\mathrm{Spec}(R), \mathbb{Z}] \to 0.$$

3.18 Show that the following sequence is exact:

$$1 \to \mathrm{Pic}(R) \to \mathrm{Pic}(R[t]) \times \mathrm{Pic}(R[t^{-1}]) \to \mathrm{Pic}(R[t, t^{-1}]).$$

Hint: If R is finitely generated, construct a diagram whose rows are Units-Pic sequences 3.10, and whose first column is the naturally split sequence of Ex. 3.17.

3.19 (NPic) Let $\mathrm{NPic}(R)$ denote the cokernel of the natural map $\mathrm{Pic}(R) \to \mathrm{Pic}(R[t])$. Show that $\mathrm{Pic}(R[t]) \cong \mathrm{Pic}(R) \times \mathrm{NPic}(R)$, and that $\mathrm{NPic}(R) = 0$ if and only if R_{red} is a seminormal ring. If R is an algebra over a field k, prove that:
 (a) If $\mathrm{char}(k) = p > 0$ then $\mathrm{NPic}(R)$ is a p-group;
 (b) If $\mathrm{char}(k) = 0$ then $\mathrm{NPic}(R)$ is a uniquely divisible abelian group.
To do this, first reduce to the case when R is finitely generated, and proceed by induction on $\dim(R)$ using conductor squares.

§4. Topological Vector Bundles and Chern Classes

Because so much of the theory of projective modules is based on analogy with the theory of topological vector bundles, it is instructive to review the main aspects of the structure of vector bundles. Details and further information may be found in [MSt], [Atiyah] or [Huse]. We will work with vector spaces over $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Let X be a topological space. A *family of vector spaces* over X is a topological space E, together with a continuous map $\eta\colon E \to X$ and a finite dimensional vector space structure (over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) on each fiber $E_x = \eta^{-1}(x)$, $x \in X$. We require the vector space structure on E_x to be compatible with the topology on E. (This means scaling $F \times E \to E$ and the addition map $E \times_X E \to E$ are continuous.) By a *homomorphism* from one family $\eta: E \to X$ to another family $\varphi: F \to X$ we mean a continuous map $f: E \to F$ with $\eta = \varphi f$, such that each induced map $f_x: E_x \to F_x$ is a linear map of vector spaces. There is an evident category of families of vector spaces over X and their homomorphisms.

For example, if V is an n-dimensional vector space, the projection from $T^n = X \times V$ to X forms a "constant" family of vector spaces. We call such a family, and any family isomorphic to it, a *trivial vector bundle* over X.

If $Y \subseteq X$, we write $E|Y$ for the restriction $\eta^{-1}(Y)$ of E to Y; the restriction $\eta|Y: E|Y \to Y$ of η makes $E|Y$ into a family of vector spaces over Y. More generally, if $f: Y \to X$ is any continuous map then we can construct an induced family $f^*(\eta): f^*E \to Y$ as follows. The space f^*E is the subspace of $Y \times E$ consisting of all pairs (y, e) such that $f(y) = \eta(e)$, and $f^*E \to Y$ is the restriction of the projection map. Since the fiber of f^*E at $y \in Y$ is $E_{f(y)}$, f^*E is a family of vector spaces over Y.

A *vector bundle* over X is a family of vector spaces $\eta: E \to X$ such that every point $x \in X$ has a neighborhood U such that $\eta|U: E|U \to U$ is trivial. A vector bundle is also called a *locally trivial* family of vector spaces.

The most historically important example of a vector bundle is the tangent bundle $TX \to X$ of a smooth manifold X. Another famous example is the *Möbius bundle* E over S^1; E is the open Möbius strip and $E_x \cong \mathbb{R}$ for each $x \in S^1$.

Suppose that $f: X \to Y$ is continuous. If $E \to Y$ is a vector bundle, then the induced family $f^*E \to X$ is a vector bundle on X. To see this, note that if E is trivial over a neighborhood U of $f(x)$ then f^*E is trivial over $f^{-1}(U)$.

The symbol $\mathbf{VB}(X)$ denotes the category of vector bundles and homomorphisms over X. If clarification is needed, we write $\mathbf{VB}_{\mathbb{R}}(X)$, $\mathbf{VB}_{\mathbb{C}}(X)$ or $\mathbf{VB}_{\mathbb{H}}(X)$. The induced bundle construction gives rise to an additive functor $f^* : \mathbf{VB}(X) \to \mathbf{VB}(Y)$.

The *Whitney sum* $E \oplus F$ of two vector bundles $\eta: E \to X$ and $\varphi: F \to X$ is the family of all the vector spaces $E_x \oplus F_x$, topologized as a subspace of $E \times F$. Since E and F are locally trivial, so is $E \oplus F$; hence $E \oplus F$ is a vector bundle. By inspection, the Whitney sum is the product in the category $\mathbf{VB}(X)$. Since there is a natural notion of the sum $f + g$ of two homomorphisms $f, g: E \to F$, this makes $\mathbf{VB}(X)$ into an additive category with Whitney sum the direct sum operation.

A *sub-bundle* of a vector bundle $\eta: E \to X$ is a subspace F of E which is a vector bundle under the induced structure. That is, each fiber F_x is a vector subspace of E_x and the family $F \to X$ is locally trivial. The *quotient bundle* E/F is the union of all the vector spaces E_x/F_x, given the quotient topology. Since F is locally a Whitney direct summand in E, we see that E/F is locally trivial, hence a vector bundle. This gives a "short exact sequence" of vector bundles in $\mathbf{VB}(X)$:

$$0 \to F \to E \to E/F \to 0.$$

A vector bundle $E \to X$ is said to be of *finite type* if there is a finite covering $U_1, \dots, U_n$ of X such that each each $E|U_i$ is a trivial bundle. Every bundle over a compact space X must be of finite type; the same is true if X is a finite-dimensional CW complex [Huse, §3.5], or more generally if there is an integer n such that every open cover of X has a refinement $\mathcal{V}$ such that no point of X is contained in more that n elements of $\mathcal{V}$. We will see in Exercise 4.15 that the canonical line bundle on infinite dimensional projective space $\mathbb{P}^\infty$ is an example of a vector bundle which is *not* of finite type.

Riemannian Metrics. Let $E \to X$ be a real vector bundle. A *Riemannian metric* on E is a family of inner products $\beta_x: E_x \times E_x \to \mathbb{R}$, $x \in X$, which varies continuously with x (in the sense that β is a continuous function on the Whitney sum $E \oplus E$). The notion of *Hermitian metric* on a complex (or quaternionic) vector bundle is defined similarly. A fundamental result [Huse, 3.5.5 and 3.9.5] states that every vector bundle over a paracompact space X has a Riemannian (or Hermitian) metric; see Ex. 4.17 for the quaternionic case.

Dimension of vector bundles

If E is a vector bundle over X then $\dim(E_x)$ is a locally constant function on X with values in $\mathbb{N} = \{0, 1, \dots\}$. Hence $\dim(E)$ is a continuous function from X to the discrete topological space $\mathbb{N}$; it is the analogue of the rank of a projective module. An *n-dimensional vector bundle* is a bundle E such that $\dim(E) = n$ is constant; 1-dimensional vector bundles are frequently called *line bundles*. The Möbius bundle is an example of a nontrivial line bundle.

A vector bundle E is called *componentwise trivial* if we can write X as a disjoint union of (closed and open) components X_i in such a way that each $E|X_i$ is trivial. Isomorphism classes of componentwise trivial bundles are in 1-1 correspondence with the set $[X, \mathbb{N}]$ of all continuous maps from X to $\mathbb{N}$. To see this, note that any continuous map $f: X \to \mathbb{N}$ induces a decomposition of X into components $X_i = f^{-1}(i)$. Given such an f, let T^f denote the disjoint union

$$T^f = \coprod_{i \in \mathbb{N}} X_i \times F^i, \qquad F = \mathbb{R}, \mathbb{C} \text{ or } \mathbb{H}.$$

The projection $T^f \to \amalg X_i = X$ makes T^f into a componentwise trivial vector bundle with $\dim(T^f) = f$. Conversely, if E is componentwise trivial, then $E \cong T^{\dim(E)}$. Note that $T^f \oplus T^g \cong T^{f+g}$. Thus if f is bounded then by choosing $g = n - f$ we can make T^f into a summand of the trivial bundle T^n.

The following result, which we cite from [Huse, 3.5.8 and 3.9.6], illustrates some of the similarities between $\mathbf{VB}(X)$ and the category of finitely generated projective modules. It is proven using a Riemannian (or Hermitian) metric on E: $F_x^\perp$ is the subspace of E_x perpendicular to F_x. (A topological space is *paracompact* if it is Hausdorff and every open cover has a partition of unity subordinate to it.)

Subbundle Theorem 4.1. *Let $E \to X$ be a vector bundle on a paracompact topological space X. Then:*

1. *If F is a sub-bundle of E, there is a sub-bundle $F^\perp$ such that $E \cong F \oplus F^\perp$.*
2. *E has finite type if and only if E is isomorphic to a sub-bundle of a trivial bundle. That is, if and only if there is another bundle F such that $E \oplus F$ is trivial.*

Corollary 4.1.1. *Suppose that X is compact, or that X is a finite-dimensional CW complex. Then every vector bundle over X is a Whitney direct summand of a trivial bundle.*

EXAMPLE 4.1.2. If X is a smooth d-dimensional manifold, its tangent bundle $TX \to X$ is a d-dimensional real vector bundle. Embedding X in $\mathbb{R}^n$ allows us to form the *normal bundle* $NX \to X$; N_xX is the orthogonal complement of T_xX in $\mathbb{R}^n$. Clearly $TX \oplus NX$ is the trivial n-dimensional vector bundle $X \times \mathbb{R}^n \to X$ over X.

EXAMPLE 4.1.3. Consider the canonical line bundle E_1 on projective n-space; a point x of $\mathbb{P}^n$ corresponds to a line L_x in $n+1$-space, and the fiber of E_1 at x is just L_x. In fact, E_1 is a subbundle of the trivial bundle T^{n+1}. Letting F_x be the n-dimensional hyperplane perpendicular to L_x, the family of vector spaces F forms a vector bundle such that $E_1 \oplus F = T^{n+1}$.

EXAMPLE 4.1.4. (Global sections) A *global section* of a vector bundle $\eta: E \to X$ is a continuous map $s: X \to E$ such that $\eta s = 1_X$. It is *nowhere zero* if $s(x) \neq 0$ for all $x \in X$. Every global section s determines a map from the trivial line bundle T^1 to E; if s is nowhere zero then the image is a line subbundle of E. If X is paracompact the Subbundle Theorem determines a splitting $E \cong F \oplus T^1$.

Patching vector bundles

4.2. One technique for creating vector bundles uses transition functions. The idea is to patch together a collection of vector bundles which are defined on subspaces of X. A related technique is the clutching construction discussed in 4.7 below.

Let $\eta: E \to X$ be an n-dimensional vector bundle on X over the field F (F is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). Since E is locally trivial, we can find an open covering $\{U_i\}$ of X, and isomorphisms $h_i : U_i \times F^n \cong E|U_i$. If $U_i \cap U_j \neq \varnothing$, the isomorphism

$$h_i^{-1}h_j : (U_i \cap U_j) \times F^n \cong \eta|U_i \cap U_j \cong (U_i \cap U_j) \times F^n$$

sends $(x, v) \in (U_i \cap U_j) \times F^n$ to $(x, g_{ij}(x)(v))$ for some $g_{ij}(x) \in GL_n(F)$.

Conversely, suppose we are given maps $g_{ij}: U_i \cap U_j \to GL_n(F)$ such that $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. On the disjoint union of the $U_i \times F^n$, form the equivalence relation $\sim$ which is generated by the relation that $(x, v) \in U_j \times F^n$ and $(x, g_{ij}(x)(v)) \in U_i \times F^n$ are equivalent for every $x \in U_i \cap U_j$. Let E denote the quotient space $(\coprod U_i \times F^n)/\sim$. It is not hard to see that there is an induced map $\eta: E \to X$ making E into a vector bundle over X.

We call E the vector bundle *obtained by patching* via the transition functions g_{ij}; this patching construction is the geometric motivation for open patching of projective modules in 2.5.

CONSTRUCTION 4.2.1. (Tensor product). Let E and F be real or complex vector bundles over X. There is a vector bundle $E \otimes F$ over X whose fiber over $x \in X$ is the vector space tensor product $E_x \otimes F_x$, and $\dim(E \otimes F) = \dim(E)\dim(F)$.

To construct $E \otimes F$, we first suppose that E and F are trivial bundles, *i.e.*, $E = X \times V$ and $F = X \times W$ for vector spaces V, W. In this case we let $E \otimes F$ be the trivial bundle $X \times (V \otimes W)$. In the general case, we proceed as follows. Restricting to a component of X on which $\dim(E)$ and $\dim(F)$ are constant, we may assume that E and F have constant ranks m and n respectively. Choose a covering $\{U_i\}$ and transition maps g_{ij}, g'_{ij} defining E and F by patching. Identifying $M_m(F) \otimes M_n(F)$ with $M_{mn}(F)$ gives a map $GL_m(F) \times GL_n(F) \to GL_{mn}(F)$, and the elements

$g_{ij} \otimes g'_{ij}$ give transition maps for $E \otimes F$ from $U_i \cap U_j$ to $GL_{mn}(F)$. The last assertion comes from the classical vector space formula $\dim(E_x \otimes F_x) = \dim(E_x)\dim(F_x)$.

Construction 4.2.2. (Determinant bundle). For every n-dimensional real or complex vector bundle E, there is an associated "determinant" line bundle $\det(E) = \wedge^n E$ whose fibers are the 1-dimensional vector spaces $\wedge^n(E_x)$. In fact, $\det(E)$ is a line bundle obtained by patching, the transition functions for $\det(E)$ being the determinants $\det(g_{ij})$ of the transition functions g_{ij} of E. More generally, if E is any vector bundle then this construction may be performed componentwise to form a line bundle $\det(E) = \wedge^{\dim(E)} E$. As in §3, if L is a line bundle and $E = L \oplus T^f$, then $\det(E) = L$, so E uniquely determines L. Taking E trivial, this shows that nontrivial line bundles cannot be stably trivial.

Orthogonal, unitary and symplectic structure groups 4.2.3. We say that an n-dimensional vector bundle $E \to X$ has *structure group* O_n, U_n or Sp_n if the transition functions g_{ij} map $U_i \cap U_j$ to the subgroup O_n of $GL_n(\mathbb{R})$ the subgroup U_n of $GL_n(\mathbb{C})$ or the subgroup Sp_n of $GL_n(\mathbb{H})$. If X is paracompact, this can always be arranged, because then E has a (Riemannian or Hermitian) metric. Indeed, it is easy to continuously modify the isomorphisms $h_i: U_i \times F^n \to E|U_i$ so that on each fiber the map $F^n \cong E_x$ is an isometry. But then the fiber isomorphisms $g_{ij}(x)$ are isometries, and so belong to O_n, U_n or Sp_n. Using the same continuous modification trick, any vector bundle isomorphism between vector bundles with a metric gives rise to a metric-preserving isomorphism. If X is paracompact, this implies that $\mathbf{VB}_n(X)$ is also the set of equivalence classes of vector bundles with structure group O_n, U_n or Sp_n.

The following pair of results forms the historical motivation for the Bass-Serre Cancellation Theorem 2.3. Their proofs may be found in [Huse, 8.1].

Real Cancellation Theorem 4.3. *Suppose X is a d-dimensional CW complex, and that $\eta: E \to X$ is an n-dimensional real vector bundle with $n > d$. Then*
(i) $E \cong E_0 \oplus T^{n-d}$ for some d-dimensional vector bundle E_0
(ii) If F is another bundle and $E \oplus T^k \cong F \oplus T^k$, then $E \cong F$.

Corollary 4.3.1. *Over a 1-dimensional CW complex, every real vector bundle E of rank ≥ 1 is isomorphic to $L \oplus T^f$, where $L = \det(E)$ and $f = \dim(E) - 1$.*

Complex Cancellation Theorem 4.4. *Suppose X is a d-dimensional CW complex, and that $\eta: E \to X$ is a complex vector bundle with $\dim(E) \geq d/2$.*
(i) $E \cong E_0 \oplus T^k$ for some vector bundle E_0 of dimension $\leq d/2$
(ii) If F is another bundle and $E \oplus T^k \cong F \oplus T^k$, then $E \cong F$.

Corollary 4.4.1. *Let X be a CW complex of dimension ≤ 3. Every complex vector bundle E of rank ≥ 1 is isomorphic to $L \oplus T^f$, where $L = \det(E)$ and $f = \dim(E) - 1$.*

There is also a cancellation theorem for a quaternionic vector bundle E with $\dim(E) \geq d/4$, $d = \dim(X)$. If $d \leq 3$ it implies that all quaternionic vector bundles are trivial; the splitting $E \cong L \oplus T^f$ occurs when $d \leq 7$.

Vector bundles are somewhat more tractable than projective modules, as the following result shows. Its proof may be found in [Huse, 3.4.7].

HOMOTOPY INVARIANCE THEOREM 4.5. *If $f, g: Y \to X$ are homotopic maps and Y is paracompact, then $f^*E \cong g^*E$ for every vector bundle E over X.*

COROLLARY 4.6. *If X and Y are homotopy equivalent paracompact spaces, there is a 1–1 correspondence between isomorphism classes of vector bundles on X and Y.*

APPLICATION 4.6.1. If Y is a contractible paracompact space then every vector bundle over Y is trivial.

CLUTCHING CONSTRUCTION 4.7. Here is an analogue for vector bundles of Milnor Patching 2.7 for projective modules. Suppose that X is a paracompact space, expressed as the union of two closed subspaces X_1 and X_2, with $X_1 \cap X_2 = A$. Given vector bundles $E_i \to X_i$ and an isomorphism $g: E_1|A \to E_2|A$, we form a vector bundle $E = E_1 \cup_g E_2$ over X as follows. As a topological space E is the quotient of the disjoint union $(E_1 \coprod E_2)$ by the equivalence relation identifying $e_1 \in E_1|A$ with $g(e_1) \in E_2|A$. Clearly the natural projection $\eta: E \to X$ makes E a family of vector spaces, and $E|X_i \cong E_i$. Moreover, E is locally trivial over X (see [Atiyah, p. 21]; paracompactness is needed to extend g off of A). The isomorphism $g: E_1|A \cong E_2|A$ is called the *clutching map* of the construction. As with Milnor patching, every vector bundle over X arises by this clutching construction. A new feature, however, is homotopy invariance: if f, g are homotopic clutching isomorphisms $E_1|A \cong E_2|A$, then $E_1 \cup_f E_2$ and $E_1 \cup_g E_2$ are isomorphic vector bundles over X.

PROPOSITION 4.8. *Let SX denote the suspension of a paracompact space X. A choice of basepoint for X yields a 1-1 correspondence between the set $\mathbf{VB}_n(SX)$ of isomorphism classes of n-dimensional (resp., real, complex or quaternionic) vector bundles over SX and the respective set of based homotopy classes of maps*

$$[X, O_n]_*, \qquad [X, U_n]_* \quad \textit{or} \quad [X, Sp_n]_*$$

from X to the orthogonal group O_n, unitary group U_n or symplectic group Sp_n.

SKETCH OF PROOF. SX is the union of two contractible cones C_1 and C_2 whose intersection is X. As every vector bundle on the cones C_i is trivial, every vector bundle on SX is obtained from an isomorphism of trivial bundles over X via the clutching construction. Such an isomorphism is given by a homotopy class of maps from X to GL_n, or equivalently to the appropriate deformation retract (O_n, U_n or Sp_n) of GL_n. The indeterminacy in the resulting map from $[X, GL_n]$ to classes of vector bundles is eliminated by insisting that the basepoint of X map to $1 \in GL_n$.

Vector Bundles on Spheres

Proposition 4.8 allows us to use homotopy theory to determine the vector bundles on the sphere S^d, because S^d is the suspension of S^{d-1}. Hence n-dimensional (real, complex or symplectic) bundles on S^d are in 1-1 correspondence with elements of $\pi_{d-1}(O_n)$, $\pi_{d-1}(U_n)$ and $\pi_{d-1}(Sp_n)$, respectively. For example, every real or complex line bundle over S^d is trivial if $d \geq 3$, because the appropriate homotopy groups of $O_1 \cong S^0$ and $U_1 \cong S^1$ vanish. This is not true for $Sp_1 \cong S^3$; for example there are infinitely many symplectic line bundles on S^4 because $\pi_3 Sp_1 = \mathbb{Z}$. The classical calculation of the homotopy groups of O_n, U_n and Sp_n (see [Huse, 7.12]) yields the following facts:

(4.9.1) On S^1, there are $|\pi_0(O_n)| = 2$ real vector bundles of dimension n for all $n \geq 1$. The nontrivial line bundle on S^1 is the Möbius bundle. The Whitney sum of the Möbius bundle with trivial bundles yields all the other nontrivial bundles. Since $|\pi_0(U_n)| = 1$ for all n, every complex vector bundle on S^1 is trivial.

(4.9.2) On S^2, the situation is more complicated. Since $\pi_1(O_1) = 0$ there are no nontrivial real line bundles on S^2. There are infinitely many 2-dimensional real vector bundles on S^2 (indexed by the degree d of their clutching functions), because $\pi_1(O_2) = \mathbb{Z}$. However, there is only one nontrivial n-dimensional real vector bundle for each $n \geq 3$, because $\pi_1(O_n) = \mathbb{Z}/2$. A real 2-dimensional bundle E is stably trivial (and $E \oplus T \cong T^3$) if and only if the degree d is even. The tangent bundle of S^2 has degree $d = 2$.

There are infinitely many complex line bundles L_d on S^2, indexed by the degree d (in $\pi_1(U_1) = \mathbb{Z}$) of their clutching function. The Complex Cancellation theorem (4.4) states that every other complex vector bundle on S^2 is isomorphic to a Whitney sum $L_d \oplus T^n$, and that all the $L_d \oplus T^n$ are distinct.

(4.9.3) Every vector bundle on S^3 is trivial. This is a consequence of the classical result that $\pi_2(G) = 0$ for every compact Lie group G, such as $G = O_n$, U_n or Sp_n.

(4.9.4) As noted above, every real or complex line bundle on S^4 is trivial. S^4 carries infinitely many distinct n-dimensional vector bundles for $n \geq 5$ over $\mathbb{R}$, for $n \geq 2$ over $\mathbb{C}$, and for $n \geq 1$ over $\mathbb{H}$ because $\pi_3(O_n) = \mathbb{Z}$ for $n \geq 5$, $\pi_3(U_n) = \mathbb{Z}$ for $n \geq 2$ and $\pi_3(Sp_n) = \mathbb{Z}$ for $n \geq 1$. In the intermediate range, we have $\pi_3(O_2) = 0$, $\pi_3(O_3) = \mathbb{Z}$ and $\pi_3(O_4) = \mathbb{Z} \oplus \mathbb{Z}$. Every 5-dimensional real bundle comes from a unique 3-dimensional bundle but every 4-dimensional real bundle on S^4 is stably isomorphic to infinitely many other distinct 4-dimensional vector bundles.

(4.9.5) There are no 2-dimensional real vector bundles on S^d for $d \geq 3$, because the appropriate homotopy groups of $O_2 \cong S^1 \times \mathbb{Z}/2$ vanish. This vanishing phenomenon doesn't persist though; if $d \geq 5$ the 2-dimensional complex bundles, as well as the 3-dimensional real bundles on S^d, correspond to elements of $\pi_{d-1}(O_3) \cong \pi_{d-1}(U_2) \cong \pi_{d-1}(S^3)$. This is a finite group which is rarely trivial.

Classifying Vector Bundles

One feature present in the theory of vector bundles, yet absent in the theory of projective modules, is the classification of vector bundles using Grassmannians.

If V is any finite-dimensional vector space, the set $\text{Grass}_n(V)$ of all n-dimensional linear subspaces of V is a smooth manifold, called the *Grassmann manifold* of n-planes in V. If $V \subset W$, then $\text{Grass}_n(V)$ is naturally a submanifold of $\text{Grass}_n(W)$. The *infinite Grassmannian* Grass_n is the union of the $\text{Grass}_n(V)$ as V ranges over all finite-dimensional subspaces of a fixed infinite-dimensional vector space ($\mathbb{R}^\infty$, $\mathbb{C}^\infty$ or $\mathbb{H}^\infty$); thus Grass_n is an infinite-dimensional CW complex (see [MSt]). For example, if $n = 1$ then Grass_1 is either $\mathbb{R}\mathbb{P}^\infty$, $\mathbb{C}\mathbb{P}^\infty$ or $\mathbb{H}\mathbb{P}^\infty$, depending on whether the vector spaces are over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

There is a canonical n-dimensional vector bundle $E_n(V)$ over each $\text{Grass}_n(V)$, whose fibre over each $x \in \text{Grass}_n(V)$ is the linear subspace of V corresponding to x. To topologize this family of vector spaces, and see that it is a vector bundle, we define $E_n(V)$ to be the sub-bundle of the trivial bundle $\mathrm{Grass}_n(V)\times V \to \mathrm{Grass}_n(V)$ having the prescribed fibers. For $n = 1$ this is just the canonical line bundle on projective space described in Example 4.1.3.

The union (as V varies) of the $E_n(V)$ yields an n-dimensional vector bundle $E_n \to \mathrm{Grass}_n$, called the n-dimensional *classifying bundle* because of the following theorem (see [Huse, 3.7.2]).

Classification Theorem 4.10. *Let X be a paracompact space. Then the set $\mathbf{VB}_n(X)$ of isomorphism classes of n-dimensional vector bundles over X is in 1–1 correspondence with the set $[X, \mathrm{Grass}_n]$ of homotopy classes of maps $X \to \mathrm{Grass}_n$:*

$$\mathbf{VB}_n(X) \cong [X, \mathrm{Grass}_n].$$

In more detail, every n-dimensional vector bundle $\eta\colon E \to X$ is isomorphic to $f^*(E_n)$ for some map $f\colon X \to \mathrm{Grass}_n$, and E determines f up to homotopy.

Remark 4.10.1. (Classifying Spaces) The Classification Theorem 4.10 states that the contravariant functor $\mathbf{VB}_n$ is "representable" by the infinite Grassmannian Grass_n. Because X is paracompact we may assume (by 4.2.3) that all vector bundles have structure group O_n, U_n or Sp_n, respectively. For this reason, the infinite Grassmannian Grass_n is called the *classifying space* of O_n, U_n or Sp_n (depending on the choice of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$). It is the custom to write BO_n, BU_n and BSp_n for the Grassmannians Grass_n (or any spaces homotopy equivalent to it) over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively.

In fact, there are homotopy equivalences $\Omega(BG) \simeq G$ for any Lie group G. If G is O_n, U_n or Sp_n, we can deduce this from 4.8 and 4.10: for any paracompact space X we have $[X, G]_* \cong \mathbf{VB}_n(SX) \cong [SX, BG] \cong [X, \Omega(BG)]_*$. Taking X to be G and $\Omega(BG)$ yields the homotopy equivalences.

It is well-known that there are canonical isomorphisms $[X, \mathbb{R}\mathbb{P}^\infty] \cong H^1(X;\mathbb{Z}/2)$ and $[X, \mathbb{C}\mathbb{P}^\infty] \cong H^2(X;\mathbb{Z})$ respectively. Therefore the case $n = 1$ may be reformulated as follows over $\mathbb{R}$ and $\mathbb{C}$.

Classification Theorem for Line Bundles 4.11. *If X is paracompact, there are natural isomorphisms:*

$$w_1 : \mathbf{VB}_{1,\mathbb{R}}(X) = \{\textit{real line bundles on } X\} \cong H^1(X;\mathbb{Z}/2)$$

$$c_1 : \mathbf{VB}_{1,\mathbb{C}}(X) = \{\textit{complex line bundles on } X\} \cong H^2(X;\mathbb{Z}).$$

Remark 4.11.1. Since $H^1(X)$ and $H^2(X)$ are abelian groups, it follows that the set $\mathbf{VB}_1(X)$ of isomorphism classes of line bundles is an abelian group. We can understand this group structure in a more elementary way, as follows. The tensor product $E \otimes F$ of line bundles is again a line bundle by 4.2.1, and $\otimes$ is the product in the group $\mathbf{VB}_1(X)$. The inverse of E in this group is the dual bundle $\check{E}$ of Ex. 4.3, because $\check{E} \otimes E$ is a trivial line bundle (see Ex. 4.4).

Riemann Surfaces 4.11.2. Here is a complete classification of complex vector bundles on a Riemann surface X. Recall that a Riemann surface is a compact 2-dimensional oriented manifold; the orientation gives a canonical isomorphism $H^2(X;\mathbb{Z}) = \mathbb{Z}$. If $\mathcal{L}$ is a complex line bundle, the *degree* of $\mathcal{L}$ is that integer d such that $c_1(\mathcal{L}) = d$. By Theorem 4.11, there is a unique complex line bundle $\mathcal{O}(d)$ of each degree on X. By Corollary 4.4.1, every complex vector bundle of rank r on X is isomorphic to $\mathcal{O}(d) \oplus T^{r-1}$ for some d. Therefore complex vector bundles on a Riemann surface are completely classified by their rank and degree.

For example, the tangent bundle $\mathcal{T}_X$ of a Riemann surface X has the structure of a complex line bundle, because every Riemann surface has the structure of a 1-dimensional complex manifold. The Riemann-Roch Theorem states that $\mathcal{T}_X$ has degree $2-2g$, where g is the *genus* of X. (Riemann surfaces are completely classified by their genus $g \geq 0$, a Riemann surface of genus g being a surface with g “handles.”)

In contrast, there are 2^{2g} distinct real line bundles on X, because $H^1(X;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2g}$. The Real Cancellation Theorem 4.3 shows that every real vector bundle is the sum of a trivial bundle and a bundle of dimension ≤ 2, but there are infinitely many 2-dimensional bundles over X. For example, the complex line bundles $\mathcal{O}(d)$ all give distinct oriented 2-dimensional real vector bundles on X; they are distinguished by an invariant called the *Euler class* (see [MSt]).

Characteristic Classes

By Theorem 4.11, the determinant line bundle $\det(E)$ of a vector bundle E yields a cohomology class: if E is a real vector bundle, it is the first Stiefel-Whitney class $w_1(E)$ in $H^1(X;\mathbb{Z}/2)$; if E is a complex vector bundle, it is the first Chern class $c_1(E)$ in $H^2(X;\mathbb{Z})$. These classes fit into a more general theory of characteristic classes, which are constructed and described in the book [MSt]. Here is an axiomatic description of these classes.

Axioms for Stiefel-Whitney classes 4.12. The *Stiefel–Whitney classes* of a real vector bundle E over X are elements $w_i(E) \in H^i(X;\mathbb{Z}/2)$, which satisfy the following axioms. By convention $w_0(E) = 1$.

(SW1) (Dimension) If $i > \dim(E)$ then $w_i(E) = 0$.

(SW2) (Naturality) If $f\colon Y \to X$ is continuous then $f^*\colon H^i(X;\mathbb{Z}/2) \to H^i(Y;\mathbb{Z}/2)$ sends $w_i(E)$ to $w_i(f^*E)$. If E and E' are isomorphic bundles then $w_i(E) = w_i(E')$.

(SW3) (Whitney sum formula) If E and F are bundles, then in the graded cohomology ring $H^*(X;\mathbb{Z}/2)$ we have:

$$w_n(E \oplus F) = \sum w_i(E)w_{n-i}(F) = w_n(E) + w_{n-1}(E)w_1(F) + \cdots + w_n(F).$$

(SW4) (Normalization) For the canonical line bundle E_1 over $\mathbb{R}\mathbb{P}^\infty$, $w_1(E_1)$ is the unique nonzero element of $H^1(\mathbb{R}\mathbb{P}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

Axioms (SW2) and (SW4), together with the Classification Theorem 4.10, show that w_1 classifies real line bundles in the sense that it gives the isomorphism $\mathbf{VB}_1(X) \cong H^1(X;\mathbb{Z}/2)$ of Theorem 4.11. The fact that $w_1(E) = w_1(\det E)$ is a consequence of the “Splitting Principle” for vector bundles, and is left to the exercises.

Since trivial bundles are induced from the map $X \to \{*\}$, it follows from (SW1) and (SW2) that $w_i(T^n) = 0$ for every trivial bundle T^n (and $i \neq 0$). The same is true for componentwise trivial bundles; see Ex. 4.2. From (SW3) it follows that $w_i(E \oplus T^n) = w_i(E)$ for every bundle E and every trivial bundle T^n.

The *total Stiefel-Whitney class* $w(E)$ of E is defined to be the formal sum

$$w(E) = 1 + w_1(E) + \cdots + w_i(E) + \cdots$$

in the complete cohomology ring $\hat{H}^*(X;\mathbb{Z}/2) = \prod_i H^i(X;\mathbb{Z}/2)$, which consists of all formal infinite series $a_0 + a_1 + \cdots$ with $a_i \in H^i(X;\mathbb{Z}/2)$. With this formalism, the Whitney sum formula becomes a product formula: $w(E \oplus F) = w(E)w(F)$. Now the collection U of all formal sums $1 + a_1 + \cdots$ in $\hat{H}^*(X;\mathbb{Z}/2)$ forms an abelian group under multiplication (the group of units of $\hat{H}^*(X;\mathbb{Z}/2)$ if X is connected). Therefore if $E \oplus F$ is trivial we can compute $w(F)$ via the formula $w(F) = w(E)^{-1}$.

For example, consider the canonical line bundle $E_1(\mathbb{R}^n)$ over $\mathbb{RP}^n$. By axiom (SW4) we have $w(E_1) = 1 + x$ in the ring $H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{F}_2[x]/(x^{n+1})$. We saw in Example 4.1.3 that there is an n-dimensional vector bundle F with $F \oplus E_1 = T^{n+1}$. Using the Whitney Sum formula (SW3), we compute that $w(F) = 1 + x + \cdots + x^n$. Thus $w_i(F) = x^i$ for $i \leq n$ and $w_i(F) = 0$ for $i > n$.

Stiefel-Whitney classes were named for E. Stiefel and H. Whitney, who discovered the w_i independently in 1935, and used them to study the tangent bundle of a smooth manifold.

Axioms for Chern classes 4.13. If E is a complex vector bundle over X, the *Chern classes* of E are certain elements $c_i(E) \in H^{2i}(X;\mathbb{Z})$, with $c_0(E) = 1$. They satisfy the following axioms. Note that the natural inclusion of $S^2 \cong \mathbb{CP}^1$ in $\mathbb{CP}^\infty$ induces a canonical isomorphism $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$.

(C1) (Dimension) If $i > \dim(E)$ then $c_i(E) = 0$

(C2) (Naturality) If $f\colon Y \to X$ is continuous then $f^*\colon H^{2i}(X;\mathbb{Z}) \to H^{2i}(Y;\mathbb{Z})$ sends $c_i(E)$ to $c_i(f^*E)$. If $E \cong E'$ then $c_i(E) = c_i(E')$.

(C3) (Whitney sum formula) If E and F are bundles then

$$c_n(E \oplus F) = \sum c_i(E)c_{n-i}(F) = c_n(E) + c_{n-1}(E)c_1(F) + \cdots + c_n(F).$$

(C4) (Normalization) For the canonical line bundle E_1 over $\mathbb{CP}^\infty$, $c_1(E_1)$ is the canonical generator x of $H^2(\mathbb{CP}^\infty;\mathbb{Z}) \cong \mathbb{Z}$.

Axioms (C2) and (C4) and the Classification Theorem 4.10 imply that the first Chern class c_1 classifies complex line bundles; it gives the isomorphism $\mathbf{VB}_1(X) \cong H^2(X;\mathbb{Z})$ of Theorem 4.11. The identity $c_1(E) = c_1(\det E)$ is left to the exercises.

The total *Chern class* $c(E)$ of E is defined to be the formal sum

$$c(E) = 1 + c_1(E) + \cdots + c_i(E) + \cdots .$$

in the complete cohomology ring $\hat{H}^*(X;\mathbb{Z}) = \prod_i H^i(X;\mathbb{Z})$. With this formalism, the Whitney sum formula becomes $c(E \oplus F) = c(E)c(F)$. As with Stiefel-Whitney classes, axioms (C1) and (C2) imply that for a trivial bundle T^n we have $c_i(T^n) = 0$ $(i \neq 0)$, and axiom (C3) implies that for all E

$$c_i(E \oplus T^n) = c_i(E).$$

For example, consider the canonical line bundle $E_1(\mathbb{C}^n)$ over $\mathbb{CP}^n$. By axiom (C4), $c(E_1) = 1 + x$ in the truncated polynomial ring $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$. We saw in Example 4.1.3 that there is a canonical n-dimensional vector bundle F with $F \oplus E_1 = T^{n+1}$. Using the Whitney Sum Formula (C3), we compute that $c(F) = \sum(-1)^i x^i$. Thus $c_i(F) = (-1)^i x^i$ for all $i \leq n$.

Chern classes are named for S.-S. Chern, who discovered them in 1946 while studying L. Pontrjagin's 1942 construction of cohomology classes $p_i(E) \in H^{4i}(X;\mathbb{Z})$ associated to a real vector bundle E. In fact, $p_i(E)$ is $(-1)^i c_{2i}(E \otimes \mathbb{C})$, where $E \otimes \mathbb{C}$ is the complexification of E (see Ex. 4.5). However, the Whitney sum formula for Pontrjagin classes only holds up to elements of order 2 in $H^{4n}(X;\mathbb{Z})$; see Ex. 4.13.

EXERCISES

4.1 Let $\eta\colon E \to X$ and $\varphi\colon F \to X$ be two vector bundles, and form the induced bundle η^*F over E. Show that the Whitney sum $E \oplus F \to X$ is η^*F, considered as a bundle over X by the map $\eta^*F \to E \to X$.

4.2 Show that all of the uncountably many vector bundles on the discrete space $X = \mathbb{N}$ are componentwise trivial. Let $T^{\mathbb{N}} \to \mathbb{N}$ be the bundle with $\dim(T^{\mathbb{N}}_n) = n$ for all n. Show that every componentwise trivial vector bundle $T^f \to Y$ over every space Y is isomorphic to $f^*T^{\mathbb{N}}$. Use this to show that the Stiefel-Whitney and Chern classes vanish for componentwise trivial vector bundles.

4.3 If E and F are vector bundles over X, show that there are vector bundles $\mathrm{Hom}(E,F)$, $\check{E}$ and $\wedge^k E$ over X whose fibers are, respectively: $\mathrm{Hom}(E_x,F_x)$, the dual space $(\check{E_x})$ and the exterior power $\wedge^k(E_x)$. Then show that there are natural isomorphisms $(E \oplus F)\check{} \cong \check{E} \oplus \check{F}$, $\check{E} \otimes F \cong \mathrm{Hom}(E,F)$, $\wedge^1 E \cong E$ and

$$\wedge^k(E \oplus F) \cong \wedge^k E \oplus (\wedge^{k-1}E \otimes F) \oplus \cdots \oplus (\wedge^i E \otimes \wedge^{k-i}F) \oplus \cdots \oplus \wedge^k F.$$

4.4 Show that the global sections of the bundle $\mathrm{Hom}(E,F)$ of Ex. 4.3 are in 1-1 correspondence with vector bundle maps $E \to F$. (Cf. 4.1.4.) If E is a line bundle, show that the vector bundle $\check{E} \otimes E \cong \mathrm{Hom}(E,E)$ is trivial.

4.5 *Complexification.* Let $E \to X$ be a real vector bundle. Show that there is a complex vector bundle $E_{\mathbb{C}} \to X$ with fibers $E_x \otimes_{\mathbb{R}} \mathbb{C}$ and that there is a natural isomorphism $(E \oplus F)_{\mathbb{C}} \cong (E_{\mathbb{C}}) \oplus (F_{\mathbb{C}})$. Then show that $E_{\mathbb{C}} \to X$, considered as a real vector bundle, is isomorphic to the Whitney sum $E \oplus E$.

4.6 *Complex conjugate bundle.* If $F \to X$ is a complex vector bundle on a paracompact space, given by transition functions g_{ij}, let $\bar{F}$ denote the complex vector bundle obtained by using the complex conjugates $\bar{g}_{ij}$ for transition functions; $\bar{F}$ is called the *complex conjugate bundle* of F. Show that F and $\bar{F}$ are isomorphic as real vector bundles, and that the complexification $F_{\mathbb{C}} \to X$ of Ex. 4.5 is isomorphic to the Whitney sum $F \oplus \bar{F}$. If $F = E_{\mathbb{C}}$ for some real bundle E, show that $F \cong \bar{F}$. Finally, show that for every complex line bundle L on X we have $\bar{L} \cong \check{L}$.

4.7 Use the formula $\bar{L} \cong \check{L}$ of Ex. 4.6 to show that $c_1(\bar{E}) = -c_1(E)$ in $H^2(X;\mathbb{Z})$ for every complex vector bundle E on a paracompact space.

4.8 *Global sections.* If $\eta\colon E \to X$ is a vector bundle, let $\Gamma(E)$ denote the set of all global sections of E (see 4.1.4). Show that $\Gamma(E)$ is a module over the ring $C^0(X)$ of continuous functions on X (taking values in $\mathbb{R}$ or $\mathbb{C}$). If E is an n-dimensional trivial bundle, show that $\Gamma(E)$ is a free $C^0(X)$-module of rank n.

Conclude that if X is paracompact then $\Gamma(E)$ is a locally free $C^0(X)$-module in the sense of 2.4, and that $\Gamma(E)$ is a finitely generated projective module if X is compact or if E is of finite type. This is the easy half of Swan's theorem; the rest is given in the next exercise.

4.9 *Swan's Theorem.* Let X be a compact Hausdorff space, and write R for $C^0(X)$. Show that the functor Γ of the previous exercise is a functor from $\mathbf{VB}(X)$ to the category $\mathbf{P}(R)$ of finitely generated projective modules, and that the homomorphisms

$$\Gamma\colon \mathrm{Hom}_{\mathbf{VB}(X)}(E,F) \to \mathrm{Hom}_{\mathbf{P}(R)}(\Gamma(E),\Gamma(F)) \qquad (*)$$

are isomorphisms. This proves Swan's Theorem, that Γ is an equivalence of categories $\mathbf{VB}(X) \approx \mathbf{P}(C^0(X))$. *Hint:* First show that $(*)$ holds when E and F are trivial bundles, and then use Corollary 4.1.1.

4.10 *Projective and Flag bundles.* If $E \to X$ is a vector bundle, consider the subspace $E_0 = E - X$ of E, where X lies in E as the zero section. The units $\mathbb{R}^\times$ (or $\mathbb{C}^\times$) act fiberwise on E_0, and the quotient space $\mathbb{P}(E)$ obtained by dividing out by this action is called the *projective bundle* associated to E. If $p\colon \mathbb{P}(E) \to X$ is the projection, the fibers $p^{-1}(x)$ are projective spaces.

(a) Show that there is a line sub-bundle L of p^*E over $\mathbb{P}(E)$. Use the Subbundle Theorem to conclude that $p^*E \cong E' \oplus L$.

Now suppose that $E \to X$ is an n-dimensional vector bundle, and let $\mathbb{F}(E)$ be the *flag space* $f\colon \mathbb{F}(E) \to X$ obtained by iterating the construction

$$\cdots \to \mathbb{P}(E'') \to \mathbb{P}(E') \to \mathbb{P}(E) \to X.$$

(b) Show that $f^*E \to \mathbb{F}(E)$ is a direct sum $L_1 \oplus \cdots \oplus L_n$ of line bundles.

4.11 If E is a direct sum $L_1 \oplus \cdots \oplus L_n$ of line bundles, show that $\det(E) \cong L_1 \otimes \cdots \otimes L_n$. Then use the Whitney Sum formula to show that $w_1(E) = w_1(\det(E))$, resp. $c_1(E) = c_1(\det(E))$. Prove that every $w_i(E)$, resp. $c_i(E)$ is the i^{th} elementary symmetric function of the n cohomology classes $\{w_1(L_i)\}$, resp. $\{c_1(L_i)\}$.

4.12 *Splitting Principle.* Write $H^i(X)$ for $H^i(X;\mathbb{Z}/2)$ or $H^{2i}(X;\mathbb{Z})$, depending on whether our base field is $\mathbb{R}$ or $\mathbb{C}$, and let $p : \mathbb{F}(E) \to X$ be the flag bundle of a vector bundle E over X (see Ex. 4.10). Prove that $p^*\colon H^i(X) \to H^i(\mathbb{F}(E))$ is an injection. Then use Ex. 4.11 to show that the characteristic classes $w_i(E)$ or $c_i(E)$ in $H^i(X)$ may be calculated inside $H^i(\mathbb{F}(E))$. *Hint:* For a trivial bundle this follows easily from the Künneth formula for $H^*(X \times \mathbb{F})$.

4.13 *Pontrjagin classes.* In this exercise we assume the results of Ex. 4.6 on the conjugate bundle $\bar{F}$ of a complex bundle F. Use the Splitting Principle to show that $c_i(\bar{F}) = (-1)^i c_i(F)$. Then prove the following:

(i) The Pontrjagin classes $p_n(F)$ of F (considered as a real bundle) are

$$p_n(F) = c_n(F)^2 + 2\sum_{i=1}^{n-1}(-1)^i c_{n-i}(F)c_{n+i}(F) + (-1)^n 2c_{2n}(F).$$

(ii) If $F = E \otimes \mathbb{C}$ for some real bundle E, the odd Chern classes $c_1(F), c_3(F), \dots$ all have order 2 in $H^*(X;\mathbb{Z})$.

(iii) The Whitney sum formula for Pontrjagin classes holds modulo 2:

$$p_n(E \oplus E') - \sum p_i(E)p_{n-i}(E') \text{ has order 2 in } H^{4n}(X;\mathbb{Z}).$$

4.14 *Disk with double origin.* The classification theorems 4.10 and 4.11 can fail for locally compact spaces which aren't Hausdorff. To see this, let D denote the closed unit disk in $\mathbb{R}^2$. The *disk with double origin* is the non-Hausdorff space X obtained from the disjoint union of two copies of D by identifying together the common subsets $D - \{0\}$. For all $n \geq 1$, show that $[X, BU_n] = [X, BO_n] = 0$, yet: $\mathbf{VB}_{n,\mathbb{C}}(X) \cong \mathbb{Z} \cong H^2(X;\mathbb{Z})$; $\mathbf{VB}_{2,\mathbb{R}}(X) \cong \mathbb{Z}$; and $\mathbf{VB}_{n,\mathbb{R}}(X) \cong \mathbb{Z}/2$ for $n \geq 3$.

4.15 Show that the canonical line bundles E_1 over $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$ do not have finite type. *Hint:* Use characteristic classes and the Subbundle Theorem, or II.3.7.2.

4.16 Consider the suspension SX of a paracompact space X. Show that every vector bundle E over SX has finite type. *Hint:* If $\dim(E) = n$, use 4.8 and Ex. 1.11 to construct a bundle E' such that $E \oplus E' \cong T^{2n}$.

4.17 Let V be a complex vector space. A quaternionic *structure map* on V is a complex conjugate-linear automorphism j satisfying $j^2 = -1$. A (complex) Hermitian metric β on V is said to be *quaternionic* if $\beta(jv, jw) = \overline{\beta(v,w)}$.

(a) Show that structure maps on V are in 1-1 correspondence with underlying $\mathbb{H}$-vector space structures on V in which $j \in \mathbb{H}$ acts as j.

(b) Given a structure map and a complex Hermitian metric β on V, show that the Hermitian metric $\frac{1}{2}\big(\beta(v,w) + \overline{\beta(jv,jw)}\big)$ is quaternionic. Conclude that every quaternionic vector bundle over a paracompact space has a quaternionic Hermitian metric.

(c) If V is a vector space over $\mathbb{H}$, show that its dual $\check{V} = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is also a vector space over $\mathbb{H}$. If E is a quaternionic vector bundle, show that there is a quaternionic vector bundle $\check{E}$ whose fibers are $\check{E_x}$. *Hint:* If V is a right $\mathbb{H}$-module, first construct $\check{V}$ as a left $\mathbb{H}$-module using Ex. 2.6 and then use $\mathbb{H} \cong \mathbb{H}^{op}$ to make it a right module.

4.18 Let E be a quaternionic vector bundle, and uE its underlying real vector bundle. If F is any real bundle, show that $\mathbb{H}^m \otimes_{\mathbb{R}} \mathbb{R}^n \cong \mathbb{H}^{mn}$ endows the real bundle $uE \otimes F$ with the natural structure of a quaternionic vector bundle, which we write as $E \otimes F$. Then show that $(E \otimes F_1) \otimes F_2 \cong E \otimes (F_1 \otimes F_2)$.

4.19 If E and F are quaternionic vector bundles over X, show that there are real vector bundles $E \otimes_{\mathbb{H}} F$ and $\mathrm{Hom}_{\mathbb{H}}(E, F)$, whose fibers are, respectively: $E_x \otimes_H F_x$ and $\mathrm{Hom}_{\mathbb{H}}(E_x, F_x)$. Then show that $\mathrm{Hom}_{\mathbb{H}}(E, F) \cong \check{E} \otimes_{\mathbb{H}} F$.

§5. Algebraic Vector Bundles

Modern Algebraic Geometry studies sheaves of modules over schemes. This generalizes modules over commutative rings, and has many features in common with the topological vector bundles that we considered in the last section. In this section we discuss the main aspects of the structure of algebraic vector bundles.

We will assume the reader has some rudimentary knowledge of the language of schemes, in order to get to the main points quickly. Here is a glossary of the basic concepts; details for most things may be found in [Hart], but the ultimate source is [EGA].

A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$; it is a *locally ringed space* if each $\mathcal{O}_X(U)$ is a commutative ring, and if for every $x \in X$ the stalk ring $\mathcal{O}_{X,x} = \varinjlim_{x\in U} \mathcal{O}_X(U)$ is a local ring. By definition, an *affine scheme* is a locally ringed space isomorphic to $(\text{Spec}(R), \tilde{R})$ for some commutative ring R (where $\tilde{R}$ is the canonical structure sheaf), and a *scheme* is a ringed space $(X, \mathcal{O}_X)$ which can be covered by open sets U_i such that each $(U_i, \mathcal{O}_X|U_i)$ is an affine scheme.

An $\mathcal{O}_X$*-module* is a sheaf $\mathcal{F}$ on X such that (i) for each open $U \subseteq X$ the set $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and (ii) if $V \subset U$ then the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures. A morphism $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a sheaf map such that each $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. The category $\mathcal{O}_X$-**mod** of all $\mathcal{O}_X$-modules is an abelian category.

A *global section* of an $\mathcal{O}_X$-module $\mathcal{F}$ is an element e_i of $\mathcal{F}(X)$. We say that $\mathcal{F}$ is *generated by global sections* if there is a set $\{e_i\}_{i\in I}$ of global sections of $\mathcal{F}$ whose restrictions $e_i|_U$ generate $\mathcal{F}(U)$ as an $\mathcal{O}_X(U)$-module for every open $U \subseteq X$. We can reinterpret these definitions as follows. Giving a global section e of $\mathcal{F}$ is equivalent to giving a morphism $\mathcal{O}_X \to \mathcal{F}$ of $\mathcal{O}_X$-modules, and to say that $\mathcal{F}$ is generated by the global sections $\{e_i\}$ is equivalent to saying that the corresponding morphism $\oplus_{i\in I} \mathcal{O}_X \to \mathcal{F}$ is a surjection.

FREE MODULES. We say that $\mathcal{F}$ is a *free* $\mathcal{O}_X$*-module* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. A set $\{e_i\} \subset \mathcal{F}(X)$ is called a *basis* of $\mathcal{F}$ if the restrictions $e_i|_U$ form a basis of each $\mathcal{F}(U)$, *i.e.*, if the e_i provide an explicit isomorphism $\oplus\mathcal{O}_X \cong \mathcal{F}$.

The rank of a free $\mathcal{O}_X$-module $\mathcal{F}$ is not well-defined over all ringed spaces. For example, if X is a 1-point space then $\mathcal{O}_X$ is just a ring R and an $\mathcal{O}_X$-module is just an R-module, so our remarks in §1 about the invariant basis property (IBP) apply. There is no difficulty in defining the rank of a free $\mathcal{O}_X$-module when $(X, \mathcal{O}_X)$ is a scheme, or a locally ringed space, or even more generally when any of the rings $\mathcal{O}_X(U)$ satisfy the IBP. We shall avoid these difficulties by assuming henceforth that $(X, \mathcal{O}_X)$ is a locally ringed space.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if X can be covered by open sets U for which $\mathcal{F}|_U$ is a free $\mathcal{O}_U$-module. The *rank* of a locally free module $\mathcal{F}$ is defined at each point x of X: $\text{rank}_x(\mathcal{F})$ is the rank of the free $\mathcal{O}_U$-module $\mathcal{F}|_U$, where U is a neighborhood of x on which $\mathcal{F}|_U$ is free. Since the function $x \mapsto \text{rank}_x(\mathcal{F})$ is locally constant, $\text{rank}(\mathcal{F})$ is a continuous function on X. In particular, if X is a connected space then every locally free module has constant rank.

DEFINITION 5.1 (VECTOR BUNDLES). A *vector bundle* over a ringed space X is a locally free $\mathcal{O}_X$-module whose rank is finite at every point. We will write $\mathbf{VB}(X)$

or $\mathbf{VB}(X, \mathcal{O}_X)$ for the category of vector bundles on $(X, \mathcal{O}_X)$; the morphisms in $\mathbf{VB}(X)$ are just morphisms of $\mathcal{O}_X$-modules. Since the direct sum of locally free modules is locally free, $\mathbf{VB}(X)$ is an additive category.

A *line bundle* $\mathcal{L}$ is a locally free module of constant rank 1. A line bundle is also called an *invertible sheaf* because as we shall see in 5.3 there is another sheaf $\mathcal{L}'$ such that $\mathcal{L} \otimes \mathcal{L}' = \mathcal{O}_X$.

These notions are the analogues for ringed spaces of finitely generated projective modules and algebraic line bundles, as can be seen from the discussion in 2.4 and §3. However, the analogy breaks down if X is not locally ringed; in effect locally projective modules need not be locally free.

Example 5.1.1. (Topological spaces). Fix a topological space X. Then $X_{top} = (X, \mathcal{O}_{top})$ is a locally ringed space, where $\mathcal{O}_{top}$ is the sheaf of ($\mathbb{R}$ or $\mathbb{C}$-valued) continuous functions on X: $\mathcal{O}_{top}(U) = C^0(U)$ for all $U \subseteq X$. The following constructions give an equivalence between the category $\mathbf{VB}(X_{top})$ of vector bundles over the ringed space X_{top} and the category $\mathbf{VB}(X)$ of (real or complex) topological vector bundles over X in the sense of §4. Thus our notation is consistent with the notation of §4.

If $\eta\colon E \to X$ is a topological vector bundle, then the sheaf $\mathcal{E}$ of continuous sections of E is defined by $\mathcal{E}(U) = \{s\colon U \to E : \eta s = 1_U\}$. By Ex. 4.8 we know that $\mathcal{E}$ is a locally free $\mathcal{O}_{top}$-module. Conversely, given a locally free $\mathcal{O}_{top}$-module $\mathcal{E}$, choose a cover $\{U_i\}$ and bases for the free $\mathcal{O}_{top}$-modules $\mathcal{E}|U_i$; the base change isomorphisms over the $U_i \cap U_j$ are elements g_{ij} of $GL_n(C^0(U_i \cap U_j))$. Interpreting the g_{ij} as maps $U_i \cap U_j \to GL_n(\mathbb{C})$, they become transition functions for a topological vector bundle $E \to X$ in the sense of 4.2.

Example 5.1.2 (Affine schemes). Suppose $X = \operatorname{Spec}(R)$. Every R-module M yields an $\mathcal{O}_X$-module $\tilde{M}$, and $\tilde{R} = \mathcal{O}_X$. Hence every free $\mathcal{O}_X$-module arises as $\tilde{M}$ for a free R-module M. The $\mathcal{O}_X$-module $\mathcal{F} = \tilde{P}$ associated to a finitely generated projective R-module P is locally free by 2.4, and the two rank functions agree: $\operatorname{rank}(P) = \operatorname{rank}(\mathcal{F})$. Conversely, if $\mathcal{F}$ is locally free $\mathcal{O}_X$-module, it can be made trivial on a covering by open sets of the form $U_i = D(s_i)$, *i.e.*, there are free modules M_i such that $\mathcal{F}|_{U_i} = \tilde{M}_i$. The isomorphisms between the restrictions of $\tilde{M}_i$ and $\tilde{M}_j$ to $U_i \cap U_j$ amount to open patching data defining a projective R-module P as in 2.5. In fact it is not hard to see that $\mathcal{F} \cong \tilde{P}$. Thus vector bundles on $\operatorname{Spec}(R)$ are in 1-1 correspondence with finitely generated projective R-modules. And it is no accident that the notion of an algebraic line bundle over a ring R in §3 corresponds exactly to the notion of a line bundle over the ringed space $(\operatorname{Spec}(R), \tilde{R})$.

More is true: the categories $\mathbf{VB}(X)$ and $\mathbf{P}(R)$ are equivalent when $X = \operatorname{Spec}(R)$. To see this, recall that an $\mathcal{O}_X$-module is called *quasicoherent* if it is isomorphic to some $\tilde{M}$ ([Hart, II.5.4]). The above correspondence shows that every vector bundle is quasicoherent. It turns out that the category $\mathcal{O}_X\text{-}\mathbf{mod}_{qcoh}$ of quasicoherent $\mathcal{O}_X$-modules is equivalent to the category R-**mod** of all R-modules (see [Hart, II.5.5]). Since the subcategories $\mathbf{VB}(\operatorname{Spec} R)$ and $\mathbf{P}(R)$ correspond, they are equivalent.

Definition (Coherent modules). Suppose that X is any scheme. We say that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasicoherent* if X may be covered by affine opens $U_i = \operatorname{Spec}(R_i)$ such that each $\mathcal{F}|U_i$ is $\tilde{M}_i$ for an R_i-module M_i. (If X is affine, this agrees with the definition of quasicoherent in Example 5.1.2 by [Hart, II.5.4].) We say that $\mathcal{F}$ is *coherent* if moreover each M_i is a finitely presented R_i-module.

The category of quasicoherent $\mathcal{O}_X$-modules is abelian; if X is noetherian then so is the category of coherent $\mathcal{O}_X$-modules.

If X is affine then $\mathcal{F} = \tilde{M}$ is coherent if and only if M is a finitely presented R-module by [EGA, I(1.4.3)]. In particular, if R is noetherian then "coherent" is just a synonym for "finitely generated." If X is a noetherian scheme, our definition of coherent module agrees with [Hart] and [EGA]. For general schemes, our definition is slightly stronger than in [Hart], and slightly weaker than in [EGA, $0_I(5.3.1)$]; $\mathcal{O}_X$ is always coherent in our sense, but not in the sense of [EGA].

The equivalent conditions for locally free modules in 2.4 translate into:

LEMMA 5.1.3. *For every scheme X and $\mathcal{O}_X$-module $\mathcal{F}$, the following conditions are equivalent:*

(1) *$\mathcal{F}$ is a vector bundle (i.e., is locally free of finite rank);*
(2) *$\mathcal{F}$ is quasicoherent and the stalks $\mathcal{F}_x$ are free $\mathcal{O}_{X,x}$-modules of finite rank;*
(3) *$\mathcal{F}$ is coherent and the stalks $\mathcal{F}_x$ are free $\mathcal{O}_{X,x}$-modules;*
(4) *For every affine open $U = \mathrm{Spec}(R)$ in X, $\mathcal{F}|_U$ is a finitely generated projective R-module.*

EXAMPLE 5.1.4. (Analytic spaces). Analytic spaces form another family of locally ringed spaces. To define them, one proceeds as follows. On the topological space $\mathbb{C}^n$, the subsheaf $\mathcal{O}_{an}$ of $\mathcal{O}_{top}$ consisting of analytic functions makes $(\mathbb{C}^n, \mathcal{O}_{an})$ into a locally ringed space. A *basic analytic set* in an open subset U of $\mathbb{C}^n$ is the zero locus V of a finite number of holomorphic functions, made into a locally ringed space $(V, \mathcal{O}_{V,an})$ as follows. If $\mathcal{I}_V$ is the subsheaf of $\mathcal{O}_{U,an}$ consisting of functions vanishing on V, the quotient sheaf $\mathcal{O}_{V,an} = \mathcal{O}_{U,an}/\mathcal{I}_V$ is supported on V, and is a subsheaf of the sheaf $\mathcal{O}_{V,top}$. By definition, a (reduced) *analytic space* $X_{an} = (X, \mathcal{O}_{an})$ is a ringed space which is locally isomorphic to a basic analytic set. A good reference for (reduced) analytic spaces is [GA]; the original source is Serre's [GAGA].

Let X_{an} be an analytic space. For clarity, a vector bundle over X_{an} (in the sense of Definition 5.1) is sometimes called an *analytic vector bundle*. Since finitely generated $\mathcal{O}_{an}(U)$-modules are finitely presented, there is also good a notion of coherence on an analytic space: an $\mathcal{O}_{an}$-module $\mathcal{F}$ is called *coherent* if it is locally finitely presented in the sense that in a neighborhood U of any point it is presented as a cokernel:

$$\mathcal{O}_{U,an}^n \to \mathcal{O}_{U,an}^m \to \mathcal{F}|_U \to 0.$$

One special class of analytic spaces is the class of *Stein spaces*. It is known that analytic vector bundles are the same as topological vector bundles over a Stein space. For example, any analytic subspace of $\mathbb{C}^n$ is a Stein space. See [GR].

Morphisms of ringed spaces

Here are two basic ways to construct new ringed spaces and morphisms:

(1) If $\mathcal{A}$ is a sheaf of $\mathcal{O}_X$-algebras, $(X, \mathcal{A})$ is a ringed space;
(2) If $f\colon Y \to X$ is a continuous map and $(Y, \mathcal{O}_Y)$ is a ringed space, the direct image sheaf $f_*\mathcal{O}_Y$ is a sheaf of rings on X, so $(X, f_*\mathcal{O}_Y)$ is a ringed space.

A *morphism of ringed spaces* $f:(Y,\mathcal{O}_Y)\to(X,\mathcal{O}_X)$ is a continuous map $f:Y\to X$ together with a map $f^{\#}:\mathcal{O}_X\to f_*\mathcal{O}_Y$ of sheaves of rings on X. In case (1) there is a morphism $i:(X,\mathcal{A})\to(X,\mathcal{O}_X)$; in case (2) the morphism is $(Y,\mathcal{O}_Y)\to(X,f_*\mathcal{O}_Y)$; in general, every morphism factors as $(Y,\mathcal{O}_Y)\to(X,f_*\mathcal{O}_Y)\to(X,\mathcal{O}_X)$.

A morphism of ringed spaces $f:X\to Y$ between two locally ringed spaces is a *morphism of locally ringed spaces* if in addition for each point $y\in Y$ the map of stalk rings $\mathcal{O}_{X,f(y)}\to\mathcal{O}_{Y,y}$ sends the maximal ideal $\mathfrak{m}_{f(y)}$ into the maximal ideal $\mathfrak{m}_y$. A *morphism of schemes* is a morphism of locally ringed spaces $f:Y\to X$ between schemes.

If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then the direct image sheaf $f_*\mathcal{F}$ is an $f_*\mathcal{O}_Y$-module, and hence also an $\mathcal{O}_X$-module. Thus f_* is a functor from $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules, making $\mathcal{O}_X$-**mod** covariant in X. If $\mathcal{F}$ is a vector bundle over Y then $f_*\mathcal{F}$ is a vector bundle over $(X,f_*\mathcal{O}_Y)$. However, $f_*\mathcal{F}$ will not be a vector bundle over $(X,\mathcal{O}_X)$ unless $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module of finite rank, which rarely occurs.

If $f:Y\to X$ is a *proper* morphism between noetherian schemes then Serre's "Theorem B" states that if $\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module then the direct image $f_*\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. (See [EGA, III(3.2.2)] or [Hart, III.5.2 and II.5.19].)

EXAMPLE 5.2.1 (PROJECTIVE SCHEMES). When Y is a projective scheme over a field k, the structural map $\pi:Y\to\operatorname{Spec}(k)$ is proper. In this case the direct image $\pi_*\mathcal{F}=H^0(Y,\mathcal{F})$ is a finite-dimensional vector space over k. Indeed, every coherent k-module is finitely generated. Not surprisingly, $\dim_k H^0(Y,\mathcal{F})$ gives an important invariant for coherent modules (and vector bundles) over projective schemes.

The functor f_* has a left adjoint f^* (from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules):

$$\operatorname{Hom}_{\mathcal{O}_Y}(f^*\mathcal{E},\mathcal{F})\cong\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E},f_*\mathcal{F})$$

for every $\mathcal{O}_X$-module $\mathcal{E}$ and $\mathcal{O}_Y$-module $\mathcal{F}$. The explicit construction is given in [Hart, II.5], and shows that f^* sends free $\mathcal{O}_X$-modules to free $\mathcal{O}_Y$-modules, with $f^*\mathcal{O}_X\cong\mathcal{O}_Y$. If $i:U\subset X$ is the inclusion of an open subset then $i^*\mathcal{E}$ is just $\mathcal{E}|U$; it follows that if $\mathcal{E}|U$ is free then $(f^*\mathcal{E})|f^{-1}(U)$ is free. Thus f^* sends locally free $\mathcal{O}_X$-modules to locally free $\mathcal{O}_Y$-modules, and yields a functor $f^*:\mathbf{VB}(X)\to\mathbf{VB}(Y)$, making $\mathbf{VB}(X)$ contravariant in the ringed space X.

EXAMPLE 5.2.2. If R and S are commutative rings then ring maps $f^{\#}:R\to S$ are in 1-1 correspondence with morphisms $f:\operatorname{Spec}(S)\to\operatorname{Spec}(R)$ of ringed spaces. The direct image functor f_* corresponds to the forgetful functor from S-modules to R-modules, and the functor f^* corresponds to the functor $\otimes_R S$ from R-modules to S-modules.

ASSOCIATED ANALYTIC AND TOPOLOGICAL BUNDLES 5.2.3. Suppose that X is a scheme of finite type over $\mathbb{C}$, such as a subvariety of $\mathbb{P}^n_{\mathbb{C}}$ or $\mathbb{A}^n_{\mathbb{C}}=\operatorname{Spec}(\mathbb{C}[x_1,...,x_n])$. The closed points $X(\mathbb{C})$ of X have the natural structure of an analytic space; in particular it is a locally compact topological space. Indeed, $X(\mathbb{C})$ is covered by open sets $U(\mathbb{C})$ homeomorphic to analytic subspaces of $\mathbb{A}^n(\mathbb{C})$, and $\mathbb{A}^n(\mathbb{C})\cong\mathbb{C}^n$. Note that if X is a projective variety then $X(\mathbb{C})$ is compact, because it is a closed subspace of the compact space $\mathbb{P}^n(\mathbb{C})\cong\mathbb{CP}^n$.

Considering $X(\mathbb{C})$ as topological and analytic ringed spaces as in Examples 5.1.1 and 5.1.4, the evident continuous map $\tau:X(\mathbb{C})\to X$ induces morphisms of ringed spaces $X(\mathbb{C})_{top} \to X(\mathbb{C})_{an} \to X$. This yields functors from $\mathbf{VB}(X, \mathcal{O}_X)$ to $\mathbf{VB}(X(\mathbb{C})_{an})$, and from $\mathbf{VB}(X_{an})$ to $\mathbf{VB}(X(\mathbb{C})_{top}) \cong \mathbf{VB}_{\mathbb{C}}(X(\mathbb{C}))$. Thus every vector bundle $\mathcal{E}$ over the scheme X has an associated analytic vector bundle $\mathcal{E}_{an}$, as well as an associated complex vector bundle $\tau^*\mathcal{E}$ over $X(\mathbb{C})$. In particular, every vector bundle $\mathcal{E}$ on X has topological Chern classes $c_i(\mathcal{E}) = c_i(\tau^*\mathcal{E})$ in the group $H^{2i}(X(\mathbb{C}); \mathbb{Z})$.

The main theorem of [GAGA] is that if X is a projective algebraic variety over $\mathbb{C}$ then there is an equivalence between the categories of coherent modules over X and over X_{an}. In particular, the categories of vector bundles $\mathbf{VB}(X)$ and $\mathbf{VB}(X_{an})$ are equivalent.

A similar situation arises if X is a scheme of finite type over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the closed points of X with residue field $\mathbb{R}$; it too is a locally compact space. We consider $X(\mathbb{R})$ as a ringed space, using $\mathbb{R}$-valued functions as in Example 5.1.1. There is a morphism of ringed spaces $\tau\colon X(\mathbb{R}) \to X$, and the functor τ^* sends $\mathbf{VB}(X)$ to $\mathbf{VB}_{\mathbb{R}}(X(\mathbb{R}))$. That is, every vector bundle $\mathcal{F}$ over X has an associated real vector bundle $\tau^*\mathcal{F}$ over $X(\mathbb{R})$; in particular, every vector bundle $\mathcal{F}$ over X has Stiefel-Whitney classes $w_i(\mathcal{F}) = w_i(\tau^*\mathcal{F}) \in H^i(X(\mathbb{R}); \mathbb{Z}/2)$.

Patching and Operations 5.3. Just as we built up projective modules by patching in 2.5, we can obtain a locally free sheaf $\mathcal{F}$ by patching (or *glueing*) locally free sheaves $\mathcal{F}_i$ of $\mathcal{O}_{U_i}$-modules via isomorphisms g_{ij} between $\mathcal{F}_j|U_i \cap U_j$ and $\mathcal{F}_i|U_i \cap U_j$, as long as $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ for all i, j, k.

The patching process allows us to take any natural operation on free modules and extend it to locally free modules. For example, if $\mathcal{O}_X$ is commutative we can construct tensor products $\mathcal{F} \otimes \mathcal{G}$, Hom-modules $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, dual modules $\check{\mathcal{F}}$ and exterior powers $\wedge^i\mathcal{F}$ using $P \otimes_R Q$, $\mathrm{Hom}_R(P, Q)$, $\check{P}$ and $\wedge^i P$. If $\mathcal{F}$ and $\mathcal{G}$ are vector bundles, then so are $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$, $\check{\mathcal{F}}$ and $\wedge^i\mathcal{F}$. All of the natural isomorphisms such as $\check{\mathcal{F}} \otimes \mathcal{G} \cong \mathcal{H}om(\mathcal{F}, \mathcal{G})$ hold for locally free modules, because a sheaf map is an isomorphism if it is locally an isomorphism.

The Picard group and determinant bundles

If $(X, \mathcal{O}_X)$ is a commutative ringed space, the set $\mathrm{Pic}(X)$ of isomorphism classes of line bundles forms a group, called the *Picard group* of X. To see this, we modify the proof in §3: the dual $\check{\mathcal{L}}$ of a line bundle $\mathcal{L}$ is again a line bundle and $\check{\mathcal{L}} \otimes \mathcal{L} \cong \mathcal{O}_X$ because by Lemma 3.1 this is true locally. Note that if X is $\mathrm{Spec}(R)$, we recover the definition of §3: $\mathrm{Pic}(\mathrm{Spec}(R)) = \mathrm{Pic}(R)$.

If $\mathcal{F}$ is locally free of rank n, then $\det(\mathcal{F}) = \wedge^n(\mathcal{F})$ is a line bundle. Operating componentwise as in §3, every locally free $\mathcal{O}_X$-module $\mathcal{F}$ has an associated determinant line bundle $\det(\mathcal{F})$. The natural map $\det(\mathcal{F}) \otimes \det(\mathcal{G}) \to \det(\mathcal{F} \oplus \mathcal{G})$ is an isomorphism because this is true locally by the Sum Formula in §3 (see Ex. 5.4 for a generalization). Thus det is a useful invariant of a locally free $\mathcal{O}_X$-module. We will discuss $\mathrm{Pic}(X)$ in terms of divisors at the end of this section.

Projective schemes

If X is a projective variety, maps between vector bundles are most easily described using graded modules. Following [Hart, II.2]] this trick works more generally if X is $\mathrm{Proj}(S)$ for a commutative graded ring $S = S_0 \oplus S_1 \oplus \cdots$. By definition, the scheme

$\mathrm{Proj}(S)$ is the union of the affine open sets $D_+(f) = \mathrm{Spec}\, S_{(f)}$, where $f \in S_n$ $(n \geq 1)$ and $S_{(f)}$ is the degree 0 subring of the $\mathbb{Z}$-graded ring $S[\frac{1}{f}]$. To cover $\mathrm{Proj}(S)$, it suffices to use $D_+(f)$ for a family of f's generating the ideal $S_+ = S_1 \oplus S_2 \oplus \cdots$ of S. For example, *projective n-space* over R is $\mathbb{P}^n_R = \mathrm{Proj}(R[X_0, ..., X_n])$; it is covered by the $D_+(X_i)$ and if $x_j = X_j/X_i$ then $D_+(X_i) = \mathrm{Spec}(R[x_1, ..., x_n])$.

If $M = \oplus_{i\in\mathbb{Z}} M_i$ is a graded S-module, there is an associated $\mathcal{O}_X$-module $\tilde{M}$ on $X = \mathrm{Proj}(S)$. The restriction of $\tilde{M}$ to $D_+(f)$ is the sheaf associated to $M_{(f)}$, the $S_{(f)}$-module which constitutes the degree 0 submodule of $M[\frac{1}{f}]$; more details of the construction of $\tilde{M}$ are given in [Hart, II.5.11]. Clearly $\tilde{S} = \mathcal{O}_X$. The functor $M \mapsto \tilde{M}$ is exact, and has the property that $\tilde{M} = 0$ whenever $M_i = 0$ for large i.

Example 5.3.1 (Twisting Line Bundles). The most important example of this construction is when M is $S(n)$, the module S regraded so that the degree i part is S_{n+i}; the associated sheaf $\tilde{S}(n)$ is written as $\mathcal{O}_X(n)$. If $f \in S_1$ then $S(n)_{(f)} \cong S_{(f)}$, so if S is generated by S_1 as an S_0-algebra then $\mathcal{O}_X(n)$ is a line bundle on $X = \mathrm{Proj}(S)$; it is called the n^{th} *twisting line bundle*. If $\mathcal{F}$ is any $\mathcal{O}_X$-module, we write $\mathcal{F}(n)$ for $\mathcal{F} \otimes \mathcal{O}_X(n)$, and call it "$\mathcal{F}$ twisted n times."

We will usually assume that S is generated by S_1 as an S_0-algebra, so that the $\mathcal{O}_X(n)$ are line bundles. This hypothesis ensures that every quasicoherent $\mathcal{O}_X$-module has the form $\tilde{M}$ for some M ([Hart, II.5.15]). It also ensures that the canonical maps $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to (M \otimes_S N)\tilde{}$ are isomorphisms, so if $\mathcal{F} = \tilde{M}$ then $\mathcal{F}(n)$ is the $\mathcal{O}_X$-module associated to $M(n) = M \otimes_S S(n)$. Since $S(m) \otimes S(n) \cong S(m+n)$ we have the formula

$$\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n).$$

Thus there is a homomorphism from $\mathbb{Z}$ to $\mathrm{Pic}(X)$ sending n to $\mathcal{O}_X(n)$. Operating componentwise, the same formula yields a homomorphism $[X, \mathbb{Z}] \to \mathrm{Pic}(X)$.

Here is another application of twisting line bundles. An element $x \in M_n$ gives rise to a graded map $S(-n) \to M$ and hence a sheaf map $\mathcal{O}_X(-n) \to \tilde{M}$. Taking the direct sum over a generating set for M, we see that for every quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ there is a surjection from a locally free module $\oplus\mathcal{O}_X(-n_i)$ onto $\mathcal{F}$. In contrast, there is a surjection from a free $\mathcal{O}_X$-module onto $\mathcal{F}$ if and only if $\mathcal{F}$ can be generated by global sections, which is not always the case.

If P is a graded finitely generated projective S-module, the $\mathcal{O}_X$-module $\tilde{P}$ is a vector bundle over $\mathrm{Proj}(S)$. To see this, suppose the generators of P lie in degrees $n_1, ..., n_r$ and set $F = S(-n_1) \oplus \cdots \oplus S(-n_r)$. The kernel Q of the surjection $F \to P$ is a graded S-module, and that the projective lifting property implies that $P \oplus Q \cong F$. Hence $\tilde{P} \oplus \tilde{Q}$ is the direct sum $\tilde{F}$ of the line bundles $\mathcal{O}_X(-n_i)$, proving that $\tilde{P}$ is a vector bundle.

Example 5.4 (No vector bundles are projective). Consider the projective line $\mathbb{P}^1_R = \mathrm{Proj}(S)$, $S = R[x, y]$. Associated to the "Koszul" exact sequence of graded S-modules

$$0 \to S(-2) \xrightarrow{(y,-x)} S(-1) \oplus S(-1) \xrightarrow{(x,y)} S \to R \to 0 \qquad (5.4.1)$$

is the short exact sequence of vector bundles over $\mathbb{P}^1_R$:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to 0. \qquad (5.4.2)$$

The sequence (5.4.2) cannot split, because there are no nonzero maps from $\mathcal{O}_{\mathbb{P}^1}$ to $\mathcal{O}_{\mathbb{P}^1}(-1)$ (see Ex. 5.2). This shows that the projective lifting property of §2 fails for the free module $\mathcal{O}_{\mathbb{P}^1}$. In fact, the projective lifting property fails for every vector bundle over $\mathbb{P}^1_R$; the category of $\mathcal{O}_{\mathbb{P}^1}$-modules has no "projective objects." This failure is the single biggest difference between the study of projective modules over rings and vector bundles over schemes.

The strict analogue of the Cancellation Theorem 2.3 does not hold for projective schemes. To see this, we cite the following result from [Atiy56]. A vector bundle is called *indecomposable* if it cannot be written as the sum of two proper sub-bundles. For example, every line bundle is indecomposable.

Krull-Schmidt Theorem 5.5 (Atiyah). *Let X be a projective scheme over a field k. Then the Krull-Schmidt theorem holds for vector bundles over X. That is, every vector bundle over X can be written uniquely (up to reordering) as a direct sum of indecomposable vector bundles.*

In particular, the direct sums of line bundles $\mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_r)$ are all distinct whenever $\dim(X) \neq 0$, because then all the $\mathcal{O}_X(n_i)$ are distinct.

Example 5.5.1. If X is a smooth projective curve over $\mathbb{C}$, then the associated topological space $X(\mathbb{C})$ is a Riemann surface. We saw in 4.11.2 that every topological line bundle on $X(\mathbb{C})$ is completely determined by its topological degree, and that every topological vector bundle is completely determined by its rank and degree. Now it is not hard to show that the twisting line bundle $\mathcal{O}_X(d)$ has degree d. Hence every topological vector bundle $\mathcal{E}$ of rank r and degree d is isomorphic to the direct sum $\mathcal{O}_X(d) \oplus T^{r-1}$. Moreover, the topological degree of a line bundle agrees with the usual algebraic degree one encounters in Algebraic Geometry.

The Krull-Schmidt Theorem shows that for each $r \geq 2$ and $d \in \mathbb{Z}$ there are infinitely many vector bundles over X with rank r and degree d. Indeed, there are infinitely many ways to choose integers $d_1, \dots, d_r$ so that $\sum d_i = d$, and these choices yield the vector bundles $\mathcal{O}_X(d_1) \oplus \cdots \oplus \mathcal{O}_X(d_r)$, which are all distinct with rank r and degree d.

For $X = \mathbb{P}^1_k$, the only indecomposable vector bundles are the line bundles $\mathcal{O}(n)$. This is a theorem of A. Grothendieck, proven in [Groth57]. Using the Krull-Schmidt Theorem, we obtain the following classification.

Theorem 5.6 (Classification of Vector Bundles over $\mathbb{P}^1_k$). *Let k be an algebraically closed field. Every vector bundle $\mathcal{F}$ over $X = \mathbb{P}^1_k$ is a direct sum of the line bundles $\mathcal{O}_X(n)$ in a unique way. That is, $\mathcal{F}$ determines a finite decreasing family of integers $n_1 \geq \cdots \geq n_r$ such that*

$$\mathcal{F} \cong \mathcal{O}_X(n_1) \oplus \cdots \oplus \mathcal{O}_X(n_r).$$

The classification over other spaces is much more complicated than it is for $\mathbb{P}^1$. The following example is taken from [Atiy57]. Atiyah's result holds over any algebraically closed field k, but we shall state it for $k = \mathbb{C}$ because we have not yet introduced the notion on the degree of a line bundle. (Using the Riemann-Roch theorem, we could define the degree of a line bundle $\mathcal{L}$ over an elliptic curve as the integer $\dim H^0(X, \mathcal{L}(n)) - n$ for $n >> 0$.)

Classification of vector bundles over elliptic curves 5.7. Let X be a smooth elliptic curve over $\mathbb{C}$. Every vector bundle $\mathcal{E}$ over X has two integer invariants: its rank, and its *degree*, which we saw in 5.5.1 is just the Chern class $c_1(E) \in H^2(X(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}$ of the associated topological vector bundle over the Riemann surface $X(\mathbb{C})$ of genus 1, defined in 5.2.3. Let $\mathbf{VB}^{ind}_{r,d}(X)$ denote the set of isomorphism classes of *indecomposable* vector bundles over X having rank r and degree d. Then for all $r \geq 1$ and $d \in \mathbb{Z}$:

(1) All the vector bundles in the set $\mathbf{VB}^{ind}_{r,d}(X)$ yield the same topological vector bundle E over $X(\mathbb{C})$. This follows from Example 5.5.1.
(2) There is a natural identification of each $\mathbf{VB}^{ind}_{r,d}(X)$ with the set $X(\mathbb{C})$; in particular, there are uncountably many indecomposable vector bundles of rank r and degree d.
(3) Tensoring with the twisting bundle $\mathcal{O}_X(d)$ induces a bijection between $\mathbf{VB}^{ind}_{r,0}(X)$ and $\mathbf{VB}^{ind}_{r,d}(X)$.
(4) The r^{th} exterior power $\wedge^r$ maps $\mathbf{VB}^{ind}_{r,d}(X)$ onto $\mathbf{VB}^{ind}_{1,d}(X)$. This map is a bijection if and only if r and d are relatively prime. If $(r,d) = h$ then for each line bundle $\mathcal{L}$ of degree d there are h^2 vector bundles $\mathcal{E}$ with rank r and determinant $\mathcal{L}$.

Construction 5.8 (Projective bundles). If $\mathcal{E}$ is an vector bundle over a scheme X, we can form a *projective space bundle* $\mathbb{P}(\mathcal{E})$, which is a scheme equipped with a map $\pi\colon \mathbb{P}(\mathcal{E}) \to X$ and a canonical line bundle $\mathcal{O}(1)$. To do this, we first construct $\mathbb{P}(\mathcal{E})$ when X is affine, and then glue the resulting schemes together.

If M is any module over a commutative ring R, the i^{th} *symmetric product* $Sym^i M$ is the quotient of the i-fold tensor product $M \otimes \cdots \otimes M$ by the permutation action of the symmetric group, identifying $m_1 \otimes \cdots \otimes m_i$ with $m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(i)}$ for every permutation σ. The obvious concatenation product $(Sym^i M) \otimes_R (Sym^j M) \to Sym^{i+j} M$ makes $Sym(M) = \oplus Sym^i(M)$ into a graded commutative R-algebra, called the *symmetric algebra* of M. As an example, note that if $M = R^n$ then $Sym(M)$ is the polynomial ring $R[x_1, ..., x_n]$. This construction is natural in R: if $R \to S$ is a ring homomorphism, then $Sym(M) \otimes_R S \cong Sym(M \otimes_R S)$.

If E is a finitely generated projective R-module, let $\mathbb{P}(E)$ denote the scheme $\mathrm{Proj}(Sym(E))$. This scheme comes equipped with a map $\pi\colon \mathbb{P}(E) \to \mathrm{Spec}(R)$ and a canonical line bundle $\mathcal{O}(1)$; the scheme $\mathbb{P}(E)$ with this data is called the *projective space bundle* associated to E. If $E = R^n$, then $\mathbb{P}(E)$ is just the projective space $\mathbb{P}^{n-1}_R$. In general, the fact that E is locally free implies that $\mathrm{Spec}(R)$ is covered by open sets $D(s) = \mathrm{Spec}(R[\frac{1}{s}])$ on which E is free. If $E[\frac{1}{s}]$ is free of rank n then the restriction of $\mathbb{P}(E)$ to $D(s)$ is

$$\mathbb{P}(E[\frac{1}{s}]) \cong \mathrm{Proj}(R[\frac{1}{s}][x_1, ..., x_n]) = \mathbb{P}^{n-1}_{D(s)}.$$

Hence $\mathbb{P}(E)$ is locally just a projective space over $\mathrm{Spec}(R)$. The vector bundles $\mathcal{O}(1)$ and $\pi^*\tilde{E}$ on $\mathbb{P}(E)$ are the sheaves associated to the graded S-modules $S(1)$ and $E \otimes_R S$, where S is $Sym(E)$. The concatenation $E \otimes Sym^j(E) \to Sym^{1+j}(E)$ yields an exact sequence of graded modules,

$$0 \to E_1 \to E \otimes_R S \to S(1) \to R(-1) \to 0 \qquad (5.8.1)$$

hence a natural short exact sequence of $\mathbb{P}(E)$-modules

$$0 \to \mathcal{E}_1 \to \pi^* \tilde{E} \to \mathcal{O}(1) \to 0. \tag{5.8.2}$$

Since $\pi^* \tilde{E}$ and $\mathcal{O}(1)$ are locally free, $\mathcal{E}_1$ is locally free and $\text{rank}(\mathcal{E}_1) = \text{rank}(E) - 1$. For example, if $E = R^2$ then $\mathbb{P}(E)$ is $\mathbb{P}^1_R$ and $\mathcal{E}_1$ is $\mathcal{O}(-1)$ because (5.8.1) is the sequence (5.4.1) tensored with $S(1)$. That is, (5.8.2) is just:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(+1) \to 0.$$

Having constructed $\mathbb{P}(E)$ over affine schemes, we now suppose that $\mathcal{E}$ is a vector bundle over any scheme X. We can cover X by affine open sets U and construct the projective bundles $\mathbb{P}(\mathcal{E}|U)$ over each U. By naturality of the construction of $\mathbb{P}(\mathcal{E}|U)$, the restrictions of $\mathbb{P}(\mathcal{E}|U)$ and $\mathbb{P}(\mathcal{E}|V)$ to $U \cap V$ may be identified with each other. Thus we can glue the $\mathbb{P}(\mathcal{E}|U)$ together to obtain a projective space bundle $\mathbb{P}(\mathcal{E})$ over X; a patching process similar to that in 5.3 yields a canonical line bundle $\mathcal{O}(1)$ over $\mathbb{P}(\mathcal{E})$.

By naturality of $E \otimes_R Sym(E) \to Sym(E)(1)$, we have a natural short exact sequence of vector bundles on $\mathbb{P}(\mathcal{E})$, which is locally the sequence (5.8.2):

$$0 \to \mathcal{E}_1 \to \pi^* \mathcal{E} \to \mathcal{O}(1) \to 0. \tag{5.8.3}$$

Let ρ denote the projective space bundle $\mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E})$ and let $\mathcal{E}_2$ denote the kernel of $\rho^*\mathcal{E}_1 \to \mathcal{O}(1)$. Then $(\pi\rho)^*\mathcal{E}$ has a filtration $\mathcal{E}_2 \subset \rho^*\mathcal{E}_1 \subset (\rho\pi)^*\mathcal{E}$ with filtration quotients $\mathcal{O}(1)$ and $\rho^*\mathcal{O}(1)$. This yields a projective space bundle $\mathbb{P}(\mathcal{E}_2) \to \mathbb{P}(\mathcal{E}_1)$. As long as $\mathcal{E}_i$ has rank ≥ 2 we can iterate this construction, forming a new projective space bundle $\mathbb{P}(\mathcal{E}_i)$ and a vector bundle $\mathcal{E}_{i+1}$. If $\text{rank}\,\mathcal{E} = r$, $\mathcal{E}_{r-1}$ will be a line bundle. We write $\mathbb{F}(\mathcal{E})$ for $\mathbb{P}(\mathcal{E}_{r-2})$, and call it the *flag bundle* of $\mathcal{E}$. We may summarize the results of this construction as follows.

THEOREM 5.9 (SPLITTING PRINCIPLE). *Given a vector bundle $\mathcal{E}$ of rank r on a scheme X, there exists a morphism $f\colon \mathbb{F}(\mathcal{E}) \to X$ such that $f^*\mathcal{E}$ has a filtration*

$$f^*\mathcal{E} = \mathcal{E}'_0 \supset \mathcal{E}'_1 \supset \cdots \supset \mathcal{E}'_r = 0$$

by sub-vector bundles whose successive quotients $\mathcal{E}'_i/\mathcal{E}'_{i+1}$ are all line bundles.

Cohomological classification of vector bundles

The formation of vector bundles via the patching process in 5.3 may be codified into a classification of rank n vector bundles via a Čech cohomology set $\check{H}^1(X, GL_n(\mathcal{O}_X))$ which is associated to the sheaf of groups $\mathcal{G} = GL_n(\mathcal{O}_X)$. This cohomology set is defined more generally for any sheaf of groups $\mathcal{G}$ as follows. A *Čech 1-cocycle* for an open cover $\mathcal{U} = \{U_i\}$ of X is a family of elements g_{ij} in $\mathcal{G}(U_i \cap U_j)$ such that $g_{ii} = 1$ and $g_{ij}g_{jk} = g_{ik}$ for all i, j, k. Two 1-cocycles $\{g_{ij}\}$ and $\{h_{ij}\}$ are said to be *equivalent* if there are $f_i \in \mathcal{G}(U_i)$ such that $h_{ij} = f_i g_{ij} f_j^{-1}$. The equivalence classes of 1-cocycles form the set $\check{H}^1(\mathcal{U}, \mathcal{G})$. If $\mathcal{V}$ is a refinement of a cover $\mathcal{U}$, there is a set map from $\check{H}^1(\mathcal{U}, \mathcal{G})$ to $\check{H}^1(\mathcal{V}, \mathcal{G})$. The cohomology set $\check{H}^1(X, \mathcal{G})$ is defined to be the direct limit of the $\check{H}^1(\mathcal{U}, \mathcal{G})$ as $\mathcal{U}$ ranges over the system of all open covers of X.

We saw in 5.3 that every rank n vector bundle arises from patching, using a 1-cocycle for $\mathcal{G} = GL_n(\mathcal{O}_X)$. It isn't hard to see that equivalent cocycles give isomorphic vector bundles. From this, we deduce the following result.

Classification Theorem 5.10. *For every ringed space X, the set $\mathbf{VB}_n(X)$ of isomorphism classes of vector bundles of rank n over X is in 1-1 correspondence with the cohomology set $\check{H}^1(X, GL_n(\mathcal{O}_X))$:*

$$\mathbf{VB}_n(X) \cong \check{H}^1(X, GL_n(\mathcal{O}_X)).$$

When $\mathcal{G}$ is an abelian sheaf of groups, such as $\mathcal{O}_X^\times = GL_1(\mathcal{O}_X)$, it is known that the Čech set $\check{H}^1(X, \mathcal{G})$ agrees with the usual sheaf cohomology group $H^1(X, \mathcal{G})$ (see Ex. III.4.4 of [Hart]). In particular, each $\check{H}^1(X, \mathcal{G})$ is an abelian group. A little work, detailed in [EGA, $0_I(5.6.3)$]) establishes:

Corollary 5.10.1. *For every locally ringed space X the isomorphism of Theorem 5.10 is a group isomorphism:*

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times).$$

As an application, suppose that X is the union of two open sets V_1 and V_2. Write $U(X)$ for the group $H^0(X, \mathcal{O}_X^\times) = \mathcal{O}_X^\times(X)$ of global units on X. The cohomology Mayer-Vietoris sequence translates to the following exact sequence.

$$\begin{aligned} 1 \to U(X) \to U(V_1) \times U(V_2) \to U(V_1 \cap V_2) \xrightarrow{\partial} \\ \xrightarrow{\partial} \mathrm{Pic}(X) \to \mathrm{Pic}(V_1) \times \mathrm{Pic}(V_2) \to \mathrm{Pic}(V_1 \cap V_2). \end{aligned} \tag{5.10.2}$$

To illustrate how this sequence works, consider the standard covering of $\mathbb{P}^1_R$ by $\mathrm{Spec}(R[t])$ and $\mathrm{Spec}(R[t^{-1}])$. Their intersection is $\mathrm{Spec}(R[t, t^{-1}])$. Comparing (5.10.2) with the sequences of Ex. 3.17 and Ex. 3.18 yields

Theorem 5.11. *For any commutative ring R,*

$$U(\mathbb{P}^1_R) = U(R) = R^\times \text{ and } \quad \mathrm{Pic}(\mathbb{P}^1_R) \cong \mathrm{Pic}(R) \times [\mathrm{Spec}(R), \mathbb{Z}].$$

As in 5.3.1, the continuous function $n\colon \mathrm{Spec}(R) \to \mathbb{Z}$ corresponds to the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1_R$ obtained by patching $R[t]$ and $R[t^{-1}]$ together via $t^n \in R[t, t^{-1}]^\times$.

Here is an application of Corollary 5.10.1 to nonreduced schemes. Suppose that $\mathcal{I}$ is a sheaf of nilpotent ideals, and let X_0 denote the ringed space $(X, \mathcal{O}_X/\mathcal{I})$. Writing $\mathcal{I}^\times$ for the sheaf $GL_1(\mathcal{I})$ of Ex. 1.10, we have an exact sequence of sheaves of abelian groups:

$$1 \to \mathcal{I}^\times \to \mathcal{O}_X^\times \to \mathcal{O}_{X_0}^\times \to 1.$$

The resulting long exact cohomology sequence starts with global units:

$$U(X) \to U(X_0) \to H^1(X, \mathcal{I}^\times) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X_0) \to H^2(X, \mathcal{I}^\times) \cdots . \tag{5.11.1}$$

Thus $\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$ may not be an isomorphism, as it is in the affine case (Lemma 3.9).

Invertible ideal sheaves

Suppose that X is an integral scheme, *i.e.*, that each $\mathcal{O}_X(U)$ is an integral domain. The function field $k(X)$ of X is the common quotient field of the integral domains $\mathcal{O}_X(U)$. Following the discussion in §3, we use $\mathcal{K}$ to denote the constant sheaf $U \mapsto k(X)$ and consider $\mathcal{O}_X$-submodules of $\mathcal{K}$. Those that lie in some $f\mathcal{O}_X$ we call *fractional*; a fractional ideal $\mathcal{I}$ is called *invertible* if $\mathcal{I}\mathcal{J} = \mathcal{O}_X$ for some $\mathcal{J}$. As in Proposition 3.5, invertible ideals are line bundles and $\mathcal{I} \otimes \mathcal{J} \cong \mathcal{I}\mathcal{J}$. The set $\mathrm{Cart}(X)$ of invertible ideals in $\mathcal{K}$ is therefore an abelian group.

PROPOSITION 5.12. *If X is an integral scheme, there is an exact sequence*

$$1 \to U(X) \to k(X)^\times \to \mathrm{Cart}(X) \to \mathrm{Pic}(X) \to 1. \qquad (5.12.1)$$

PROOF. The proof of 3.5 goes through to prove everything except that every line bundle $\mathcal{L}$ on X is isomorphic to an invertible ideal. On any affine open set U we have $(\mathcal{L}\otimes\mathcal{K})|U \cong \mathcal{K}|U$, a constant sheaf on U. This implies that $\mathcal{L}\otimes\mathcal{K} \cong \mathcal{K}$, because over an irreducible scheme like X any locally constant sheaf must be constant. Thus the natural inclusion of $\mathcal{L}$ in $\mathcal{L}\otimes\mathcal{K}$ expresses $\mathcal{L}$ as an $\mathcal{O}_X$-submodule of $\mathcal{K}$, and the rest of the proof of 3.5 goes through.

Here is another way to understand $\mathrm{Cart}(X)$. Let $\mathcal{K}^\times$ denote the constant sheaf of units of $\mathcal{K}$; it contains the sheaf $\mathcal{O}_X^\times$. Associated to the exact sequence

$$1 \to \mathcal{O}_X^\times \to \mathcal{K}^\times \to \mathcal{K}^\times/\mathcal{O}_X^\times \to 1$$

is a long exact cohomology sequence. Since X is irreducible and $\mathcal{K}^\times$ is constant, we have $H^0(X,\mathcal{K}^\times) = k(X)^\times$ and $H^1(X,\mathcal{K}^\times) = 0$. Since $U(X) = H^0(X,\mathcal{O}_X^\times)$ we get the exact sequence

$$1 \to U(X) \to k(X)^\times \to H^0(X,\mathcal{K}^\times/\mathcal{O}_X^\times) \to \mathrm{Pic}(X) \to 1. \qquad (5.12.2)$$

Motivated by this sequence, we use the term *Cartier divisor* for a global section of the sheaf $\mathcal{K}^\times/\mathcal{O}_X^\times$. A Cartier divisor can be described by giving an open cover $\{U_i\}$ of X and $f_i \in k(X)^\times$ such that f_i/f_j is in $\mathcal{O}_X^\times(U_i \cap U_j)$ for each i and j.

LEMMA 5.13. *Over every integral scheme X, there is a 1-1 correspondence between Cartier divisors on X and invertible ideal sheaves. Under this identification the sequences (5.12.1) and (5.12.2) are the same.*

PROOF. If $\mathcal{I} \subset \mathcal{K}$ is an invertible ideal, there is a cover $\{U_i\}$ on which it is trivial, *i.e.*, $\mathcal{I}|U_i \cong \mathcal{O}_{U_i}$. Choosing $f_i \in \mathcal{I}(U_i) \subseteq k(X)$ generating $\mathcal{I}|U_i$ gives a Cartier divisor. This gives a set map $\mathrm{Cart}(X) \to H^0(X,\mathcal{K}^\times/\mathcal{O}_X^\times)$; it is easily seen to be a group homomorphism compatible with the map from $k(X)^\times$, and with the map to $\mathrm{Pic}(X) = H^1(X,\mathcal{O}_X^\times)$. This gives a map between the sequences (5.12.1) and (5.12.2); the 5-lemma implies that $\mathrm{Cart}(X) \cong H^0(X,\mathcal{K}^\times/\mathcal{O}_X^\times)$.

VARIATION 5.13.1. Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$. Historically, the invertible ideal sheaf associated to D is the subsheaf $\mathcal{L}(D)$ of $\mathcal{K}$ defined by letting $\mathcal{L}(D)|U_i$ be the submodule of $k(X)$ generated by f_i^{-1}. Since f_i/f_j is a unit on $U_i \cap U_j$, these patch to yield an invertible ideal. If $\mathcal{I}$ is invertible and D is the Cartier divisor attached to $\mathcal{I}$ by (5.13), then $\mathcal{L}(D)$ is $\mathcal{I}^{-1}$. Under the correspondence $D \leftrightarrow \mathcal{L}(D)$ the sequences (5.12.1) and (5.12.2) differ by a minus sign.

For example if $X = \mathbb{P}^1_R$, let D be the Cartier divisor given by t^n on $\mathrm{Spec}(R[t])$ and 1 on $\mathrm{Spec}(R[t^{-1}])$. The correspondence of Lemma 5.13 sends D to $\mathcal{O}(n)$, but $\mathcal{L}(D) \cong \mathcal{O}(-n)$.

Weil divisors

There is a notion of Weil divisor corresponding to that for rings (see 3.6). We say that a scheme X is *normal* if all the local rings $\mathcal{O}_{X,x}$ are normal domains (if X is affine this is the definition of Ex. 3.14), and *Krull* if it is integral, separated and has an affine cover $\{\mathrm{Spec}(R_i)\}$ with the R_i Krull domains. For example, if X is noetherian, integral and separated, then X is Krull if and only if it is normal.

A *prime divisor* on X is a closed integral subscheme Y of codimension 1; this is the analogue of a height 1 prime ideal. A *Weil divisor* is an element of the free abelian group $D(X)$ on the set of prime divisors of X; we call a Weil divisor $D = \sum n_i Y_i$ *effective* if all the $n_i \geq 0$.

Let $k(X)$ be the function field of X. Every prime divisor Y yields a discrete valuation on $k(X)$, because the local ring $\mathcal{O}_{X,y}$ at the generic point y of Y is a DVR. Conversely, each discrete valuation on $k(X)$ determines a unique prime divisor on X, because X is separated [Hart, Ex. II(4.5)]. Having made these observations, the discussion in §3 applies to yield group homomorphisms $\nu\colon k(X)^\times \to D(X)$ and $\nu\colon \mathrm{Cart}(X) \to D(X)$. We define the *divisor class group* $Cl(X)$ to be the quotient of $D(X)$ by the subgroup of all Weil divisors $\nu(f)$, $f \in k(X)^\times$. The proof of Proposition 3.6 establishes the following result.

PROPOSITION 5.14. *Let X be Krull. Then* $\mathrm{Pic}(X)$ *is a subgroup of the divisor class group $Cl(X)$, and there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 1 \to & U(X) & \to & k(X)^\times & \to & \mathrm{Cart}(X) & \to & \mathrm{Pic}(X) & \to 1 \\ & \downarrow & & \downarrow & & \cap\ \nu & & \cap & \\ 1 \to & U(X) & \to & k(X)^\times & \to & D(X) & \to & Cl(X) & \to 1. \end{array}$$

A scheme X is called *regular* (resp. *locally factorial*) if the local rings $\mathcal{O}_{X,x}$ are all regular local rings (resp. UFD's). By 3.8, regular schemes are locally factorial. Suppose that X is locally factorial and Krull. If $\mathcal{I}_Y$ is the ideal of a prime divisor Y and $U = \mathrm{Spec}(R)$ is an affine open subset of X, $\mathcal{I}_Y|U$ is invertible by Corollary 3.8.1. Since $\nu(\mathcal{I}_Y) = Y$, this proves that $\nu : \mathrm{Cart}(X) \to D(X)$ is onto. Inspecting the diagram of Proposition 5.14, we have:

PROPOSITION 5.15. *Let X be an integral, separated and locally factorial scheme. Then*

$$\mathrm{Cart}(X) \cong D(X) \qquad \textit{and} \qquad \mathrm{Pic}(X) \cong Cl(X).$$

EXAMPLE 5.15.1. ([Hart, II(6.4)]). If X is the projective space $\mathbb{P}^n_k$ over a field k, then $\mathrm{Pic}(\mathbb{P}^n_k) \cong Cl(\mathbb{P}^n_k) \cong \mathbb{Z}$. By Theorem 5.11, $\mathrm{Pic}(\mathbb{P}^n)$ is generated by $\mathcal{O}(1)$. The class group $Cl(\mathbb{P}^n)$ is generated by the class of a hyperplane H, whose corresponding ideal sheaf $\mathcal{I}_H$ is isomorphic to $\mathcal{O}(1)$. If Y is a hypersurface defined by a homogeneous polynomial of degree d, we say $\deg(Y) = d$; $Y \sim dH$ in $D(\mathbb{P}^n)$.

The *degree* of a Weil divisor $D = \sum n_i Y_i$ is defined to be $\sum n_i \deg(Y_i)$; the degree function $D(\mathbb{P}^n) \to \mathbb{Z}$ induces the isomorphism $Cl(\mathbb{P}^n) \cong \mathbb{Z}$. We remark that when $k = \mathbb{C}$ the degree of a Weil divisor agrees with the topological degree of the associated line bundle in $H^2(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}$, defined by the first Chern class as in Example 4.11.2.

Blowing Up 5.15.2. Let X be a smooth variety over an algebraically closed field, and let Y be a smooth subvariety of codimension ≥ 2. If the ideal sheaf of Y is $\mathcal{I}$, then $\mathcal{I}/\mathcal{I}^2$ is a vector bundle on Y. The *blowing up of X along Y* is a nonsingular variety $\tilde{X}$, containing a prime divisor $\tilde{Y} \cong \mathbb{P}(\mathcal{I}/\mathcal{I}^2)$, together with a map $\pi: \tilde{X} \to X$ such that $\pi^{-1}(Y) = \tilde{Y}$ and $\tilde{X} - \tilde{Y} \cong X - Y$ (see [Hart, II.7]). For example, the blowing up of a smooth surface X at a point x is a smooth surface $\tilde{X}$, and the smooth curve $\tilde{Y} \cong \mathbb{P}^1$ is called the *exceptional divisor*.

The maps $\pi^*: \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ and $\mathbb{Z} \to \mathrm{Pic}(\tilde{X})$ sending n to $n[\tilde{Y}]$ give rise to an isomorphism (see [Hart, Ex. II.8.5 or V.3.2]):

$$\mathrm{Pic}(\tilde{X}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}.$$

Example 5.15.3. Consider the rational ruled surface S in $\mathbb{P}^1 \times \mathbb{P}^2$, defined by $X_iY_j = X_jY_i$ $(i, j = 1, 2)$, and the smooth quadric surface Q in $\mathbb{P}^3$, defined by $xy = zw$. Now S is obtained by blowing up $\mathbb{P}^2_k$ at a point [Hart, V.2.11.5], while Q is obtained from $\mathbb{P}^2_k$ by first blowing up two points, and then blowing down the line between them [Hart, Ex. V.4.1]. Thus $\mathrm{Pic}(S) = Cl(S)$ and $\mathrm{Pic}(Q) = Cl(Q)$ are both isomorphic to $\mathbb{Z} \times \mathbb{Z}$. For both surfaces, divisors are classified by a pair (a, b) of integers (see [Hart, II.6.6.1]).

Example 5.16. Let X be a smooth projective curve over an algebraically closed field k. In this case a Weil divisor is a formal sum of closed points on X: $D = \sum n_i x_i$. The *degree* of D is defined to be $\sum n_i$; a point has degree 1. Since the divisor of a function has degree 0 [Hart, II(6.4)], the degree induces a surjective homomorphism $\mathrm{Pic}(X) \to \mathbb{Z}$. Writing $\mathrm{Pic}^0(X)$ for the kernel, the choice of a basepoint $\infty \in X$ determines a splitting $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(X)$. The group $\mathrm{Pic}^0(X)$ is divisible and has the same cardinality as k; its torsion subgroup is $(\mathbb{Q}/\mathbb{Z})^{2g}$ if $\mathrm{char}(k) = 0$. If k is perfect of characteristic $p > 0$, the torsion subgroup lies between $(\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])^{2g}$ and $(\mathbb{Q}/\mathbb{Z})^{2g}$. These facts are established in [MmAB, II].

If X has genus 0, then $X \cong \mathbb{P}^1$ and $\mathrm{Pic}^0(X) = 0$. If X has genus 1, the map $x \mapsto x - \infty$ gives a canonical bijection $X(k) \cong \mathrm{Pic}^0(X)$. In general, if X has genus g there is an abelian variety $J(X)$ of dimension g, called the *Jacobian variety* of X [Hart, IV.4.10] such that the closed points of $J(X)$ are in 1–1 correspondence with the elements of $\mathrm{Pic}^0(X)$. The Jacobian variety is a generalization of the Picard variety of Exercise 5.9 below.

Example 5.17. Let X be a smooth projective curve over a finite field $\mathbb{F} = \mathbb{F}_q$. As observed in [Hart, 4.10.4], the elements of the kernel $\mathrm{Pic}^0(X)$ of the degree map $\mathrm{Pic}(X) \to \mathbb{Z}$ are in 1–1 correspondence with the set $J(X)(\mathbb{F})$ of closed points of the Jacobian variety $J(X)$ whose coordinates belong to $\mathbb{F}$. Since $J(X)$ is contained in some projective space, the set $J(X)(\mathbb{F})$ is finite. Thus $\mathrm{Pic}(X)$ is the direct sum of $\mathbb{Z}$ and a finite group. We may assume that $H^0(X, \mathcal{O}_X) = \mathbb{F}$, so that $U(X) = \mathbb{F}^\times$.

Now let S be any nonempty set of closed points of X and consider the affine curve $X - S$; the coordinate ring R of $X - S$ is called the *ring of S-integers* in the function field $\mathbb{F}(X)$. Comparing the sequences of Proposition 5.14 for X and $X - S$ yields the exact sequence

$$1 \to \mathbb{F}^\times \to R^\times \to \mathbb{Z}^S \to \mathrm{Pic}(X) \to \mathrm{Pic}(R) \to 0. \tag{5.17.1}$$

(See Ex. 5.12.) The image of the map $\mathbb{Z}^S \to \mathrm{Pic}(X)$ is the subgroup generated by the lines bundles associated to the points of S via the identification of Weil divisors with Cartier divisors given by Proposition 5.15 (compare with Ex. 3.8.)

When $S = \{s\}$ is a single point, the map $\mathbb{Z} \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$ is multiplication by the degree of the field extension $[\mathbb{F}(s) : \mathbb{F}]$, so $\mathrm{Pic}(R)$ is a finite group: it is an extension of $\mathrm{Pic}^0(X)$ by a cyclic group of order $[k(s) : \mathbb{F}]$. (See Ex. 5.12(b).) From the exact sequence, we see that $R^\times = \mathbb{F}^\times$. By induction on $|S|$, it follows easily from (5.17.1) that $\mathrm{Pic}(R)$ is finite and $R^\times \cong \mathbb{F}^\times \oplus \mathbb{Z}^{|S|-1}$.

Historical Note 5.18. The term "Picard group" (of a scheme or commutative ring), and the notation $\mathrm{Pic}(X)$, was introduced by Grothendieck around 1960. Of course the construction itself was familiar to the topologists of the early 1950's, and the connection to invertible ideals was clear from to framework of Serre's 1954 paper "Faisceaux algébriques cohérents," [S-FAC], but had not been given a name.

Grothendieck's choice of terminology followed André Weil's usage of the term *Picard variety* in his 1950 paper *Variétés Abéliennes.* Weil says that, "accidentally enough," his choice coincided with the introduction by Castelnuovo in 1905 of the "Picard variety associated with continuous systems of curves" on a surface X (*Sugli integrali semplici appartenenti ad una superficie irregolare*, Rend. Accad. dei Lincei, vol XIV, 1905). In turn, Castelnuovo named it in honor of Picard's paper *Sur la théorie des groupes et des surfaces algébriques* (Rend. Circolo Mat. Palermo, IX, 1895), which studied the number of integrals of the first kind attached to algebraic surfaces. (I am grateful to Serre and Pedrini for the historical information.)

EXERCISES

5.1 Give an example of a ringed space $(X, \mathcal{O}_X)$ such that the rank of $\mathcal{O}_X(X)$ is well-defined, but such that the rank of $\mathcal{O}_X(U)$ is not well-defined for any proper open $U \subseteq X$.

5.2 Show that the global sections of the vector bundle $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ are in 1-1 correspondence with vector bundle maps $\mathcal{E} \to \mathcal{F}$. Conclude that there is a non-zero map $\mathcal{O}(m) \to \mathcal{O}(n)$ over $\mathbb{P}^1_R$ only if $m \le n$.

5.3 *Projection Formula.* If $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, show that there is a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}$.

5.4 Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of locally free sheaves. Show that each $\wedge^n\mathcal{F}$ has a finite filtration

$$\wedge^n\mathcal{F} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^{n+1} = 0$$

with successive quotients $F^i/F^{i+1} \cong (\wedge^i\mathcal{E}) \otimes (\wedge^{n-i}\mathcal{G})$. In particular, show that $\det(\mathcal{F}) \cong \det(\mathcal{E}) \otimes \det(\mathcal{G})$.

5.5 Let S be a graded ring generated by S_1 and set $X = \mathrm{Proj}(S)$. Show that $\mathcal{O}_X(n)\check{} \cong \mathcal{O}_X(-n)$ and $\mathcal{H}om(\mathcal{O}_X(m), \mathcal{O}_X(n)) \cong \mathcal{O}_X(n-m)$.

5.6 *Serre's "Theorem A."* Suppose that X is Proj(S) for a graded ring S which is finitely generated as an S_0-algebra by S_1. Recall from 5.3.1 (or [Hart, II.5.15]) that every quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is isomorphic to $\tilde{M}$ for some graded S-module M. In fact, we can take M_n to be $H^0(X, \mathcal{F}(n))$.

(a) If M is generated by M_0 and the M_i with $i < 0$, show that the sheaf $\tilde{M}$ is generated by global sections. *Hint:* consider $M_0 \oplus M_1 \oplus \cdots$.

(b) By (a), $\mathcal{O}_X(n)$ is generated by global sections if $n \geq 0$. Is the converse true?

(c) If M is a finitely generated S-module, show that $\tilde{M}(n)$ is generated by global sections for all large n (*i.e.*, for all $n \geq n_0$ for some n_0).

(d) If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, show that $\mathcal{F}(n)$ is generated by global sections for all large n. This result is known as Serre's "Theorem A," and it implies that $\mathcal{O}_X(1)$ is an *ample line bundle* in the sense of [EGA, II(4.5.5)].

5.7 Let X be a d-dimensional quasi-projective variety, *i.e.*, a locally closed integral subscheme of some $\mathbb{P}^n_k$, where k is an algebraically closed field.

(a) Suppose that $\mathcal{E}$ is a vector bundle generated by global sections. If rank($\mathcal{E}$) $> d$, Bertini's Theorem implies that $\mathcal{E}$ has a global section s such that $s_x \notin \mathfrak{m}_x\mathcal{E}_x$ for each $x \in X$. Establish the analogue of the Serre Cancellation Theorem 2.3(a), that there is a short exact sequence of vector bundles

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{E} \to \mathcal{F} \to 0.$$

(b) Now suppose that X is a curve. Show that every vector bundle $\mathcal{E}$ is a successive extension of invertible sheaves in the sense that there is a filtration of $\mathcal{E}$

$$\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_r = 0.$$

by sub-bundles such that each $\mathcal{E}_i/\mathcal{E}_{i+1}$ is a line bundle. *Hint*: by Ex. 5.6(d), $\mathcal{E}(n)$ is generated by global sections for large n.

5.8 *Complex analytic spaces.* Recall from Example 5.1.4 that a *complex analytic space* is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to a basic analytic subset of $\mathbb{C}^n$.

(a) Use Example 5.2.3 to show that every analytic vector bundle on $\mathbb{C}^n$ is free, *i.e.*, $\mathcal{O}^r_{an}$ for some r. What about $\mathbb{C}^n - 0$?

(b) Let X be the complex affine node defined by the equation $y^2 = x^3 - x^2$. We saw in 3.10.2 that $\text{Pic}(X) \cong \mathbb{C}^\times$. Use (4.9.1) to show that $\text{Pic}(X(\mathbb{C})_{an}) = 0$.

(c) (Serre) Let X be the scheme $\text{Spec}(\mathbb{C}[x,y]) - \{0\}$, 0 being the origin. Using the affine cover of X by $D(x)$ and $D(y)$, show that $\text{Pic}(X) = 0$ but $\text{Pic}(X_{an}) \neq 0$.

5.9 *Picard Variety.* Let X be a scheme over $\mathbb{C}$ and $X_{an} = X(\mathbb{C})_{an}$ the associated complex analytic space of Example 5.1.4. There is an exact sequence of sheaves of abelian groups on the topological space $X(\mathbb{C})$ underlying X_{an}:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_{X_{an}} \xrightarrow{\exp} \mathcal{O}^\times_{X_{an}} \to 0, \qquad (*)$$

where $\mathbb{Z}$ is the constant sheaf on $X(\mathbb{C})$.

(a) Show that the Chern class $c_1\colon \text{Pic}(X_{an}) \to H^2(X(\mathbb{C})_{top}; \mathbb{Z})$ of Example 5.2.3 is naturally isomorphic to the composite map

$$\text{Pic}(X_{an}) \cong H^1(X_{an}, \mathcal{O}^\times_{X_{an}}) \cong H^1(X(\mathbb{C})_{top}; \mathcal{O}^\times_{X_{an}}) \xrightarrow{\partial} H^2(X(\mathbb{C})_{top}; \mathbb{Z})$$

coming from Corollary 5.10.1, the map $X_{an} \to X(\mathbb{C})_{top}$ of Example 5.1.4, and the boundary map of $(*)$.

Now suppose that X is projective. The image of $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_{an})$ in $H^2(X(\mathbb{C});\mathbb{Z})$ is called the *Néron-Severi group* $NS(X)$ and the kernel of $\mathrm{Pic}(X) \to NS(X)$ is written as $\mathrm{Pic}^0(X)$. Since $H^2(X(\mathbb{C});\mathbb{Z})$ is a finitely generated abelian group, so is $NS(X)$. It turns out that $H^1(X(\mathbb{C}),\mathcal{O}_{X_{an}}) \cong \mathbb{C}^n$ for some n, and that $H^1(X(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}^{2n}$ is a lattice in $H^1(X,\mathcal{O}_{X_{an}})$.

(b) Show that $\mathrm{Pic}^0(X)$ is isomorphic to $H^1(X,\mathcal{O}_X)/H^1(X(\mathbb{C});\mathbb{Z})$. Thus $\mathrm{Pic}^0(X)$ is a complex analytic torus; in fact it is the set of closed points of an abelian variety, called the *Picard variety* of X.

5.10 If E and F are finitely generated projective R-modules, show that their projective bundles $\mathbb{P}(E)$ and $\mathbb{P}(F)$ are isomorphic as schemes over R if and only if $E \cong F \otimes_R L$ for some line bundle L on R.

5.11 Let X be a Krull scheme and Z an irreducible closed subset with complement U. Define a map $\rho\colon Cl(X) \to Cl(U)$ of class groups by sending the Weil divisor $\sum n_iY_i$ to $\sum n_i(Y_i \cap U)$, ignoring terms n_iY_i for which $Y_i \cap U = \phi$. (Cf. Ex. 3.8.)

(a) If Z has codimension ≥ 2, show that $\rho\colon Cl(X) \cong Cl(U)$.

(b) If Z has codimension 1, show that there is an exact sequence

$$\mathbb{Z} \xrightarrow{[Z]} Cl(X) \xrightarrow{\rho} Cl(U) \to 0.$$

5.12 Let X be a smooth curve over a field k, and let S be a finite nonempty set of closed points in X. By Riemann-Roch, the complement $U = X - S$ is affine; set $R = H^0(U,\mathcal{O})$ so that $U = \mathrm{Spec}(R)$.

(a) Using Propositions 5.12 and 5.14, show that there is an exact sequence

$$1 \to H^0(X,\mathcal{O}_X^\times) \to R^\times \to \mathbb{Z}^S \to \mathrm{Pic}(X) \to \mathrm{Pic}(R) \to 0.$$

(b) If X is a smooth projective curve over k and $s \in X$ is a closed point, show that the map $\mathbb{Z} \xrightarrow{[s]} \mathrm{Pic}(X) = Cl(X)$ in (a) is injective. (It is the map of Ex. 5.11(b).) If $k(x) = k$, conclude that $\mathrm{Pic}(X) \cong \mathrm{Pic}(U) \times \mathbb{Z}$. What happens if $k(x) \neq k$?