CHAPTER II

THE GROTHENDIECK GROUP K_0

There are several ways to construct the "Grothendieck group" of a mathematical object. We begin with the group completion version, because it has been the most historically important. After giving the applications to rings and topological spaces, we discuss λ-operations in §4. In sections 6 and 7 we describe the Grothendieck group of an "exact category," and apply it to the K-theory of schemes in §8. This construction is generalized to the Grothendieck group of a "Waldhausen category" in §9.

§1. The Group Completion of a monoid

Both $K_0(R)$ and $K^0(X)$ are formed by taking the group completion of an abelian monoid—the monoid $\mathbf{P}(R)$ of finitely generated projective R-modules and the monoid $\mathbf{VB}(X)$ of vector bundles over X, respectively. We begin with a description of this construction.

Recall that an *abelian monoid* is a set M together with an associative, commutative operation $+$ and an "additive" identity element 0. A monoid map $f\colon M \to N$ is a set map such that $f(0) = 0$ and $f(m + m') = f(m) + f(m')$. The most famous example of an abelian monoid is $\mathbb{N} = \{0, 1, 2, ...\}$, the natural numbers with additive identity zero. If A is an abelian group then not only is A an abelian monoid, but so is any additively closed subset of A containing 0.

The *group completion* of an abelian monoid M is an abelian group $M^{-1}M$, together with a monoid map $[\;]\colon M \to M^{-1}M$ which is universal in the sense that, for every abelian group A and every monoid map $\alpha\colon M \to A$, there is a unique abelian group homomorphism $\tilde{\alpha}\colon M^{-1}M \to A$ such that $\tilde{\alpha}([m]) = \alpha(m)$ for all $m \in M$.

For example, the group completion of $\mathbb{N}$ is $\mathbb{Z}$. If A is an abelian group then clearly $A^{-1}A = A$; if M is a submonoid of A (additively closed subset containing 0), then $M^{-1}M$ is the subgroup of A generated by M.

Every abelian monoid M has a group completion. One way to contruct it is to form the free abelian group $F(M)$ on symbols $[m]$, $m \in M$, and then factor out by the subgroup $R(M)$ generated by the relations $[m+n]-[m]-[n]$. By universality, if $M \to N$ is a monoid map, the map $M \to N \to N^{-1}N$ extends uniquely to a homomorphism from $M^{-1}M$ to $N^{-1}N$. Thus group completion is a functor from abelian monoids to abelian groups. A little decoding shows that in fact it is left adjoint to the forgetful functor, because of the natural isomorphism

$$\operatorname{Hom}_{\substack{\text{abelian}\\\text{monoids}}}(M, A) \cong \operatorname{Hom}_{\substack{\text{abelian}\\\text{groups}}}(M^{-1}M, A).$$

PROPOSITION 1.1. *Let M be an abelian monoid. Then:*

(a) *Every element of $M^{-1}M$ is of the form $[m]-[n]$ for some $m,n \in M$;*
(b) *If $m,n \in M$ then $[m]=[n]$ in $M^{-1}M$ if and only if $m+p=n+p$ for some $p \in M$;*
(c) *The monoid map $M \times M \to M^{-1}M$ sending (m,n) to $[m]-[n]$ is surjective.*
(d) *Hence $M^{-1}M$ is the set-theoretic quotient of $M \times M$ by the equivalence relation generated by $(m,n) \sim (m+p,n+p)$.*

PROOF. Every element of a free abelian group is a difference of sums of generators, and in $F(M)$ we have $([m_1]+[m_2]+\cdots) \equiv [m_1+m_2+\cdots]$ modulo $R(M)$. Hence every element of $M^{-1}M$ is a difference of generators. This establishes (a) and (c). For (b), suppose that $[m]-[n]=0$ in $M^{-1}M$. Then in the free abelian group $F(M)$ we have

$$[m]-[n]=\sum([a_i+b_i]-[a_i]-[b_i])-\sum([c_j+d_j]-[c_j]-[d_j]).$$

Translating negative terms to the other side yields the following equation:

$$(*) \qquad [m]+\sum([a_i]+[b_i])+\sum[c_j+d_j]=[n]+\sum[a_i+b_i]+\sum([c_j]+[d_j]).$$

Now in a free abelian group two sums of generators $\sum[x_i]$ and $\sum[y_j]$ can only be equal if they have the same number of terms, and the generators differ by a permutation σ in the sense that $y_i = x_{\sigma(i)}$. Hence the generators on the left and right of $(*)$ differ only by a permutation. This means that in M the sum of the terms on the left and right of $(*)$ are the same, *i.e.,*

$$m+\sum(a_i+b_i)+\sum(c_j+d_j)=n+\sum(a_i+b_i)+\sum(c_j+d_j)$$

in M. This yields (b), and part (d) follows from (a) and (b). $\square$

The two corollaries below are immediate from Proposition 1.1, given the following definitions. An (abelian) *cancellation monoid* is an abelian monoid M such that for all $m,n,p \in M$, $m+p=n+p$ implies $m=n$. A submonoid L of an abelian monoid M is called *cofinal* if for every $m \in M$ there is an $m' \in M$ so that $m+m' \in L$.

COROLLARY 1.2. *M injects into $M^{-1}M$ if and only if M is a cancellation monoid.*

COROLLARY 1.3. *If L is cofinal in an abelian monoid M, then:*

(a) *$L^{-1}L$ is a subgroup of $M^{-1}M$;*
(b) *Every element of $M^{-1}M$ is of the form $[m]-[\ell]$ for some $m \in M$, $\ell \in L$;*
(c) *If $[m]=[m']$ in $M^{-1}M$ then $m+\ell=m'+\ell$ for some $\ell \in L$.*

A *semiring* is an abelian monoid $(M,+)$, together with an associative product $\cdot$ which distributes over $+$, and a 2-sided multiplicative identity element 1. That is, a semiring satisfies all the axioms for a ring except for the existence of subtraction. The prototype semiring is $\mathbb{N}$.

The group completion $M^{-1}M$ (with respect to $+$) of a semiring M is a ring, the product on $M^{-1}M$ being extended from the product on M using 1.1. If $M \to N$ is a semiring map, then the induced map $M^{-1}M \to N^{-1}N$ is a ring homomorphism. Hence group completion is also a functor from semirings to rings, and from commutative semirings to commutative rings.

Example 1.4. Let X be a topological space. The set $[X, \mathbb{N}]$ of continuous maps $X \to \mathbb{N}$ is a semiring under pointwise $+$ and $\cdot$. The group completion of $[X, \mathbb{N}]$ is the ring $[X, \mathbb{Z}]$ of all continuous maps $X \to \mathbb{Z}$.

If X is (quasi-)compact, $[X, \mathbb{Z}]$ is a free abelian group. Indeed, $[X, \mathbb{Z}]$ is a subgroup of the group S of all bounded set functions from X to $\mathbb{Z}$, and S is a free abelian group (S is a "Specker group"; see [Fuchs]).

Example 1.5 (Burnside Ring). Let G be a finite group. The set M of (isomorphism classes of) finite G-sets is an abelian monoid under disjoint union, '0' being the empty set $\emptyset$. Suppose there are c distinct G-orbits. Since every G-set is a disjoint union of orbits, M is the free abelian monoid $\mathbb{N}^c$, a basis of M being the classes of the c distinct orbits of G. Each orbit is isomorphic to a coset G/H, where H is the stabilizer of an element, and $G/H \cong G/H'$ if and only if H and H' are conjugate subgroups of G, so c is the number of conjugacy classes of subgroups of G. Therefore the group completion $A(G)$ of M is the free abelian group $\mathbb{Z}^c$, a basis being the set of all c coset spaces $[G/H]$.

The direct product of two G-sets is again a G-set, so M is a semiring with '1' the 1-element G-set. Therefore $A(G)$ is a commutative ring; it is called the *Burnside ring* of G. The forgetful functor from G-sets to sets induces a map $M \to \mathbb{N}$ and hence an augmentation map $\epsilon: A(G) \to \mathbb{Z}$. For example, if G is cyclic of prime order p, then $A(G)$ is the ring $\mathbb{Z}[x]/(x^2 = px)$ and $x = [G]$ has $\epsilon(x) = p$.

Example 1.6. (Representation ring). Let G be a finite group. The set $Rep_{\mathbb{C}}(G)$ of finite-dimensional representations $\rho: G \to GL_n\mathbb{C}$ (up to isomorphism) is an abelian monoid under $\oplus$. By Maschke's Theorem, $\mathbb{C}G$ is semisimple and $Rep_{\mathbb{C}}(G) \cong \mathbb{N}^r$, where r is the number of conjugacy classes of elements of G. Therefore the group completion $R(G)$ of $Rep_{\mathbb{C}}(G)$ is isomorphic to $\mathbb{Z}^r$ as an abelian group.

The tensor product $V \otimes_{\mathbb{C}} W$ of two representations is also a representation, so $Rep_{\mathbb{C}}(G)$ is a semiring (the element 1 is the 1-dimensional trivial representation). Therefore $R(G)$ is a commutative ring; it is called the *Representation ring* of G. For example, if G is cyclic of prime order p then $R(G)$ is isomorphic to the group ring $\mathbb{Z}[G]$, a subring of $\mathbb{Q}[G] = \mathbb{Q} \times \mathbb{Q}(\zeta)$, $\zeta^p = 1$.

Every representation is determined by its character $\chi: G \to \mathbb{C}$, and irreducible representations have linearly independent characters. Therefore $R(G)$ is isomorphic to the ring of all complex characters $\chi: G \to \mathbb{C}$, a subring of $Map(G, \mathbb{C})$.

Definition. A (connected) *partially ordered abelian group* (A, P) is an abelian group A, together with a submonoid P of A which generates A (so $A = P^{-1}P$) and $P \cap (-P) = \{0\}$. This structure induces a translation-invariant partial ordering $\geq$ on A: $a \geq b$ if $a - b \in P$. Conversely, given a translation-invariant partial order on A, let P be $\{a \in A : a \geq 0\}$. If $a, b \geq 0$ then $a + b \geq a \geq 0$, so P is a submonoid of A. If P generates A then (A, P) is a partially ordered abelian group.

If M is an abelian monoid, $M^{-1}M$ need not be partially ordered (by the image of M), because we may have $[a] + [b] = 0$ for $a, b \in M$. However, interesting examples are often partially ordered. For example, the Burnside ring $A(G)$ and Representation ring $R(G)$ are partially ordered (by G-sets and representations).

When it exists, the ordering on $M^{-1}M$ is an extra piece of structure. For example, $\mathbb{Z}^r$ is the group completion of both $\mathbb{N}^r$ and $M = \{0\} \cup \{(n_1, ..., n_r) \in \mathbb{N}^r : n_1, ..., n_r > 0\}$. However, the two partially ordered structures on $\mathbb{Z}^r$ are different.

EXERCISES

1.1 The group completion of a non-abelian monoid M is a group $\widehat{M}$, together with a monoid map $M \to \widehat{M}$ which is universal for maps from M to groups. Show that every monoid has a group completion in this sense, and that if M is abelian then $\widehat{M} = M^{-1}M$. If M is the free monoid on a set X, show that the group completion of M is the free group on the set X.

Note: The results in this section fail for non-abelian monoids. Proposition 1.1 fails for the free monoid on X. Corollary 1.2 can also fail: an example of a cancellation monoid M which does not inject into $\widehat{M}$ was given by Mal'cev in 1937.

1.2 If $M = M_1 \times M_2$, show that $M^{-1}M$ is the product group $(M_1^{-1}M_1) \times (M_2^{-1}M_2)$.

1.3 If M is the filtered colimit of abelian monoids M_α, show that $M^{-1}M$ is the filtered colimit of the abelian groups $M_\alpha^{-1}M_\alpha$.

1.4 *Mayer-Vietoris for group completions.* Suppose that a sequence $L \to M_1 \times M_2 \to N$ of abelian monoids is "exact" in the sense that whenever $m_1 \in M_1$ and $m_2 \in M_2$ agree in N then m_1 and m_2 are the image of a common $\ell \in L$. If L is cofinal in M_1, M_2 and N, show that there is an exact sequence of groups $L^{-1}L \to (M_1^{-1}M_1) \oplus (M_2^{-1}M_2) \to N^{-1}N$, where the first map is the diagonal inclusion and the second map is the difference map $(m_1, m_2) \mapsto \bar{m}_1 - \bar{m}_2$.

1.5 Classify all abelian monoids which are quotients of $\mathbb{N} = \{0, 1, \dots\}$ and show that they are all finite. How many quotient monoids $M = \mathbb{N}/\!\sim$ of $\mathbb{N}$ have m elements and group completion $\widehat{M} = \mathbb{Z}/n\mathbb{Z}$?

1.6 Here is another description of the Burnside ring $A(G)$ of a finite group G. For each subgroup H, and finite G-set X, let $\chi_H(X)$ denote the cardinality of X^H.

- (a) Show that χ_H defines a ring homomorphism $A(G) \to \mathbb{Z}$, and $\epsilon = \chi_1$.
- (b) Deduce that the product χ of the χ_H (over the c conjugacy classes of subgroups) induces an injection of $A(G)$ into the product ring $\prod_1^c \mathbb{Z}$.
- (c) Conclude that $A(G) \otimes \mathbb{Q} \cong \prod_1^c \mathbb{Q}$.

1.7 (T-Y Lam) Let $\phi : G \to H$ be a homomorphism of finite groups. Show that the restriction functor from H-sets to G-sets ($gx = \phi(g)x$) induces a ring homomorphism $\phi^* : A(H) \to A(G)$. If X is a G-set, we can form the H-set $H \times_G X = H \times X/\{(h, gx) \sim (h\phi(g), x))\}$. Show that $H \times_G$ induces a group homomorphism $\phi_* : A(G) \to A(H)$. If ϕ is an injection, show that the *Frobenius Reciprocity* formula holds: $\phi_*(\phi^*(x) \cdot y) = x \cdot \phi_*(y)$ for all $x \in A(H)$, $y \in A(G)$.

§2. K_0 of a ring

Let R be a ring. The set $\mathbf{P}(R)$ of isomorphism classes of finitely generated projective R-modules, together with direct sum $\oplus$ and identity 0, forms an abelian monoid. The *Grothendieck group of* R, $K_0(R)$, is the group completion $\mathbf{P}^{-1}\mathbf{P}$ of $\mathbf{P}(R)$.

When R is commutative, $K_0(R)$ is a commutative ring with $1 = [R]$, because the monoid $\mathbf{P}(R)$ is a commutative semiring with product $\otimes_R$. This follows from the following facts: $\otimes$ distributes over $\oplus$; $P \otimes_R Q \cong Q \otimes_R P$ and $P \otimes_R R \cong P$; if P, Q are finitely generated projective modules then so is $P \otimes_R Q$ (by Ex. I.2.7).

For example, let k be a field or division ring. Then the abelian monoid $\mathbf{P}(k)$ is isomorphic to $\mathbb{N} = \{0, 1, 2, \dots\}$, so $K_0(k) = \mathbb{Z}$. The same argument applies to show that $K_0(R) = \mathbb{Z}$ for every local ring R by (I.2.2), and also for every PID (by the Structure Theorem for modules over a PID). In particular, $K_0(\mathbb{Z}) = \mathbb{Z}$.

The *Eilenberg Swindle* I.2.8 shows why we restrict to finitely generated projectives. If we included R^∞ (defined in Ex. I.1.7), then the formula $P \oplus R^\infty \cong R^\infty$ would imply that $[P] = 0$ for every finitely generated projective R-module, and we would have $K_0(R) = 0$.

K_0 is a functor from rings to abelian groups, and from commutative rings to commutative rings. To see this, suppose that $R \to S$ is a ring homomorphism. The functor $\otimes_R S \colon \mathbf{P}(R) \to \mathbf{P}(S)$ (sending P to $P \otimes_R S$) yields a monoid map $\mathbf{P}(R) \to \mathbf{P}(S)$, hence a group homomorphism $K_0(R) \to K_0(S)$. If R, S are commutative rings then $\otimes_R S \colon K_0(R) \to K_0(S)$ is a ring homomorphism, because $\otimes_R S \colon \mathbf{P}(R) \to \mathbf{P}(S)$ is a semiring map:

$$(P \otimes_R Q) \otimes_R S \cong (P \otimes_R S) \otimes_S (Q \otimes_R S).$$

The free modules play a special role in understanding $K_0(R)$ because they are cofinal in $\mathbf{P}(R)$. By Corollary 1.3 every element of $K_0(R)$ can be written as $[P] - [R^n]$ for some P and n. Moreover, $[P] = [Q]$ in $K_0(R)$ if and only if P, Q are stably isomorphic: $P \oplus R^m \cong Q \oplus R^m$ for some m. In particular, $[P] = [R^n]$ if and only if P is stably free. The monoid L of isomorphism classes of free modules is $\mathbb{N}$ if and only if R satisfies the Invariant Basis Property of Chapter I, §1. This yields the following information about $K_0(R)$.

Lemma 2.1. *The monoid map $\mathbb{N} \to \mathbf{P}(R)$ sending n to R^n induces a group homomorphism $\mathbb{Z} \to K_0(R)$. We have:*

1. $\mathbb{Z} \to K_0(R)$ *is injective if and only if R satisfies the Invariant Basis Property (IBP);*
2. *Suppose that R satisfies the IBP (e.g., R is commutative). Then*

$$K_0(R) \cong \mathbb{Z} \quad \Longleftrightarrow \quad \text{every finitely generated projective } R\text{-module is stably free.}$$

Example 2.1.1. Suppose that R is commutative, or more generally that there is a ring map $R \to F$ to a field F. In this case $\mathbb{Z}$ is a direct summand of $K_0(R)$, because the map $K_0(R) \to K_0(F) \cong \mathbb{Z}$ takes $[R]$ to 1. A ring with $K_0(R) = Q$ is given in Exercise 2.12 below.

EXAMPLE 2.1.2 (SIMPLE RINGS). Consider the matrix ring $R = M_n(F)$ over a field F. We saw in Example I.1.1 that every R-module is projective (because it is a sum of copies of the projective module $V \cong F^n$), and that length is an invariant of finitely generated R-modules. Thus *length* is an abelian group isomorphism $K_0(M_n(F)) \xrightarrow{\cong} \mathbb{Z}$ sending $[V]$ to 1. Since R has length n, the subgroup of $K_0(R) \cong \mathbb{Z}$ generated by the free modules has index n. In particular, the inclusion $\mathbb{Z} \subset K_0(R)$ of Lemma 2.1 does not split.

EXAMPLE 2.1.3. (Karoubi) We say a ring R is *flasque* if there is an R-bimodule M, finitely generated projective as a right module, and a bimodule isomorphism $\theta : R \oplus M \cong M$. If R is flasque then $K_0(R) = 0$. This is because for every P we have a natural isomorphism $P \oplus (P \otimes_R M) \cong P \otimes_R (R \oplus M) \cong (P \otimes_R M)$.

If R is flasque and the underlying right R-module structure on M is R, we say that R is an *infinite sum ring*. The right module isomorphism $R^2 \cong R$ underlying θ makes R a direct sum ring (Ex. I.1.7). The Cone Rings of Ex. I.1.8, and the rings $\mathrm{End}_R(R^\infty)$ of Ex. I.1.7, are examples of infinite sum rings, and hence flasque rings; see Exercise 2.15.

If $R = R_1 \times R_2$ then $\mathbf{P}(R) \cong \mathbf{P}(R_1) \times \mathbf{P}(R_2)$. As in Exercise 1.2, this implies that $K_0(R) \cong K_0(R_1) \times K_0(R_2)$. Thus K_0 may be computed componentwise.

EXAMPLE 2.1.4 (SEMISIMPLE RINGS). Let R be a semisimple ring, with simple modules $V_1, ..., V_r$ (see Ex. I.1.1). Schur's Lemma states that each $D_i = \mathrm{Hom}_R(V_i, V_i)$ is a division ring; the Artin-Wedderburn Theorem states that

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $\dim_{D_i}(V_i) = n_i$. By (2.1.2), $K_0(R) \cong \prod K_0(M_{n_i}(D_i)) \cong \mathbb{Z}^r$.

Another way to see that $K_0(R) \cong \mathbb{Z}^r$ is to use the fact that $\mathbf{P}(R) \cong \mathbb{N}^r$: the Krull-Schmidt Theorem states that every finitely generated (projective) module M is $V_1^{\ell_1} \times \cdots \times V_r^{\ell_r}$ for well-defined integers $\ell_1, ..., \ell_r$.

EXAMPLE 2.1.5 (VON NEUMANN REGULAR RINGS). A ring R is said to be *von Neumann regular* if for every $r \in R$ there is an $x \in R$ such that $rxr = r$. Since $rxrx = rx$, the element $e = rx$ is idempotent, and the ideal $rR = eR$ is a projective module. In fact, every finitely generated right ideal of R is of the form eR for some idempotent, and these form a lattice. Declaring $e \simeq e'$ if $eR = e'R$, the equivalence classes of idempotents in R form a lattice: $(e_1 \wedge e_2)$ and $(e_1 \vee e_2)$ are defined to be the idempotents generating $e_1R + e_2R$ and $e_1R \cap e_2R$, respectively. Kaplansky proved in [Kap58] that every projective R-module is a direct sum of the modules eR. It follows that $K_0(R)$ is determined by the lattice of idempotents (modulo $\simeq$) in R. We will see several examples of von Neumann regular rings in the exercises.

Many von Neumann regular rings do not satisfy the (IBP), the ring $\mathrm{End}_F(F^\infty)$ of Ex. I.1.7 being a case in point.

We call a ring R *unit-regular* if for every $r \in R$ there is a unit $x \in R$ such that $rxr = r$. Every unit-regular ring is Von Neumann regular, has stable range 1, and satisfies the (IBP) (Ex. I.1.13). In particular, $\mathbb{Z} \subseteq K_0(R)$. It is unknown whether for every simple unit-regular ring R the group $K_0(R)$ is *strictly unperforated*, meaning that whenever $x \in K_0(R)$ and $nx = [Q]$ for some Q, then $x = [P]$ for some P.

Goodearl [Gdrl1] has given examples of simple unit-regular rings R in which the group $K_0(R)$ is strictly unperforated, but has torsion.

An example of a von Neumann regular ring R having the IBP and stable range 2, and $K_0(R) = \mathbb{Z} \oplus \mathbb{Z}/n$ is given in [MM82].

2.1.6. Suppose that R is the direct limit of a filtered system $\{R_i\}$ of rings. Then every finitely generated projective R-module is of the form $P_i \otimes_{R_i} R$ for some i and some finitely generated projective R_i-module P_i. Any isomorphism $P_i \otimes_{R_i} R \cong P_i' \otimes_{R_i} R$ may be expressed using finitely many elements of R, and hence $P_i \otimes_{R_i} R_j \cong P_i' \otimes_{R_i} R_j$ for some j. That is, $\mathbf{P}(R)$ is the filtered colimit of the $\mathbf{P}(R_i)$. By Ex. 1.3 we have

$$K_0(R) \cong \varinjlim K_0(R_i).$$

This observation is useful when studying $K_0(R)$ of a commutative ring R, because R is the direct limit of its finitely generated subrings. As finitely generated commutative rings are noetherian with finite normalization, properties of $K_0(R)$ may be deduced from properties of K_0 of these nice subrings. If R is integrally closed we may restrict to finitely generated normal subrings, so $K_0(R)$ is determined by K_0 of noetherian integrally closed domains.

Here is another useful reduction; it follows immediately from the observation that if I is nilpotent (or complete) then idempotent lifting (Ex. I.2.2) yields a monoid isomorphism $\mathbf{P}(R) \cong \mathbf{P}(R/I)$. Recall that an ideal I is said to be *complete* if every Cauchy sequence $\sum_{n=1}^{\infty} x_n$ with $x_n \in I^n$ converges to a unique element of I.

LEMMA 2.2. *If I is a nilpotent ideal of R, or more generally a complete ideal, then*

$$K_0(R) \cong K_0(R/I).$$

In particular, if R is commutative then $K_0(R) \cong K_0(R_{\text{red}})$.

EXAMPLE 2.2.1 (0-DIMENSIONAL COMMUTATIVE RINGS). Let R be a commutative ring. It is elementary that R_{red} is Artinian if and only if $\text{Spec}(R)$ is finite and discrete. More generally, it is known (see Ex. I.1.13 and [AM, Ex. 3.11]) that the following are equivalent:

(i) R_{red} is a commutative von Neumann regular ring (2.1.5);
(ii) R has Krull dimension 0;
(iii) $X = \text{Spec}(R)$ is compact, Hausdorff and totally disconnected.

(For example, to see that a commutative von Neumann regular R must be reduced, observe that if $r^2 = 0$ then $r = rxr = 0$.)

When R is a commutative von Neumann regular ring, the modules eR are componentwise free; Kaplansky's result states that every projective module is componentwise free. By I.2, the monoid $\mathbf{P}(R)$ is just $[X, \mathbb{N}]$, $X = \text{Spec}(R)$. By (1.4) this yields $K_0(R) = [X, \mathbb{Z}]$. By Lemma 2.2, this proves

PIERCE'S THEOREM 2.2.2. *For every 0-dimensional commutative ring R:*

$$K_0(R) = [\text{Spec}(R), \mathbb{Z}].$$

Example 2.2.3 (K_0 does not commute with infinite products). Let $R = \prod F_i$ be an infinite product of fields. Then R is von Neumann regular, so $X = \mathrm{Spec}(R)$ is an uncountable totally disconnected compact Hausdorff space. By Pierce's Theorem, $K_0(R) \cong [X, \mathbb{Z}]$. This is contained in but not equal to the product $\prod K_0(F_i) \cong \prod \mathbb{Z}$.

Rank and H_0

Definition. When R is commutative, we write $H_0(R)$ for $[\mathrm{Spec}(R), \mathbb{Z}]$, the ring of all continuous maps from $\mathrm{Spec}(R)$ to $\mathbb{Z}$. Since $\mathrm{Spec}(R)$ is quasi-compact, we know by (1.4) that $H_0(R)$ is always a free abelian group. If R is a noetherian ring, then $\mathrm{Spec}(R)$ has only finitely many (say c) components, and $H_0(R) \cong \mathbb{Z}^c$. If R is a domain, or more generally if $\mathrm{Spec}(R)$ is connected, then $H_0(R) = \mathbb{Z}$.

$H_0(R)$ is a subring of $K_0(R)$. To see this, consider the submonoid L of $\mathbf{P}(R)$ consisting of componentwise free modules R^f. Not only is L cofinal in $\mathbf{P}(R)$, but $L \to \mathbf{P}(R)$ is a semiring map: $R^f \otimes R^g \cong R^{fg}$; by (1.3), $L^{-1}L$ is a subring of $K_0(R)$. Finally, L is isomorphic to $[\mathrm{Spec}(R), \mathbb{N}]$, so as in (1.4) we have $L^{-1}L \cong H_0(R)$. For example, Pierce's theorem (2.2.2) states that if $\dim(R) = 0$ then $K_0(R) \cong H_0(R)$.

Recall from I.2 that the rank of a projective module gives a map from $\mathbf{P}(R)$ to $[\mathrm{Spec}(R), \mathbb{N}]$. Since $\mathrm{rank}(P \oplus Q) = \mathrm{rank}(P) + \mathrm{rank}(Q)$ and $\mathrm{rank}(P \otimes Q) = \mathrm{rank}(P)\,\mathrm{rank}(Q)$ (by Ex. I.2.7), this is a semiring map. As such it induces a ring map

$$\mathrm{rank}\colon K_0(R) \to H_0(R).$$

Since $\mathrm{rank}(R^f) = f$ for every componentwise free module, the composition $H_0(R) \subset K_0(R) \to H_0(R)$ is the identity. Thus $H_0(R)$ is a direct summand of $K_0(R)$.

Definition 2.3. The ideal $\widetilde{K}_0(R)$ of the ring $K_0(R)$ is defined as the kernel of the rank map. By the above remarks, there is a natural decomposition

$$K_0(R) \cong H_0(R) \oplus \widetilde{K}_0(R).$$

We will see later (in §4, §6) that $\widetilde{K}_0(R)$ is a nil ideal. Since $H_0(R)$ is visibly a reduced ring, $\widetilde{K}_0(R)$ is the nilradical of $K_0(R)$.

Lemma 2.3.1. *If R is commutative, let $\mathbf{P}_n(R)$ denote the subset of $\mathbf{P}(R)$ consisting of projective modules of constant rank n. There is a map $\mathbf{P}_n(R) \to K_0(R)$ sending P to $[P] - [R^n]$. This map is compatible with the stabilization map $\mathbf{P}_n(R) \to \mathbf{P}_{n+1}(R)$ sending P to $P \oplus R$, and the induced map is an isomorphism:*

$$\varinjlim \mathbf{P}_n(R) \cong \widetilde{K}_0(R).$$

Proof. This follows easily from (1.3). □

Corollary 2.3.2. *Let R be a commutative noetherian ring of Krull dimension d — or more generally any commutative ring of stable range $d+1$ (Ex. I.1.5). For every $n > d$ the above maps are bijections: $\mathbf{P}_n(R) \cong \widetilde{K}_0(R)$.*

PROOF. If P and Q are finitely generated projective modules of rank $> d$, then by Bass Cancellation (I.2.3b) we may conclude that

$$[P] = [Q] \text{ in } K_0(R) \quad \text{if and only if} \quad P \cong Q. \qquad \square$$

Here is another interpretation of $\widetilde{K}_0(R)$: it is the intersection of the kernels of $K_0(R) \to K_0(F)$ over all maps $R \to F$, F a field. This follows from naturality of rank and the observation that $\widetilde{K}_0(F) = 0$ for every field F.

This motivates the following definition for a noncommutative ring R: let $\widetilde{K}_0(R)$ denote the intersection of the kernels of $K_0(R) \to K_0(S)$ over all maps $R \to S$, where S is a simple artinian ring. If no such map $R \to S$ exists, we set $\widetilde{K}_0(R) = K_0(R)$. We define $H_0(R)$ to be the quotient of $K_0(R)$ by $\widetilde{K}_0(R)$. When R is commutative, this agrees with the above definitions of H_0 and $\widetilde{K}_0$, because the maximal commutative subrings of a simple artinian ring S are finite products of 0-dimensional local rings.

$H_0(R)$ is a torsionfree abelian group for every ring R. To see this, note that there is a set X of maps $R \to S_x$ through which every other $R \to S'$ factors. Since each $K_0(S_x) \to K_0(S')$ is an isomorphism, $\widetilde{K}_0(R)$ is the intersection of the kernels of the maps $K_0(R) \to K_0(S_x)$, $x \in X$. Hence $H_0(R)$ is the image of $K_0(R)$ in the torsionfree group $\prod_{x \in X} K_0(S_x) \cong \prod_x \mathbb{Z} \cong Map(X, \mathbb{Z})$.

EXAMPLE 2.4 (WHITEHEAD GROUP Wh_0). If R is the group ring $\mathbb{Z}[G]$ of a group G, the (zero-th) *Whitehead group* $Wh_0(G)$ is the quotient of $K_0(\mathbb{Z}[G])$ by the subgroup $K_0(\mathbb{Z}) = \mathbb{Z}$. The augmentation map $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ sending G to 1 induces a decomposition $K_0(\mathbb{Z}[G]) \cong \mathbb{Z} \oplus Wh_0(G)$, and clearly $\widetilde{K}_0(\mathbb{Z}[G]) \subseteq Wh_0(G)$. It follows from a theorem of Swan ([Bass, XI(5.2)]) that if G is finite then $\widetilde{K}_0(\mathbb{Z}G) = Wh_0(G)$ and $H_0(\mathbb{Z}G) = \mathbb{Z}$. I do not know whether $\widetilde{K}_0(\mathbb{Z}G) = Wh_0(G)$ for every group.

The group $Wh_0(G)$ arose in topology via the following result of C.T.C. Wall. We say that a CW complex X is *dominated* by a complex K if there is a map $f\colon K \to X$ having a right homotopy inverse; this says that X is a retract of K in the homotopy category.

THEOREM 2.4.1 (WALL FINITENESS OBSTRUCTION). *Suppose that X is dominated by a finite CW complex, with fundamental group $G = \pi_1(X)$. This data determines an element $w(X)$ of $Wh_0(G)$ such that $w(X) = 0$ if and only if X is homotopy equivalent to a finite CW complex.*

Hattori-Stallings trace map

For any associative ring R, let $[R, R]$ denote the subgroup of R generated by the elements $[r, s] = rs - sr$, $r, s \in R$.

For each n, the trace of an $n \times n$ matrix provides an additive map from $M_n(R)$ to $R/[R, R]$ invariant under conjugation; the inclusion of $M_n(R)$ in $M_{n+1}(R)$ via $g \mapsto \left(\begin{smallmatrix} g & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is compatible with the trace map. It is not hard to show that the trace $M_n(R) \to R/[R, R]$ induces an isomorphism:

$$M_n(R)/[M_n(R), M_n(R)] \cong R/[R, R].$$

If P is a finitely generated projective, choosing an isomorphism $P \oplus Q \cong R^n$ yields an idempotent e in $M_n(R)$ such that $P = e(R^n)$ and $\text{End}(P) = eM_n(R)e$. By Ex. I.2.3, any other choice yields an e_1 which is conjugate to e in some larger $M_m(R)$. Therefore the trace of an endomorphism of P is a well-defined element of $R/[R,R]$, independent of the choice of e. This gives the *trace map* $\text{End}(P) \to R/[R,R]$. In particular, the trace of the identity map of P is the trace of e; we call it the *trace* of P.

If P' is represented by an idempotent matrix f then $P \oplus P'$ is represented by the idempotent matrix $\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)$ so the trace of $P \oplus P'$ is $\text{trace}(P) + \text{trace}(P')$. Therefore the trace is an additive map on the monoid $\mathbf{P}(R)$. The map $K_0(R) \to R/[R,R]$ induced by universality is called the *Hattori-Stallings trace map*, after the two individuals who first studied it.

When R is commutative, we can provide a direct description of the ring map $H_0(R) \to R$ obtained by restricting the trace map to the subring $H_0(R)$ of $K_0(R)$. Any continuous map $f\colon \text{Spec}(R) \to \mathbb{Z}$ induces a decomposition $R = R_1 \times \cdots \times R_c$ by Ex. I.2.4; the coordinate idempotents $e_1, ..., e_c$ are elements of R. Since $\text{trace}(e_i R)$ is e_i, it follows immediately that $\text{trace}(f)$ is $\sum f(i) e_i$. The identity $\text{trace}(fg) = \text{trace}(f)\text{trace}(g)$ which follows immediately from this formula shows that trace is a ring map.

PROPOSITION 2.5. *If R is commutative then the Hattori-Stallings trace factors as*

$$K_0(R) \xrightarrow{\text{rank}} H_0(R) \to R.$$

PROOF. The product over all $\mathfrak{p}$ in $\text{Spec}(R)$ yields the commutative diagram:

$$\begin{array}{ccc} K_0(R) & \longrightarrow & \prod K_0(R_{\mathfrak{p}}) \\ {\scriptstyle\text{trace}}\downarrow & & \downarrow{\scriptstyle\text{trace}} \\ R & \xrightarrow[\text{inclusion}]{\text{diagonal}} & \prod R_{\mathfrak{p}}. \end{array}$$

The kernel of the top arrow is $\widetilde{K}_0(R)$, so the left arrow factors as claimed. $\square$

EXAMPLE 2.5.1 (GROUP RINGS). Let k be a commutative ring, and suppose that R is the group ring $k[G]$ of a group G. If g and h are conjugate elements of G then $h - g \in [R,R]$ because $xgx^{-1} - g = [xg, x^{-1}]$. From this it is not hard to see that $R/[R,R]$ is isomorphic to the free k-module $\oplus k[g]$ whose basis is the set $G/\sim$ of conjugacy classes of elements of G. Relative to this basis, we may write

$$\text{trace}(P) = \sum r_P(g)[g].$$

Clearly, the coefficients $r_P(g)$ of $\text{trace}(P)$ are functions on the set $G/\sim$ for each P.

If G is finite, then any finitely generated projective k[G]-module P is also a projective k-module, and we may also form the trace map $\text{End}_k(P) \to k$ and hence the "character" $\chi_P\colon G \to k$ by the formula $\chi_P(g) = \text{trace}(g)$. Hattori proved that if $Z_G(g)$ denotes the centralizer of $g \in G$ then Hattori's formula holds (see [Bass76, 5.8]):

$$\chi_P(g) = |Z_G(g)|\ r_P(g^{-1}). \tag{2.5.2}$$

COROLLARY 2.5.3. *If G is a finite group, the ring $\mathbb{Z}[G]$ has no idempotents except 0 and 1.*

PROOF. Let e be an idempotent element of $\mathbb{Z}[G]$. $\chi_P(1)$ is the rank of the $\mathbb{Z}$-module $P = e\mathbb{Z}[G]$, which must be less than the rank $|G|$ of $\mathbb{Z}[G]$. Since $r_P(1) \in \mathbb{Z}$, this contradicts Hattori's formula $\chi_P(1) = |G|\, r_P(1)$. $\square$

Bass has conjectured that for every group G and every finitely generated projective $\mathbb{Z}[G]$-module P we have $r_P(g) = 0$ for $g \neq 1$ and $r_P(1) = \mathrm{rank}_{\mathbb{Z}}(P \otimes_{\mathbb{Z}[G]} \mathbb{Z})$. For G finite, this follows from Hattori's formula and Swan's theorem (cited in 2.4) that $\widetilde{K}_0 = Wh_0$. See [Bass76].

EXAMPLE 2.5.4. Suppose that k is a field of characteristic 0 and $kG = k[G]$ is the group ring of a finite group G with c conjugacy classes. By Maschke's theorem, kG is a product of simple k-algebras: $S_1 \times \cdots \times S_c$ so $kG/[kG, kG]$ is k^c. By (2.1.4) $K_0(kG) \cong \mathbb{Z}^c$. Hattori's formula (and some classical representation theory) shows that the trace map from $K_0(kG)$ to $kG/[kG, kG]$ is isomorphic to the natural inclusion of $\mathbb{Z}^c$ in k^c.

Determinant

Suppose now that R is a commutative ring. Recall from I.3 that the determinant of a finitely generated projective module P is an element of the Picard group $\mathrm{Pic}(R)$.

PROPOSITION 2.6. *The determinant induces a surjective group homomorphism*

$$\det\colon K_0(R) \to \mathrm{Pic}(R)$$

PROOF. By the universal property of K_0, it suffices to show that $\det(P \oplus Q) \cong \det(P) \otimes_R \det(Q)$. We may assume that P and Q have constant rank m and n, respectively. Then $\wedge^{m+n}(P \oplus Q)$ is the sum over all i, j such that $i + j = m + n$ of $(\wedge^i Q) \otimes (\wedge^j P)$. If $i > m$ or $j > n$ we have $\wedge^i P = 0$ or $\wedge^j Q = 0$, respectively. Hence $\wedge^{m+n}(P \oplus Q) = (\wedge^m P) \otimes (\wedge^n Q)$, as asserted. $\square$

DEFINITION 2.6.1. Let $SK_0(R)$ denote the subset of $K_0(R)$ consisting of the classes $x = [P] - [R^m]$, where P has constant rank m and $\wedge^m P \cong R$. This is the kernel of $\det\colon \widetilde{K}_0(R) \to \mathrm{Pic}(R)$, by Lemma 2.3.1 and Proposition 2.6.

$SK_0(R)$ is an ideal of $K_0(R)$. To see this, we use Ex. I.3.4: if $x = [P]-[R^m]$ is in $SK_0(R)$ and Q has rank n then $\det(x \cdot Q) = (\det P)^{\otimes n} (\det Q)^{\otimes m} (\det Q)^{\otimes -m} = R$.

COROLLARY 2.6.2. *For every commutative ring R, $H_0(R) \oplus \mathrm{Pic}(R)$ is a ring with square-zero ideal $\mathrm{Pic}(R)$, and there is a surjective ring homomorphism with kernel $SK_0(R)$:*

$$\mathrm{rank} \oplus \det\colon K_0(R) \to H_0(R) \oplus \mathrm{Pic}(R)$$

COROLLARY 2.6.3. *If R is a 1-dimensional commutative noetherian ring, then the classification of finitely generated projective R-modules in I.3.4 induces an isomorphism:*

$$K_0(R) \cong H_0(R) \oplus \mathrm{Pic}(R).$$

Morita Equivalence

We say that two rings R and S are *Morita equivalent* if **mod**-R and **mod**-S are equivalent as abelian categories, that is, if there exist additive functors T and U

$$\mathbf{mod}\text{-}R \quad \underset{U}{\overset{T}{\rightleftarrows}} \quad \mathbf{mod}\text{-}S$$

such that $UT \cong id_R$ and $TU \cong id_S$. This implies that T and U preserve filtered colimits. Set $P = T(R)$ and $Q = U(S)$; P is an R-S bimodule and Q is a S–R bimodule via the maps $R = \mathrm{End}_R(R) \xrightarrow{T} \mathrm{End}_S(P)$ and $S = \mathrm{End}_S(S) \xrightarrow{U} \mathrm{End}_R(Q)$. Since $T(\oplus R) = \oplus P$ and $U(\oplus S) = \oplus Q$ it follows that we have $T(M) \cong M \otimes_R P$ and $U(N) \cong N \otimes_S Q$ for all M, N. Both $UT(R) \cong P \otimes_S Q \cong R$ and $TU(S) \cong Q \otimes_R P \cong S$ are bimodule isomorphisms. Here is the main structure theorem, taken from [Bass, II.3].

STRUCTURE THEOREM FOR MORITA EQUIVALENCE 2.7. *If R and S are Morita equivalent, and P, Q are as above, then:*

(a) *P and Q are finitely generated projective, both as R-modules and as S-modules;*
(b) *$End_S(P) \cong R \cong End_S(Q)^{op}$ and $End_R(Q) \cong S \cong End_R(P)^{op}$;*
(c) *P and Q are dual S-modules: $P \cong \mathrm{Hom}_S(Q, S)$ and $Q \cong \mathrm{Hom}_S(P, S)$;*
(d) *$T(M) \cong M \otimes_R P$ and $U(N) \cong N \otimes_S Q$ for every M and N;*
(e) *P is a "faithful" S-module in the sense that the functor $\mathrm{Hom}_S(P, -)$ from **mod**-S to abelian groups is a faithful functor. (If S is commutative then P is faithful if and only if $\mathrm{rank}(P) \geq 1$.) Similarly, Q is a "faithful" R-module.*

Since P and Q are finitely generated projective, the Morita functors T and U also induce an equivalence between the categories $\mathbf{P}(R)$ and $\mathbf{P}(S)$. This implies the following:

COROLLARY 2.7.1. *If R and S are Morita equivalent then $K_0(R) \cong K_0(S)$.*

EXAMPLE 2.7.2. $R = M_n(S)$ is always Morita equivalent to S; P is the bimodule S^n of "column vectors" and Q is the bimodule $(S^n)^t$ of "row vectors." More generally suppose that P is a "faithful" finitely generated projective S-module. Then $R = \mathrm{End}_S(P)$ is Morita equivalent to S, the bimodules being P and $Q = \mathrm{Hom}_S(P, S)$. By 2.7.1, we see that $K_0(S) \cong K_0(M_n(S))$.

ADDITIVE FUNCTORS 2.8. Any R-S bimodule P which is finitely generated projective as a right S-module, induces an additive (hence exact) functor $T(M) = M \otimes_R P$ from $\mathbf{P}(R)$ to $\mathbf{P}(S)$, and therefore induces a map $K_0(R) \to K_0(S)$. If all we want is an additive functor T from $\mathbf{P}(R)$ to $\mathbf{P}(S)$, we do not need the full strength of Morita equivalence. Given T, set $P = T(R)$. By additivity we have $T(R^n) = P^n \cong R^n \otimes_R P$; from this it is not hard to see that $T(M) \cong M \otimes_R P$ for every finitely generated projective M, and that T is isomorphic to $- \otimes_R P$. See Ex. 2.14 for more details.

A bimodule map (resp., isomorphism) $P \to P'$ induces an additive natural transformation (resp., isomorphism) $T \to T'$. This is the case, for example, with the bimodule isomorphism $R \oplus M \cong M$ defining a flasque ring (2.1.3).

EXAMPLE 2.8.1 (BASE CHANGE AND TRANSFER MAPS). Suppose that $f\colon R \to S$ is a ring map. Then S is an R–S bimodule, and it represents the *base change functor* $f^*\colon K_0(R) \to K_0(S)$ sending P to $P \otimes_R S$. If in addition S is finitely generated projective as a right R-module then there is a forgetful functor from $\mathbf{P}(S)$ to $\mathbf{P}(R)$; it is represented by S as a S–R bimodule because it sends Q to $Q \otimes_S S$. The induced map $f_*\colon K_0(S) \to K_0(R)$ is called the *transfer map*. We will return to this point in 7.9 below, explaining why we have selected the contravariant notation f^* and f_*.

Mayer-Vietoris sequences

For any ring R with unit, we can include $GL_n(R)$ in $GL_{n+1}(R)$ as the matrices $\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The group $GL(R)$ is the union of the groups $GL_n(R)$. Now suppose we are given a Milnor square of rings, as in I.2:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar f} & S/I \end{array}$$

Define $\partial_n\colon GL_n(S/I) \to K_0(R)$ by Milnor patching: $\partial_n(g)$ is $[P] - [R^n]$, where P is the projective R-module obtained by patching free modules along g as in (I.2.6). The formulas of Ex. I.2.9 imply that $\partial_n(g) = \partial_{n+1}\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\partial_n(g)+\partial_n(h) = \partial_n(gh)$. Therefore the $\{\partial_n\}$ assemble to give a group homomorphism ∂ from $GL(S/I)$ to $K_0(R)$. The following result now follows from (I.2.6) and Ex. 1.4.

THEOREM 2.9 (MAYER-VIETORIS). *Given a Milnor square as above, the sequence*

$$GL(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I)$$

is exact. The image of ∂ is the double coset space

$$GL(S)\backslash GL(S/I)/GL(R/I) = GL(S/I)/\sim$$

where $x \sim gxh$ for $x \in GL(S/I), g \in GL(S)$ and $h \in GL(R/I)$.

EXAMPLE 2.9.1. If R is the coordinate ring of the node over a field k (I.3.10.2) then $K_0(R) \cong \mathbb{Z} \oplus k^\times$. If R is the coordinate ring of the cusp over k (I.3.10.1) then $K_0(R) \cong \mathbb{Z} \oplus k$. Indeed, the coordinate rings of the node and the cusp are 1-dimensional noetherian rings, so 2.6.3 reduces the Mayer-Vietoris sequence to the Units-Pic sequence I.3.10.

We conclude with a useful construction, anticipating several later developments.

DEFINITION 2.10. Let $T : \mathbf{P}(R) \to \mathbf{P}(S)$ be an additive functor, such as the base change or transfer of 2.8.1. $\mathbf{P}(T)$ is the category whose objects are triples (P, α, Q), where $P, Q \in \mathbf{P}(R)$ and $\alpha : T(P) \to T(Q)$ is an isomorphism. A morphism $(P, \alpha, Q) \to (P', \alpha', Q')$ is a pair of R-module maps $p : P \to P'$, $q : Q \to Q'$ such that $\alpha' T(p) = T(q)\alpha$. An *exact sequence* in $\mathbf{P}(T)$ is a sequence

(2.10.1) $$0 \to (P', \alpha', Q') \to (P, \alpha, Q) \to (P'', \alpha'', Q'') \to 0$$

whose underlying sequences $0 \to P' \to P \to P'' \to 0$ and $0 \to Q' \to Q \to Q'' \to 0$ are exact. We define $K_0(T)$ to be the abelian group with generators the objects of $\mathbf{P}(T)$ and relations:

(a) $[(P, \alpha, Q)] = [(P', \alpha', Q')] + [(P'', \alpha'', Q'')]$ for every exact sequence (2.10.1);
(b) $[(P_1, \alpha, P_2)] + [(P_2, \beta, P_3)] = [(P_1, \beta\alpha, P_3)]$.

If T is the base change f^*, we write $K_0(f)$ for $K_0(T)$.

It is easy to see that there is a map $K_0(T) \to K_0(R)$ sending $[(P, \alpha, Q)]$ to $[P] - [Q]$. If T is a base change functor f^* associated to $f : R \to S$, or more generally if the $T(R^n)$ are cofinal in $\mathbf{P}(S)$, then there is an exact sequence:

(2.10.2) $$GL(S) \xrightarrow{\partial} K_0(T) \to K_0(R) \to K_0(S).$$

The construction of ∂ and verification of exactness is not hard, but lengthy enough to relegate to exercise 2.17. If $f : R \to R/I$ then $K_0(f^*)$ is the group $K_0(I)$ of Ex. 2.4; see Ex. 2.4(e).

EXERCISES

2.1 Let R be a commutative ring. If A is an R-algebra, show that the functor $\otimes_R : \mathbf{P}(A) \times \mathbf{P}(R) \to \mathbf{P}(A)$ yields a map $K_0(A) \otimes_{\mathbb{Z}} K_0(R) \to K_0(A)$ making $K_0(A)$ into a $K_0(R)$-module. If $A \to B$ is an algebra map, show that $K_0(A) \to K_0(B)$ is a $K_0(R)$-module homomorphism.

2.2 *Projection Formula.* Let R be a commutative ring, and A an R-algebra which as an R-module is finitely generated projective of rank n. By Ex. 2.1, $K_0(A)$ is a $K_0(R)$-module, and the base change map $f^* : K_0(R) \to K_0(A)$ is a module homomorphism. We shall write $x \cdot f^*y$ for the product in $K_0(A)$ of $x \in K_0(A)$ and $y \in K_0(R)$; this is an abuse of notation when A is noncommutative.

(a) Show that the transfer map $f_* : K_0(A) \to K_0(R)$ of Example 2.8.1 is a $K_0(R)$-module homomorphism, *i.e.,* that the *projection formula* holds:
$$f_*(x \cdot f^*y) = f_*(x) \cdot y \qquad \text{for every } \ x \in K_0(A),\ y \in K_0(R).$$
(b) Show that both compositions f^*f_* and f_*f^* are multiplication by $[A]$.
(c) Show that the kernels of f^*f_* and f_*f^* are annihilated by a power of n.

2.3 *Excision for K_0.* If I is an ideal in a ring R, form the augmented ring $R \oplus I$ and let $K_0(I) = K_0(R, I)$ denote the kernel of $K_0(R \oplus I) \to K_0(R)$.

(a) If $R \to S$ is a ring map sending I isomorphically onto an ideal of S, show that $K_0(R, I) \cong K_0(S, I)$. Thus $K_0(I)$ is independent of R. *Hint.* Show that $GL(S)/GL(S \oplus I) = 1$.
(b) If $I \cap J = 0$, show that $K_0(I + J) \cong K_0(I) \oplus K_0(J)$.
(c) *Ideal sequence.* Show that there is an exact sequence
$$GL(R) \to GL(R/I) \xrightarrow{\partial} K_0(I) \to K_0(R) \to K_0(R/I).$$

(d) If R is commutative, use Ex. I.3.6 to show that there is a commutative diagram with exact rows, the vertical maps being determinants:

$$\begin{array}{ccccccccc} GL(R) & \longrightarrow & GL(R/I) & \xrightarrow{\partial} & K_0(I) & \longrightarrow & K_0(R) & \longrightarrow & K_0(R/I) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R^\times & \longrightarrow & (R/I)^\times & \xrightarrow{\partial} & \mathrm{Pic}(I) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & \mathrm{Pic}(R/I). \end{array}$$

2.4 *K_0I.* If I is a ring without unit, we define $K_0(I)$ as follows. Let R be a ring with unit acting upon I, form the augmented ring $R \oplus I$, and let $K_0(I)$ be the kernel of $K_0(R \oplus I) \to K_0(R)$. Thus $K_0(R \oplus I) \cong K_0(R) \oplus K_0(I)$ by definition.

(a) If I has a unit, show that $R \oplus I \cong R \times I$ as rings with unit. Since $K_0(R \times I) = K_0(R) \times K_0(I)$, this shows that the definition of Ex. 2.3 agrees with the usual definition of $K_0(I)$.

(b) Show that a map $I \to J$ of rings without unit induces a map $K_0(I) \to K_0(J)$

(c) Let $M_\infty(R)$ denote the union $\cup M_n(R)$ of the matrix groups, where $M_n(R)$ is included in $M_{n+1}(R)$ as the matrices $\left(\begin{smallmatrix} g & 0 \\ 0 & 0 \end{smallmatrix}\right)$. $M_\infty(R)$ is a ring without unit. Show that the inclusion of $R = M_1(R)$ in $M_\infty(R)$ induces an isomorphism

$$K_0(R) \cong K_0(M_\infty(R)).$$

(d) If k is a field, show that $R = k \oplus M_\infty(k)$ is a von Neumann regular ring. Then show that $H_0(R) = \mathbb{Z}$ and $K_0(R) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(e) If $f : R \to R/I$, show that $K_0(I)$ is the group $K_0(f^*)$ of 2.10. *Hint:* Use $f_0 : R \oplus I \to R$ and Ex. 2.3(c).

2.5 *Radical ideals.* Let I be a radical ideal in a ring R (see Ex. I.1.12, I.2.1).

(a) Show that $K_0(I) = 0$, and that $K_0(R) \to K_0(R/I)$ is an injection.

(b) If I is a complete ideal, $K_0(R) \cong K_0(R/I)$ by Lemma 2.2. If R is a semilocal but not local domain, show that $K_0(R) \to K_0(R/I)$ is not an isomorphism.

2.6 *Semilocal rings.* A ring R is called *semilocal* if R/J is semisimple for some radical ideal J. Show that if R is semilocal then $K_0(R) \cong \mathbb{Z}^n$ for some $n > 0$.

2.7 Show that if $f : R \to S$ is a map of commutative rings, then:

$$\ker(f) \text{ contains no idempotents}(\neq 0) \Leftrightarrow H_0(R) \to H_0(S) \text{ is an injection.}$$

Conclude that $H_0(R) = H_0(R[t]) = H_0(R[t, t^{-1}])$.

2.8 Consider the following conditions on a ring R (cf. Ex. I.1.2):

(IBP) R satisfies the Invariant Basis Property (IBP);

(PO) $K_0(R)$ is a partially ordered abelian group (see §1);

(III) For all n, if $R^n \cong R^n \oplus P$ then $P = 0$.

Show that $(III) \Rightarrow (PO) \Rightarrow (IBP)$. This implies that $K_0(R)$ is a partially ordered abelian group if R is either commutative or noetherian. (See Ex. I.1.4.)

2.9 *Rim squares.* Let G be a cyclic group of prime order p, and $\zeta = e^{2\pi i/p}$ a primitive p^{th} root of unity. Show that the map $\mathbb{Z}[G] \to \mathbb{Z}[\zeta]$ sending a generator of G to ζ induces an isomorphism $K_0(\mathbb{Z}[G]) \cong K_0(\mathbb{Z}[\zeta])$ and hence $Wh_0(G) \cong$

$\mathrm{Pic}(\mathbb{Z}[\zeta])$. *Hint:* Form a Milnor square with $\mathbb{Z}G/I = \mathbb{Z}$, $\mathbb{Z}[\zeta]/I = \mathbb{F}_p$, and consider the cyclotomic units $u = \frac{\zeta^i-1}{\zeta-1}$, $1 \le i < p$.

2.10 Let R be a commutative ring. Prove that

(a) If $\mathrm{rank}(x) > 0$ for some $x \in K_0(R)$, then there is an $n > 0$ and a finitely generated projective module P so that $nx = [P]$. (This says that the partially ordered group $K_0(R)$ is "unperforated" in the sense of [Gdearl].)
(b) If P, Q are finitely generated projectives such that $[P] = [Q]$ in $K_0(R)$, then there is an $n > 0$ such that $P \oplus \cdots \oplus P \cong Q \oplus \cdots \oplus Q$ (n copies of P, n copies of Q).

Hint: First assume that R is noetherian of Krull dimension $d < \infty$, and use Bass-Serre Cancellation. In the general case, write R as a direct limit.

2.11 A (normalized) *dimension function* for a von Neumann regular ring R is a group homomorphism $d : K_0(R) \to \mathbb{R}$ so that $d(R^n) = n$ and $d(P) > 0$ for every nonzero finitely generated projective P.

(a) If $P \subseteq Q$, show that any dimension function must have $d(P) \le d(Q)$
(b) If R has a dimension function, show that the formula $\rho(r) = d(rR)$ defines a rank function $\rho\colon R \to [0,1]$ in the sense of Ex. I.1.13. Then show that this gives a 1-1 correspondence between rank functions on R and dimension functions on $K_0(R)$.

2.12 Let R be the union of the matrix rings $M_{n!}(F)$ constructed in Ex. I.1.13. Show that the inclusion $\mathbb{Z} \subset K_0(R)$ extends to an isomorphism $K_0(R) \cong \mathbb{Q}$.

2.13 Let R be the infinite product of the matrix rings $M_i(\mathbb{C})$, $i = 1, 2, \ldots$

(a) Show that every finitely generated projective R-module P is componentwise trivial in the sense that $P \cong \prod P_i$, the P_i being finitely generated projective $M_i(\mathbb{C})$-modules.
(b) Show that the map from $K_0(R)$ to the group $\prod K_0(M_i(\mathbb{C})) = \prod \mathbb{Z}$ of infinite sequences $(n_1, n_2, \ldots)$ of integers is an injection, and that $K_0(R) = H_0(R)$ is isomorphic to the group of bounded sequences.
(c) Show that $K_0(R)$ is not a free abelian group, even though it is torsionfree. *Hint:* Consider the subgroup S of sequences $(n_1, \ldots)$ such that the power of 2 dividing n_i approaches ∞ as $i \to \infty$; show that S is uncountable but that $S/2S$ is countable.

2.14 *Bivariant* K_0. If R and R' are rings, let $Rep(R, R')$ denote the set of isomorphism classes of R–R' bimodules M such that M is finitely generated projective as a right R'-module. Each M gives a functor $\otimes_R M$ from $\mathbf{P}(R)$ to $\mathbf{P}(R')$ sending P to $P \otimes_R M$. This induces a monoid map $\mathbf{P}(R) \to \mathbf{P}(R')$ and hence a homomorphism from $K_0(R)$ to $K_0(R')$. For example, if $f\colon R \to R'$ is a ring homomorphism and R' is considered as an element of $Rep(R, R')$, we obtain the map $\otimes_R R'$. Show that:

(a) Every additive functor $\mathbf{P}(R) \to \mathbf{P}(R')$ is induced from an M in $Rep(R, R')$;
(b) If $K_0(R, R')$ denotes the group completion of $Rep(R, R')$, then $M \otimes_{R'} N$ induces a bilinear map from $K_0(R, R') \otimes K_0(R', R'')$ to $K_0(R, R'')$;
(c) $K_0(\mathbb{Z}, R)$ is $K_0(R)$, and if $M \in Rep(R, R')$ then the map $\otimes_R M : K_0(R) \to K_0(R')$ is induced from the product of (b).

(d) If R and R' are Morita equivalent, and P is the R-R' bimodule giving the isomorphism **mod**-$R \cong$ **mod**-R', the class of P in $K_0(R, R')$ gives the Morita isomorphism $K_0(R) \cong K_0(R')$.

2.15 In this exercise, we connect the definition 2.1.3 of infinite sum ring with a more elementary description due to Wagoner. If R is a direct sum ring, the isomorphism $R^2 \cong R$ induces a ring homomorphism $\oplus : R \times R \subset \mathrm{End}_R(R^2) \cong \mathrm{End}_R(R) = R$.

(a) Suppose that R is an infinite sum ring with bimodule M, and write $r \mapsto r^\infty$ for the ring homomorphism $R \to \mathrm{End}_R(M) \cong R$ arising from the left action of R on the right R-module M. Show that $r \oplus r^\infty = r^\infty$ for all $r \in R$.

(b) Conversely, suppose that R is a direct sum ring, and $R \xrightarrow{\infty} R$ is a ring map so that $r \oplus r^\infty = r^\infty$ for all $r \in R$. Show that R is an infinite sum ring.

(c) (Wagoner) Show that the Cone Rings of Ex. I.1.8, and the rings $\mathrm{End}_R(R^\infty)$ of Ex. I.1.7, are infinite sum rings. *Hint:* $R^\infty \cong \coprod_{i=1}^\infty R^\infty$, so a version of the Eilenberg Swindle I.2.8 applies.

2.16 For any ring R, let J be the (nonunital) subring of $E = \mathrm{End}_R(R^\infty)$ of all f such that $f(R^\infty)$ is finitely generated (Ex. I.1.7). Show that $M_\infty(R) \subset J_n$ induces an isomorphism $K_0(R) \cong K_0(J)$. *Hint:* For the projection $e_n : R^\infty \to R^n$, $J_n = e_n E$ maps onto $M_n(R) = e_n E e_n$ with nilpotent kernel. But $J = \cup J_n$.

2.17 This exercise shows that there is an exact sequence (2.10.2) when T is cofinal.

(a) Show that $[(P, \alpha, Q)] + [(Q, -\alpha^{-1}, P)] = 0$ and $[(P, T(\gamma), Q)] = 0$ in $K_0(T)$.
(b) Show that every element of $K_0(T)$ has the form $[(P, \alpha, R^n)]$.
(c) Use cofinality and the maps $\partial(\alpha) = [(R^n, \alpha, R^n)]$ of (2.10.2), from $\mathrm{Aut}(TR^n)$ to $K_0(T)$, to show that there is a homomorphism $\partial : GL(S) \to K_0(T)$.
(d) Use (a), (b) and (c) to show that (2.10.2) is exact at $K_0(T)$.
(e) Show that (2.10.2) is exact at $K_0(R)$.

§3. $K(X)$, $KO(X)$ and $KU(X)$ of a topological space

Let X be a paracompact topological space. The sets $\mathbf{VB}_{\mathbb{R}}(X)$ and $\mathbf{VB}_{\mathbb{C}}(X)$ of isomorphism classes of real and complex vector bundles over X are abelian monoids under Whitney sum. By Construction I.4.2, they are commutative semirings under $\otimes$. Hence the group completions $KO(X)$ of $\mathbf{VB}_{\mathbb{R}}(X)$ and $KU(X)$ of $\mathbf{VB}_{\mathbb{C}}(X)$ are commutative rings with identity $1 = [T^1]$. If the choice of $\mathbb{R}$ or $\mathbb{C}$ is understood, we will just write $K(X)$ for simplicity.

Similarly, the set $\mathbf{VB}_{\mathbb{H}}(X)$ is an abelian monoid under $\oplus$, and we write $KSp(X)$ for its group completion. Although it has no natural ring structure, the construction of Ex. I.4.18 endows $KSp(X)$ with the structure of a module over the ring $KO(X)$.

For example if $*$ denotes a 1-point space then $K(*) = \mathbb{Z}$. If X is contractible, then $KO(X) = KU(X) = \mathbb{Z}$ by I.4.6.1. More generally, $K(X) \cong K(Y)$ whenever X and Y are homotopy equivalent by I.4.6.

The functor $K(X)$ is contravariant in X. Indeed, if $f\colon Y \to X$ is continuous, the induced bundle construction $E \mapsto f^*E$ yields a monoid map $f^*\colon \mathbf{VB}(X) \to \mathbf{VB}(Y)$ and hence a ring homomorphism $f^*\colon K(X) \to K(Y)$. By the Homotopy Invariance Theorem I.4.5, the map f^* depends only upon the homotopy class of f in $[Y, X]$.

For example, the universal map $X \to *$ induces a ring map from $\mathbb{Z} = K(*)$ into $K(X)$, sending $n > 0$ to the class of the trivial bundle T^n over X. If $X \neq \emptyset$ then any point of X yields a map $* \to X$ splitting the universal map $X \to *$. Thus the subring $\mathbb{Z}$ is a direct summand of $K(X)$ when $X \neq \emptyset$. (But if $X = \emptyset$ then $K(\emptyset) = 0$.) For the rest of this section, we will assume $X \neq \emptyset$ in order to avoid awkward hypotheses.

The trivial vector bundles T^n and the componentwise trivial vector bundles T^f form sub-semirings of $\mathbf{VB}(X)$, naturally isomorphic to $\mathbb{N}$ and $[X, \mathbb{N}]$, respectively. When X is compact, the semirings $\mathbb{N}$ and $[X, \mathbb{N}]$ are cofinal in $\mathbf{VB}(X)$ by the Subbundle Theorem I.4.1, so by Corollary 1.3 we have subrings

$$\mathbb{Z} \subset [X, \mathbb{Z}] \subset K(X).$$

More generally, it follows from Construction I.4.2 that $\dim\colon \mathbf{VB}(X) \to [X, \mathbb{N}]$ is a semiring map splitting the inclusion $[X, \mathbb{N}] \subset \mathbf{VB}(X)$. Passing to Group Completions, we get a natural ring map

$$\dim\colon K(X) \to [X, \mathbb{Z}]$$

splitting the inclusion of $[X, \mathbb{Z}]$ in $K(X)$.

The kernel of dim will be written as $\widetilde{K}(X)$, or as $\widetilde{KO}(X)$ or $\widetilde{KU}(X)$ if we wish to emphasize the choice of $\mathbb{R}$ or $\mathbb{C}$. Thus $\widetilde{K}(X)$ is an ideal in $K(X)$, and there is a natural decomposition

$$K(X) \cong \widetilde{K}(X) \oplus [X, \mathbb{Z}].$$

Warning. If X is not connected, our group $\widetilde{K}(X)$ differs slightly from the notation in the literature. However, most applications will involve connected spaces, where the notation is the same. This will be clarified by Theorem 3.2 below.

Consider the set map $\mathbf{VB}_n(X) \to \widetilde{K}(X)$ sending E to $[E]-n$. This map is compatible with the stabilization map $\mathbf{VB}_n(X) \to \mathbf{VB}_{n+1}(X)$ sending E to $E\oplus T$, giving a map

$$\varinjlim \mathbf{VB}_n(X) \to \widetilde{K}(X). \tag{3.1.0}$$

We can interpret this in terms of maps between the infinite Grassmannian spaces G_n ($= BO_n$, BU_n or BSp_n) as follows. Recall from the Classification Theorem I.4.10 that the set $\mathbf{VB}_n(X)$ is isomorphic to the set $[X,G_n]$ of homotopy classes of maps. Adding a trivial bundle T to the universal bundle E_n over G_n gives a vector bundle over G_n, so again by the Classification Theorem there is a map $i_n\colon G_n \to G_{n+1}$ such that $E_n\oplus T \cong i_n^*(E_{n+1})$. By Cellular Approximation there is no harm in assuming i_n is cellular. Using I.4.10.1, the map $\Omega i_n\colon \Omega G_n \to \Omega G_{n+1}$ is homotopic to the standard inclusion $O_n \hookrightarrow O_{n+1}$ (resp. $U_n \hookrightarrow U_{n+1}$ or $Sp_n \hookrightarrow Sp_{n+1}$), which sends an $n\times n$ matrix g to the $n+1\times n+1$ matrix $\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}$. By construction, the resulting map $i_n\colon [X,G_n] \to [X,G_{n+1}]$ corresponds to the stabilization map. The direct limit $\varinjlim [X,G_n]$ is then in 1-1 correspondence with the direct limit $\varinjlim \mathbf{VB}_n(X)$ of (3.1.0).

Stabilization Theorem 3.1. *Let X be either a compact space or a finite dimensional connected CW complex. Then the map (3.1.0) induces an isomorphism $\widetilde{K}(X) \cong \varinjlim \mathbf{VB}(X) \cong \varinjlim [X,G_n]$. In particular,*

$$\widetilde{KO}(X) \cong \varinjlim [X,BO_n],\quad \widetilde{KU}(X) \cong \varinjlim [X,BU_n] \text{ and } \widetilde{KSp}(X) \cong \varinjlim [X,BSp_n].$$

Proof. We argue as in Lemma 2.3.1. Since the monoid of (componentwise) trivial vector bundles T^f is cofinal in $\mathbf{VB}(X)$ (I.4.1), we see from Corollary 1.3 that every element of $\widetilde{K}(X)$ is represented by an element $[E]-n$ of some $\mathbf{VB}_n(X)$, and if $[E]-n=[F]-n$ then $E\oplus T^\ell \cong F\oplus T^\ell$ in some $\mathbf{VB}_{n+\ell}(X)$. Thus $\widetilde{K}(X) \cong \varinjlim \mathbf{VB}_n(X)$, as claimed. $\square$

Examples 3.1.1 (Spheres). From I(4.9) we see that $KO(S^1) \cong \mathbb{Z}\oplus\mathbb{Z}/2$ but $KU(S^1)=\mathbb{Z}$, $KO(S^2)=\mathbb{Z}\oplus\mathbb{Z}/2$ but $KU(S^2)=\mathbb{Z}\oplus\mathbb{Z}$, $KO(S^3)=KU(S^3)=\mathbb{Z}$ and $KO(S^4)\cong KU(S^4)=\mathbb{Z}\oplus\mathbb{Z}$.

By Prop. I.4.8, the n-dimensional ($\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) vector bundles on S^d are classified by the homotopy groups $\pi_{d-1}(O_n)$, $\pi_{d-1}(U_n)$ and $\pi_{d-1}(Sp_n)$, respectively. By the Stabilization Theorem, $\widetilde{KO}(S^d) = \lim_{n\to\infty} \pi_{d-1}(O_n)$ and $\widetilde{KU}(S^d) = \lim_{n\to\infty} \pi_{d-1}(U_n)$.

Now Bott Periodicity says that the homotopy groups groups of O_n, U_n and Sp_n stabilize for $n\gg 0$. Moreover, if $n\ge d/2$ then $\pi_{d-1}(U_n)$ is 0 for d odd and $\mathbb{Z}$ for d even. Thus $KU(S^d)=\mathbb{Z}\oplus\widetilde{KU}(S^d)$ is periodic of order 2 in $d>0$: the ideal $\widetilde{KU}(S^d)$ is 0 for d odd and $\mathbb{Z}$ for d even, $d\ne 0$.

Similarly, the $\pi_{d-1}(O_n)$ and $\pi_{d-1}(Sp_n)$ stabilize for $n\ge d$ and $n\ge d/4$; both are periodic of order 8. Thus $KO(S^d)=\mathbb{Z}\oplus\widetilde{KO}(S^d)$ and $KSp(S^d)=\mathbb{Z}\oplus\widetilde{KSp}(S^d)$ are periodic of order 8 in $d>0$, with the groups $\widetilde{KO}(S^d)=\pi_{d-1}(O)$ and $\widetilde{KSp}(S^d)=\pi_{d-1}(Sp)$ being tabulated in the following table.

$d \pmod 8$	1	2	3	4	5	6	7	8
$\widetilde{KO}(S^d)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\widetilde{KSp}(S^d)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

Both of the ideals $\widetilde{KO}(S^d)$ and $\widetilde{KU}(S^d)$ are of square zero.

REMARK 3.1.2. The complexification maps $\mathbb{Z} \cong \widetilde{KO}(S^{4k}) \to \widetilde{KU}(S^{4k}) \cong \mathbb{Z}$ are multiplication by 2 if k is odd, and by 1 if k is even. (The forgetful maps $\widetilde{KU}(S^{4k}) \to \widetilde{KO}(S^{4k})$ have the opposite parity.) Similarly, the maps $\mathbb{Z} \cong \widetilde{KU}(S^{4k}) \to \widetilde{KSp}(S^{4k}) \cong \mathbb{Z}$ are multiplication by 2 if k is odd, and by 1 if k is even. (The forgetful maps $\widetilde{KSp}(S^{4k}) \to \widetilde{KU}(S^{4k})$ have the opposite parity.) These calculations are taken from [MimTo, IV.5.12 and IV.6.1].

Let BO (resp. BU, BSp) denote the direct limit of the Grassmannians Grass_n. As noted after (3.1.0) and in I.4.10.1, the notation reflects the fact that $\Omega\,\mathrm{Grass}_n$ is O_n (resp. U_n, Sp_n), and the maps in the direct limit correspond to the standard inclusions, so that we have $\Omega BO \simeq O = \bigcup O_n$, $\Omega BU \simeq U = \bigcup U_n$ and $\Omega BSp \simeq Sp = \bigcup Sp_n$.

THEOREM 3.2. *For every compact space X:*

$$KO(X) \cong [X, \mathbb{Z} \times BO] \quad \textit{and} \quad \widetilde{KO}(X) \cong [X, BO];$$
$$KU(X) \cong [X, \mathbb{Z} \times BU] \quad \textit{and} \quad \widetilde{KU}(X) \cong [X, BU];$$
$$KSp(X) \cong [X, \mathbb{Z} \times BSp] \quad \textit{and} \quad \widetilde{KSp}(X) \cong [X, BSp].$$

In particular, the homotopy groups $\pi_n(BO) = \widetilde{KO}(S^n)$, $\pi_n(BU) = \widetilde{KU}(S^n)$ and $\pi_n(BSp) = \widetilde{KSp}(S^n)$ are periodic and given in Example 3.1.1.

PROOF. If X is compact then we have $[X, BO] = \varinjlim [X, BO_n]$ and similarly for $[X, BU]$ and $[X, BSp]$. The result now follows from Theorem 3.1 for connected X. For non-connected compact spaces, we only need to show that the maps $[X, BO] \to \widetilde{KO}(X)$, $[X, BU] \to \widetilde{KU}(X)$ and $[X, BSp] \to \widetilde{KSp}(X)$ of Theorem 3.1 are still isomorphisms.

Since X is compact, every continuous map $X \to \mathbb{Z}$ is bounded. Hence the rank of every vector bundle E is bounded, say $\operatorname{rank} E \le n$ for some $n \in \mathbb{N}$. If $f = n - \operatorname{rank} E$ then $F = E \oplus T^f$ has constant rank n, and $[E] - \operatorname{rank} E = [F] - n$. Hence every element of $\widetilde{K}(X)$ comes from some $\mathbf{VB}_n(X)$.

To see that these maps are injective, suppose that $E, F \in \mathbf{VB}_n(X)$ are such that $[E] - n = [F] - n$. By (1.3) we have $E \oplus T^f = F \oplus T^f$ in $\mathbf{VB}_{n+f}(X)$ for some $f \in [X, \mathbb{N}]$. If $f \le p$, $p \in \mathbb{N}$, then adding T^{p-f} yields $E \oplus T^p = F \oplus T^p$. Hence E and F agree in $\mathbf{VB}_{n+p}(X)$. $\square$

DEFINITION 3.2.1 (K^0). For every paracompact X we write $KO^0(X)$ for $[X, \mathbb{Z} \times BO]$, $KU^0(X)$ for $[X, \mathbb{Z} \times BU]$ and $KSp^0(X)$ for $[X, \mathbb{Z} \times BSp]$. By Theorem 3.2, we have $KO^0(X) \cong KO(X)$, $KU^0(X) \cong KU(X)$ and $KSp^0(X) \cong KSp(X)$ for every compact X. Similarly, we shall write $\widetilde{KO}^0(X)$, $\widetilde{KU}^0(X)$ and $\widetilde{KSp}^0(X)$ for $[X, BO]$, $[X, BU]$ and $[X, BSp]$. When the choice of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is clear, we will just write $K^0(X)$ and $\widetilde{K}^0(X)$.

If Y is a subcomplex of X, we define relative groups $K^0(X,Y) = K^0(X/Y)/\mathbb{Z}$ and $\widetilde{K}^0(X,Y) = \widetilde{K}^0(X/Y)$.

When X is paracompact but not compact, $\tilde{K}^0(X)$ and $\tilde{K}(X)$ are connected by stabilization and the map (3.1.0):

$$\widetilde{KO}(X) \leftarrow \varinjlim \mathbf{VB}_n(X) \cong \varinjlim [X, BO_n] \to [X, BO] = \widetilde{KO}^0(X)$$

and similarly for $\widetilde{KU}(X)$ and $\widetilde{KSp}(X)$. We will see in Example 3.7.2 and Ex. 3.2 that the left map need not be an isomorphism. Here is an example showing that the right map need not be an isomorphism either.

Example 3.2.2. (McGibbon) Let X be the infinite bouquet of odd-dimensional spheres $S^3 \vee S^5 \vee S^7 \vee \cdots$. By homotopy theory, there is a map $f\colon X \to BO_3$ whose restriction to S^{2p+1} is essential of order p for each odd prime p. If E denotes the 3-dimensional vector bundle f^*E_3 on X, then the class of f in $\varinjlim [X, BO_n]$ corresponds to $[E] - 3 \in KO(X)$. In fact, since X is a suspension, we have $\varinjlim [X, BO_n] \cong \widetilde{KO}(X)$ by Ex. 3.8.

Each $(n+3)$-dimensional vector bundle $E \oplus T^n$ is nontrivial, since its restriction to S^{2p+1} is nontrivial whenever $2p > n + 3$ (again by homotopy theory). Hence $[E] - 3$ is a nontrivial element of $\widetilde{KO}(X)$. However, the corresponding element in $\widetilde{KO}^0(X) = [X, BO]$ is zero, because the homotopy groups of BO have no odd torsion.

Proposition 3.3. *If Y is a subcomplex of a CW complex X, the following sequences are exact:*

$$\widetilde{K}^0(X/Y) \to \widetilde{K}^0(X) \to \widetilde{K}^0(Y),$$
$$K^0(X,Y) \to K^0(X) \to K^0(Y).$$

Proof. Since $Y \subset X$ is a cofibration, we have an exact sequence $[X/Y, B] \to [X, B] \to [Y, B]$ for every connected space B; see III(6.3) in [Wh]. This yields the first sequence (B is BO, BU or BSp). The second follows from this and the classical exact sequence $\widetilde{H}^0(X/Y;\mathbb{Z}) \to H^0(X;\mathbb{Z}) \to H^0(Y;\mathbb{Z})$. □

Change of structure field 3.4. If X is any space, the monoid (or semiring) map $\mathbf{VB}_{\mathbb{R}}(X) \to \mathbf{VB}_{\mathbb{C}}(X)$ sending $[E]$ to $[E \otimes \mathbb{C}]$ (see Ex. I.4.5) extends by universality to a ring homomorphism $KO(X) \to KU(X)$. For example, $KO(S^{8n}) \to KU(S^{8n})$ is an isomorphism but $\widetilde{KO}(S^{8n+4}) \cong \mathbb{Z}$ embeds in $\widetilde{KU}(S^{8n+4}) \cong \mathbb{Z}$ as a subgroup of index 2.

Similarly, the forgetful map $\mathbf{VB}_{\mathbb{C}}(X) \to \mathbf{VB}_{\mathbb{R}}(X)$ extends to a group homomorphism $KU(X) \to KO(X)$. As $\dim_{\mathbb{R}}(V) = 2 \cdot \dim_{\mathbb{C}}(V)$, the summand $[X, \mathbb{Z}]$ of $KU(X)$ embeds as $2[X, \mathbb{Z}]$ in the summand $[X, \mathbb{Z}]$ of $KO(X)$. Since $E \otimes \mathbb{C} \cong E \oplus E$ as real vector bundles (by Ex. I.4.5), the composition $KO(X) \to KU(X) \to KO(X)$ is multiplication by 2. The composition in the other direction is more complicated; see Exercise 3.1. For example, it is the zero map on $\widetilde{KU}(S^{8n+4}) \cong \mathbb{Z}$ but is multiplication by 2 on $\widetilde{KU}(S^{8n}) \cong \mathbb{Z}$.

There are analogous maps $KU(X) \to KSp(X)$ and $KSp(X) \to KU(X)$, whose properties we leave to the exercises.

Higher Topological K-theory

Once we have a representable functor such as K^0, standard techniques in infinite loop space theory all us to expand it into a generalized cohomology theory. Rather than get distracted by infinite loop spaces now, we choose to adopt a rather pedestrian approach, ignoring the groups K^n for $n > 0$. For this we use the suspensions S^nX of X, which are all connected paracompact spaces.

Definition 3.5. For each integer $n > 0$, we define $\widetilde{KO}^{-n}(X)$ and $KO^{-n}(X)$ by:

$$\widetilde{KO}^{-n}(X) = \widetilde{KO}^0(S^nX) = [S^nX, BO]; \quad KO^{-n}(X) = \widetilde{KO}^{-n}(X) \oplus \widetilde{KO}(S^n).$$

Replacing 'O' by 'U' yields definitions $\widetilde{KU}^{-n}(X) = \widetilde{KU}^0(S^nX) = [S^nX, BU]$ and $KU^{-n}(X) = \widetilde{KU}^{-n}(X) \oplus \widetilde{KU}(S^n)$; replacing '$O$' by '$Sp$' yields definitions for $\widetilde{KSp}^{-n}(X)$ and $KSp^{-n}(X)$. When the choice of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ is clear, we shall drop the 'O,' 'U' and 'Sp,', writing simply $\widetilde{K}^{-n}(X)$ and $K^{-n}(X)$.

We shall also define relative groups as follows. If Y is a subcomplex of X, and $n > 0$, we set $K^{-n}(X, Y) = \widetilde{K}^{-n}(X/Y)$.

Based maps 3.5.1. Note that our definitions do not assume X to have a basepoint. If X has a nondegenerate basepoint and Y is an H-space with homotopy inverse (such as BO, BU or BSp), then the group $[X, Y]$ is isomorphic to the group $\pi_0(Y) \times [X, Y]_*$, where the second term denotes homotopy classes of *based* maps from X to Y; see pp. 100 and 119 of [Wh]. For such spaces X we can interpret the formulas for $KO^{-n}(X)$, $KU^{-n}(X)$ and $KSp^{-n}(X)$ in terms of based maps, as is done in [Atiyah, p.68].

If X_* denotes the disjoint union of X and a basepoint $*$, then we have the usual formula for an unreduced cohomology theory: $K^{-n}(X) = \widetilde{K}(S^n(X_*))$. This easily leads (see Ex. 3.11) to the formulas for $n \geq 1$:

$$KO^{-n}(X) \cong [X, \Omega^n BO],\ KU^{-n}(X) \cong [X, \Omega^n BU] \text{ and } KSp^{-n}(X) \cong [X, \Omega^n BSp].$$

Theorem 3.6. *If Y is a subcomplex of a CW complex X, we have the exact sequences (infinite to the left):*

$$\cdots \to \widetilde{K}^{-2}(Y) \to \widetilde{K}^{-1}(X/Y) \to \widetilde{K}^{-1}(X) \to \widetilde{K}^{-1}(Y) \to \widetilde{K}^0(X/Y) \to \widetilde{K}^0(X) \to \widetilde{K}^0(Y),$$

$$\cdots \to K^{-2}(Y) \to K^{-1}(X, Y) \to K^{-1}(X) \to K^{-1}(Y) \to K^0(X, Y) \to K^0(X) \to K^0(Y).$$

Proof. Exactness at $K^0(X)$ was proven in Proposition 3.3. The mapping cone $\mathrm{cone}(i)$ of $i: Y \subset X$ is homotopy equivalent to X/Y, and $j: X \subset \mathrm{cone}(i)$ induces $\mathrm{cone}(i)/X \simeq SY$. This gives exactness at $K^0(X, Y)$. Similarly, $\mathrm{cone}(j) \simeq SY$ and $\mathrm{cone}(j)/\mathrm{cone}(i) \simeq SX$, giving exactness at $K^{-1}(Y)$. The long exact sequences follows by replacing $Y \subset X$ by $SY \subset SX$. □

Characteristic Classes 3.7. The total Stiefel-Whitney class $w(E)$ of a real vector bundle E was defined in chapter I, §4. By (SW3) it satisfies the product formula: $w(E \oplus F) = w(E)w(F)$. Therefore if we interpret $w(E)$ as an element of the abelian group U of all formal sums $1+a_1+\cdots$ in $\hat{H}^*(X;\mathbb{Z}/2)$ we get a get a group homomorphism $w\colon KO(X) \to U$. It follows that each Stiefel-Whitney class induces a well-defined set map $w_i\colon KO(X) \to H^i(X;\mathbb{Z}/2)$. In fact, since w vanishes on each componentwise trivial bundle T^f it follows that $w([E]-[T^f]) = w(E)$. Hence each Stiefel-Whitney class w_i factors through the projection $KO(X) \to \widetilde{KO}(X)$.

Similarly, the total Chern class $c(E) = 1 + c_1(E) + \cdots$ satisfies $c(E \oplus F) = c(E)c(F)$, so we may think of it as a group homomorphism from $KU(X)$ to the abelian group U of all formal sums $1 + a_2 + a_4 + \cdots$ in $\hat{H}^*(X;\mathbb{Z})$. It follows that the Chern classes $c_i(E) \in H^{2i}(X;\mathbb{Z})$ of a complex vector bundle define set maps $c_i\colon KU(X) \longrightarrow H^{2i}(X;\mathbb{Z})$. Again, since c vanishes on componentwise trivial bundles, each Chern class c_i factors through the projection $KU(X) \to \widetilde{KU}(X)$.

Example 3.7.1. For even spheres the Chern class $c_n\colon \widetilde{KU}(S^{2n}) \to H^{2n}(S^n;\mathbb{Z})$ is an isomorphism. We will return to this point in Ex. 3.6 and in §4.

Example 3.7.2. The map $\varinjlim[\mathbb{R}\mathbb{P}^\infty, BO_n] \to \widetilde{KO}(\mathbb{R}\mathbb{P}^\infty)$ of (3.1.0) cannot be onto. To see this, consider the element $\eta = 1 - [E_1]$ of $\widetilde{KO}(\mathbb{R}\mathbb{P}^\infty)$, where E_1 is the canonical line bundle. Since $w(-\eta) = w(E_1) = 1 + x$ we have $w(\eta) = (1+x)^{-1} = \sum_{i=0}^\infty x^i$, and $w_i(\eta) \neq 0$ for every $i \geq 0$. Axiom (SW1) implies that η cannot equal $[F] - \dim(F)$ for any bundle F.

Similarly, $\varinjlim[\mathbb{C}\mathbb{P}^\infty, BU_n] \to \widetilde{KU}(\mathbb{C}\mathbb{P}^\infty)$ cannot be onto; the argument is similar, again using the canonical line bundle: $c_i(1 - [E_1]) \neq 0$ for every $i \geq 0$.

EXERCISES

3.1 Let X be a topological space. Show that there is an involution of $\mathbf{VB}_{\mathbb{C}}(X)$ sending $[E]$ to the complex conjugate bundle $[\bar{E}]$ of Ex. I.4.6. The corresponding involution c on $KU(X)$ can be nontrivial; use I(4.9.2) to show that c is multiplication by -1 on $\widetilde{KU}(S^2) \cong \mathbb{Z}$. (By Bott periodicity, this implies that c is multiplication by $(-1)^k$ on $\widetilde{KU}(S^2) \cong \mathbb{Z}$.) Finally, show that the composite $KU(X) \to KO(X) \to KU(X)$ is the map $1 + c$ sending $[E]$ to $[E] + [\bar{E}]$.

3.2 If $\amalg X_i$ is the disjoint union of spaces X_i, show that $K(\amalg X_i) \cong \prod K(X_i)$. Then construct a space X such that the map $\varinjlim \mathbf{VB}_n(X) \to \widetilde{K}(X)$ of (3.1.0) is not onto.

3.3 *External products.* Show that there is a bilinear map $K(X_1) \otimes K(X_2) \to K(X_1 \times X_2)$ for every X_1 and X_2, sending $[E_1] \otimes [E_2]$ to $[\pi_1^*(E_1) \otimes \pi_2^*(E_2)]$, where $pi_i\colon X_1 \times X_2 \to X_i$ is the projection. Then show that if $X_1 = X_2 = X$ the composition with the diagonal map $\Delta^*\colon K(X \times X) \to K(X)$ yields the usual product in the ring $K(X)$, sending $[E_1] \otimes [E_2]$ to $[E_1 \otimes E_2]$.

3.4 Recall that the smash product $X \wedge Y$ of two based spaces is the quotient $X \times Y/X \vee Y$, where $X \vee Y$ is the union of $X \times \{*\}$ and $\{*\} \times Y$. Show that

$$\widetilde{K}^{-n}(X \times Y) \cong \widetilde{K}^{-n}(X \wedge Y) \oplus \widetilde{K}^{-n}(X) \oplus \widetilde{K}^{-n}(Y).$$

3.5 Show that $KU^{-2}(*) \otimes KU^{-n}(X) \to KU^{-n-2}(X)$ induces a "periodicity" isomorphism $\beta\colon KU^{-n}(X) \xrightarrow{\sim} KU^{-n-2}(X)$ for all n. *Hint:* $S^2 \wedge S^n X \simeq S^{n+2}X$.

3.6 Let X be a finite CW complex with only even-dimensional cells, such as $\mathbb{CP}^n$. Show that $KU(X)$ is a free abelian group on the set of cells of X, and that $KU(SX) = \mathbb{Z}$, so that $KU^{-1}(X) = 0$. Then use Example 3.7.1 to show that the total Chern class injects the group $\widetilde{KU}(X)$ into $\prod H^{2i}(X;\mathbb{Z})$. *Hint:* Use induction of $\dim(X)$ and the fact that X_{2n}/X_{2n-2} is a bouquet of $2n$-spheres.

3.7 *Chern character for* $\mathbb{CP}^n$*.* Let E_1 be the canonical line bundle on $\mathbb{CP}^n$, and let x denote the class $[E_1]-1$ in $KU(\mathbb{CP}^n)$. Use Chern classes and the previous exercise to show that $\{1, [E_1], [E_1 \otimes E_1], \dots, [E_1^{\otimes n}]\}$, and hence $\{1, x, x^2, \dots, x^n\}$, forms a basis of the free abelian group $KU(\mathbb{CP}^n)$. Then show that $x^{n+1} = 0$, so that the ring $KU(\mathbb{CP}^n)$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$. We will see in Ex. 4.11 below that the Chern character ch maps the ring $KU(\mathbb{CP}^n)$ isomorphically onto the subring $\mathbb{Z}[t]/(t^{n+1})$ of $H^*(\mathbb{CP}^n;\mathbb{Q})$ generated by $t = e^{c_1(x)} - 1$.

3.8 Consider the suspension $X = SY$ of a paracompact space Y. Use Ex. I.4.16 to show that $\varinjlim [X, BO_n] \cong \widetilde{KO}(X)$.

3.9 If X is a finite CW complex, show by induction on the number of cells that both $KO(X)$ and $KU(X)$ are finitely generated abelian groups.

3.10 Show that $KU(\mathbb{RP}^{2n}) = KU(\mathbb{RP}^{2n+1}) = \mathbb{Z} \oplus \mathbb{Z}/2^n$. *Hint:* Try the total Stiefel-Whitney class, using 3.3.

3.11 Let X be a compact space with a nondegenerate basepoint. Show that $KO^{-n}(X) \cong [X, \Omega^n BO] \cong [X, \Omega^{n-1}O]$ and $KU^{-n}(X) \cong [X, \Omega^n BU] \cong [X, \Omega^{n-1}U]$ for all $n \geq 1$. In particular, $KU^{-1}(X) \cong [X, U]$ and $KO^{-1}(X) \cong [X, O]$.

3.12 Let X be a compact space with a nondegenerate basepoint. Show that the homotopy groups of the topological groups $GL(\mathbb{R}^X) = \mathrm{Hom}(X, GL(\mathbb{R}))$ and $GL(\mathbb{C}^X) = \mathrm{Hom}(X, GL(\mathbb{C}))$ are (for $n > 0$):

$$\pi_{n-1}GL(\mathbb{R}^X) = KO^{-n}(X) \quad \text{and} \quad \pi_{n-1}GL(\mathbb{C}^X) = KU^{-n}(X).$$

3.13 If $E \to X$ is a complex bundle, there is a quaternionic vector bundle $E_{\mathbb{H}} \to X$ with fibers $E_x \otimes_{\mathbb{C}} \mathbb{H}$, as in Ex. I.4.5; this induces the map $KU(X) \to KSp(X)$ mentioned in 3.4. Show that $E_{\mathbb{H}} \to X$, considered as a complex vector bundle, is isomorphic to the Whitney sum $E \oplus E$. Deduce that the composition $KU(X) \to KSp(X) \to KU(X)$ is multiplication by 2.

3.14 Show that $\mathbb{H} \otimes_{\mathbb{C}} \mathbb{H}$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ as an $\mathbb{H}$-bimodule, on generators $1 \otimes 1 \pm j \otimes j$. This induces a natural isomorphism $V \otimes_{\mathbb{C}} \mathbb{H} \cong V \oplus V$ of vector spaces over $\mathbb{H}$. If $E \to X$ is a quaternionic vector bundle, with underlying complex bundle $uE \to X$, show that there is a natural isomorphism $(uE)_{\mathbb{H}} \cong E \oplus E$. Conclude that the composition $KSp(X) \to KU(X) \to KSp(X)$ is multiplication by 2.

3.15 Let $\bar{E}$ be the complex conjugate bundle of a complex vector bundle $E \to X$; see Ex. I.4.6. Show that $\bar{E}_{\mathbb{H}} \cong E_{\mathbb{H}}$ as quaternionic vector bundles. This shows that $KU(X) \to KSp(X)$ commutes with the involution c of Ex. 3.1.

Using exercises 3.1 and 3.14, show that the composition $KSp(X) \to KO(X) \to KSp(X)$ is multiplication by 4.

§4. Lambda and Adams Operations

A commutative ring K is called a *λ-ring* if we are given a family of set operations $\lambda^k\colon K \to K$ for $k \geq 0$ such that for all $x, y \in K$:

- $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all $x \in K$;
- $\lambda^k(x+y) = \sum_{i=0}^k \lambda^i(x)\lambda^{k-i}(y) = \lambda^k(x) + \lambda^{k-1}(x)\lambda^1 y + \cdots + \lambda^k(y)$.

This last condition is equivalent to the assertion that there is a group homomorphism λ_t from the additive group of K to the multiplicative group $W(K) = 1 + tK[[t]]$ given by the formula $\lambda_t(x) = \sum \lambda^k(x)t^k$. (*Warning:* Our notation of λ-ring follows Atiyah; Grothendieck and other authors call this a *pre-λ-ring*, reserving the term λ-ring for what we call a *special λ-ring*; see Definition 4.3.1 below.)

Example 4.1.1 (Binomial Rings). . The integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$ are λ-rings with $\lambda^k(n) = \binom{n}{k}$. If K is any $\mathbb{Q}$-algebra, we define $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ for $x \in K$ and $k \geq 1$; again the formula $\lambda^k(x) = \binom{x}{k}$ makes K into a λ-ring.

More generally, a *binomial ring* is a subring K of a $\mathbb{Q}$-algebra $K_{\mathbb{Q}}$ such that for all $x \in K$ and $k \geq 1$, $\binom{x}{k} \in K$. We make a binomial ring into a λ-ring by setting $\lambda^k(x) = \binom{x}{k}$. If K is a binomial ring then formally λ_t is given by the formula $\lambda_t(x) = (1+t)^x$. For example, if X is a topological space, then the ring $[X, \mathbb{Z}]$ is a λ-ring with $\lambda^k(f) = \binom{f}{k}$, the function sending x to $\binom{f(x)}{k}$.

The notion of λ-semiring is very useful in constructing λ-rings. Let M be a semiring (see §1); we know that the group completion $M^{-1}M$ of M is a ring. We call M a *λ–semiring* if it is equipped with operations $\lambda^k\colon M \to M$ such that $\lambda^0(x) = 1$, $\lambda^1(x) = x$ and $\lambda^k(x+y) = \sum \lambda^i(x)\lambda^{k-i}(y)$.

If M is a λ-semiring then the group completion $K = M^{-1}M$ is a λ-ring. To see this, note that sending $x \in M$ to the power series $\sum \lambda^k(x)t^k$ defines a monoid map $\lambda_t\colon M \to 1 + tK[[t]]$. By universality of K, this extends to a group homomorphism λ_t from K to $1 + tK[[t]]$, and the coefficients of $\lambda_t(x)$ define the operations $\lambda^k(x)$.

Example 4.1.2 (Algebraic K_0). Let R be a commutative ring and set $K = K_0(R)$. If P is a finitely generated projective R-module, consider the formula $\lambda^k(P) = [\wedge^k P]$. The decomposition $\wedge^k(P \oplus Q) \cong \sum (\wedge^i P) \otimes (\wedge^{k-i} Q)$ given in ch.I, §3 shows that $\mathbf{P}(R)$ is a λ-semiring. Hence $K_0(R)$ is a λ-ring.

Since $\operatorname{rank}(\wedge^k P) = \binom{\operatorname{rank} P}{k}$, it follows that the map rank: $K_0(R) \to H_0(R)$ of 2.3 is a morphism of λ-rings, and hence that $\widetilde{K}_0(R)$ is a λ-ideal of $K_0(R)$.

Example 4.1.3 (Topological K^0). Let X be a topological space and let K be either $KO(X)$ or $KU(X)$. If $E \to X$ is a vector bundle, let $\lambda^k(E)$ be the exterior power bundle $\wedge^k E$ of Ex. I.4.3. The decomposition of $\wedge^k(E \oplus F)$ given in Ex. I.4.3 shows that the monoid $\mathbf{VB}(X)$ is a λ-semiring. Hence $KO(X)$ and $KU(X)$ are λ-rings, and $KO(X) \to KU(X)$ is a morphism of λ-rings.

Since $\dim(\wedge^k E) = \binom{\dim E}{k}$, it follows that $KO(X) \to [X, \mathbb{Z}]$ and $KU(X) \to [X, \mathbb{Z}]$ are λ-ring morphisms, and that $\widetilde{KO}(X)$ and $\widetilde{KU}(X)$ are λ-ideals.

Example 4.1.4 (Representation Ring). Let G be a finite group, and consider the complex representation ring $R(G)$, constructed in Example 1.6 as the group completion of $Rep_{\mathbb{C}}(G)$, the semiring of finite dimensional representations of G; as an abelian group $R(G) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of elements in G. The exterior powers $\Lambda^i(V)$ of a representation V are also G-modules, and the decomposition of $\Lambda^k(V \oplus W)$ as complex vector spaces used in (4.1.2) shows that $Rep_{\mathbb{C}}(G)$ is a λ-semiring. Hence $R(G)$ is a λ-ring. (It is true, but harder to show, that $R(G)$ is a special λ-ring; see Ex4.2.)

If $d = \dim_{\mathbb{C}}(V)$ then $\dim_{\mathbb{C}}(\Lambda^k V) = \binom{d}{k}$, so $\dim_{\mathbb{C}}$ is a λ-ring map from $R(G)$ to $\mathbb{Z}$. The kernel $\widetilde{R}(G)$ of this map is a λ-ideal of $R(G)$.

Example 4.1.5. Let X be a scheme, or more generally a locally ringed space (Ch. I, §5). We will define a ring $K_0(X)$ in §7 below, using the category $\mathbf{VB}(X)$. As an abelian group it is generated by the classes of vector bundles on X. We will see in Proposition 8.8 that the operations $\lambda^k[\mathcal{E}] = [\wedge^k \mathcal{E}]$ are well-defined on $K_0(X)$ and make it into a λ-ring. (The formula for $\lambda^k(x+y)$ will follow from Ex. I.5.4.)

Positive structures

Not every λ-ring is well-behaved. In order to avoid pathologies, we introduce a further condition, satisfied by the above examples: the λ-ring K must have a positive structure and satisfy the Splitting Principle.

Definition 4.2.1. By a *positive structure* on a λ-ring K we mean: 1) a λ-subring H^0 of K which is a binomial ring; 2) a λ-ring surjection $\varepsilon\colon K \to H^0$ which is the identity on H^0 (ε is called the *augmentation*); and 3) a subset $P \subset K$ (the *positive* elements), such that

(1) $\mathbb{N} = \{0, 1, 2, \cdots\}$ is contained in P.
(2) P is a λ-sub-semiring of K. That is, P is closed under addition, multiplication, and the operations λ^k.
(3) Every element of the kernel $\widetilde{K}$ of ε can be written as $p-q$ for some p, $q \in P$.
(4) If $p \in P$ then $\varepsilon(p) = n \in \mathbb{N}$. Moreover, $\lambda^i(p) = 0$ for $i > n$ and $\lambda^n(p)$ is a unit of K.

Condition (2) states that the group completion $P^{-1}P$ of P is a λ-subring of K; by (3) we have $P^{-1}P = \mathbb{Z} \oplus \widetilde{K}$. By (4), $\varepsilon(p) > 0$ for $p \neq 0$, so $P \cap (-P) = 0$; therefore $P^{-1}P$ is a partially ordered abelian group in the sense of §1. An element $\ell \in P$ with $\varepsilon(\ell) = 1$ is called a *line element*; by (4), $\lambda^1(\ell) = \ell$ and ℓ is a unit of K. That is, the line elements form a subgroup L of the units of K.

The λ-rings in examples (4.1.2)–(4.1.5) all have positive structures. The λ-ring $K_0(R)$ has a positive structure with

$$H^0 = H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}] \quad \text{and} \quad P = \{[P] : \mathrm{rank}(P) \text{ is constant}\};$$

the line elements are the classes of line bundles, so $L = \mathrm{Pic}(R)$. Similarly, the λ-rings $KO(X)$ and $KU(X)$ have a positive structure in which H^0 is $H^0(X, \mathbb{Z}) = [X, \mathbb{Z}]$ and P is $\{[E] : \dim(E) \text{ is constant}\}$, as long as we restrict to compact spaces or spaces with $\pi_0(X)$ finite, so that (I.4.1.1) applies. Again, line elements are the classes of line bundles; for $KO(X)$ and $KU(X)$ we have $L = H^1(X; \mathbb{Z}/2)$ and $L = H^2(X; \mathbb{Z})$, respectively. For $R(G)$, the classes $[V]$ of representations V are the positive elements; H^0 is $\mathbb{Z}$, and L is the set of 1-dimensional representations of G. Finally, if X is a scheme (or locally ringed space) then in the positive structure on $K_0(X)$ we have $H^0 = H^0(X; \mathbb{Z})$ and P is $\{[\mathcal{E}] : \mathrm{rank}(\mathcal{E}) \text{ is constant}\}$; see I.5.1. The line bundles are again the line elements, so $L = \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^{\times})$ by I.5.10.1.

There is a natural group homomorphism "det" from K to L, which vanishes on H^0. If $p \in P$ we define $\det(p) = \lambda^n(p)$, where $\varepsilon(p) = n$. The formula for $\lambda^n(p+q)$ and the vanishing of $\lambda^i(p)$ for $i > \varepsilon(p)$ imply that $\det\colon P \to L$ is a monoid map, *i.e.,* that $\det(p+q) = \det(p)\det(q)$. Thus det extends to a map from $P^{-1}P$ to L. As $\det(n) = \binom{n}{n} = 1$ for every $n \geq 0$, $\det(\mathbb{Z}) = 1$. By (iii), defining $\det(H^0) = 1$ extends det to a map from K to L. When K is $K_0(R)$ the map det was introduced in §2. For $KO(X)$, det is the first Stiefel-Whitney class; for $KU(X)$, det is the first Chern class.

Having described what we mean by a positive structure on K, we can now state the Splitting Principle.

Definition 4.2.2. The *Splitting Principle* states that for every positive element p in K there is a extension $K \subset K'$ (of λ-rings with positive structure) such that p is a sum of line elements in K'.

The Splitting Principle for $KO(X)$ and $KU(X)$ holds by Ex. 4.12. Using algebraic geometry, we will show in 8.8 that the Splitting Principle holds for $K_0(R)$ as well as K_0 of a scheme. The Splitting Principle also holds for $R(G)$; see [AT, 1.5]. The importance of the Splitting Principle lies in its relation to "special λ-rings," a notion we shall define after citing the following motivational result from [FL, ch.I].

Theorem 4.2.3. *If K is a λ-ring with a positive structure, and* $\mathbb{N}$ *is cofinal in P, the Splitting Principle holds if and only if K is a special λ-ring.*

In order to define special λ-ring, we need the following technical example:

Example 4.3 (Witt Vectors). For every commutative ring R, the abelian group $W(R) = 1 + tR[[t]]$ has the structure of a commutative ring, natural in R; $W(R)$ is called the ring of (big) *Witt vectors* of R. The multiplicative identity of the ring $W(R)$ is $(1-t)$, and multiplication $*$ is completely determined by naturality, formal factorization of elements of $W(R)$ as $f(t) = \prod_{i=1}^{\infty}(1 - r_i t^i)$ and the formula:

$$(1 - rt) * f(t) = f(rt).$$

It is not hard to see that there are "universal" polynomials P_n in $2n$ variables so:

$$(\sum a_i t^i) * (\sum b_j t^j) = \sum c_n t^n, \text{ with } \ c_n = P_n(a_1, \dots, a_n; b_1, \dots, b_n).$$

If $\mathbb{Q} \subseteq R$ there is an isomorphism $\prod_{n=1}^{\infty} R \to W(R)$, $(r_1, \dots) \mapsto \prod \exp(1 - r_n t^n/n)$.

Grothendieck observed that there are operations λ^k on $W(R)$ making it into a λ-ring; they are defined by naturality, formal factorization and the formula

$$\lambda^k(1 - rt) = 0 \text{ for all } \ k \geq 2.$$

Another way to put it is that there are universal polynomials $P_{n,k}$ such that:

$$\lambda^k(\sum a_i t^i) = \sum b_n t^n, \text{ with } \ b_n = P_{n,k}(a_1, ..., a_{nk}).$$

DEFINITION 4.3.1. A *special λ-ring* is a λ-ring K such that the group homomorphism $\lambda_t : K \to W(K)$ is a λ-ring homomorphism. Since $\lambda_t(x) = \sum \lambda^k(x)t^k$, a special λ-ring is a λ-ring K such that

- $\lambda^k(1) = 0$ for $k \neq 0, 1$
- $\lambda^k(xy)$ is $P_k(\lambda^1(x), ..., \lambda^k(x); \lambda^1(y), ..., \lambda^k(y))$, and
- $\lambda^n(\lambda^k(x)) = P_{n,k}(\lambda^1(x), ..., \lambda^{nk}(x))$.

EXAMPLE 4.3.2. The formula $\lambda^n(s_1) = s_n$ defines a special λ-ring structure on the polynomial ring $U = \mathbb{Z}[s_1, ..., s_n, ...]$; see [AT, §2]. It is the free special λ-ring on the generator s_1, because if x is any element in any special λ-ring K then the map $U \to K$ sending s_n to $\lambda^n(x)$ is a λ-ring homomorphism. The λ-ring U cannot have a positive structure by Theorem 4.6 below, since U has no nilpotent elements except 0.

Adams operations

For every augmented λ-ring K we can define the *Adams operations* $\psi^k\colon K \to K$ for $k \geq 0$ by setting $\psi^0(x) = \varepsilon(x)$, $\psi^1(x) = x$, $\psi^2(x) = x^2 - 2\lambda^2(x)$ and inductively

$$\psi^k(x) = \lambda^1(x)\psi^{k-1}(x) - \lambda^2(x)\psi^{k-2}(x) + \cdots + (-1)^k\lambda^{k-1}(x)\psi^1(x) + (-1)^{k-1}k\lambda^k(x).$$

From this inductive definition we immediately deduce three facts:

- if ℓ is a line element then $\psi^k(\ell) = \ell^k$;
- if I is a λ-ideal with $I^2 = 0$ then $\psi^k(x) = (-1)^{k-1}k\lambda^k(x)$ for all $x \in I$;
- For every binomial ring H we have $\psi^k = 1$. Indeed, the formal identity $x\sum_{i=0}^{k-1}(-1)^i\binom{x}{i} = (-1)^{k+1}k\binom{x}{k}$ shows that $\psi^k(x) = x$ for all $x \in H$.

The operations ψ^k are named after J.F. Adams, who first introduced them in 1962 in his study of vector fields on spheres.

Here is a slicker, more formal presentation of the Adams operations. Define $\psi^k(x)$ to be the coefficient of t^k in the power series:

$$\psi_t(x) = \sum \psi^k(x)t^k = \varepsilon(x) - t\frac{d}{dt}\log\lambda_{-t}(x).$$

The proof that this agrees with the inductive definition of $\psi^k(x)$ is an exercise in formal algebra, which we relegate to Exercise 4.6 below.

PROPOSITION 4.4. *Assume K satisfies the Splitting Principle. Each ψ^k is a ring endomorphism of K, and $\psi^j\psi^k = \psi^{jk}$ for all $j, k \geq 0$.*

PROOF. The logarithm in the definition of ψ_t implies that $\psi_t(x+y) = \psi_t(x) + \psi_t(y)$, so each ψ^k is additive. The Splitting Principle and the formula $\psi^k(\ell) = \ell^k$ for line elements yield the formulas $\psi^k(pq) = \psi^k(p)\psi^k(q)$ and $\psi^j(\psi^k(p)) = \psi^{jk}(p)$ for positive p. The extension of these formulas to K is clear. □

EXAMPLE 4.4.1. Consider the λ-ring $KU(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$ of 3.1.1. On $H^0 = \mathbb{Z}$, $\psi^k = 1$, but on $\widetilde{KU}(S^n) \cong \mathbb{Z}$, ψ^k is multiplication by $k^{n/2}$. (See [Atiyah, 3.2.2].)

EXAMPLE 4.4.2. Consider $KU(\mathbb{R}\mathbb{P}^{2n})$, which by Ex. 3.10 is $\mathbb{Z}\oplus\mathbb{Z}/2^n$. I claim that for all $x\in\widetilde{KU}(X)$:

$$\psi^k(x)=\begin{cases} x \text{ if } k \text{ is odd}\\ 0 \text{ if } k \text{ is even.}\end{cases}$$

To see this, note that $\widetilde{KU}(\mathbb{R}\mathbb{P}^{2n})\cong\mathbb{Z}/2^n$ is additively generated by $(\ell-1)$, where ℓ is the nonzero element of $L=H^2(\mathbb{R}\mathbb{P}^{2n};\mathbb{Z})=\mathbb{Z}/2$. Since $\ell^2=1$, we see that $\psi^k(\ell-1)=(\ell^k-1)$ is 0 if k is even and $(\ell-1)$ if k is odd. The assertion follows.

γ-operations

Associated to the operations λ^k are the operations $\gamma^k\colon K\to K$. To construct them, we assume that $\lambda^k(1)=0$ for $k\neq 0,1$. Note that if we set $s=t/(1-t)$ then $K[[t]]=K[[s]]$ and $t=s/(1+s)$. Therefore we can rewrite $\lambda_s(x)=\sum\lambda^i(x)s^i$ as a power series $\gamma_t(x)=\sum\gamma^k(x)t^k$ in t. By definition, $\gamma^k(x)$ is the coefficient of t^k in $\gamma_t(x)$. Since $\gamma_t(x)=\lambda_s(x)$ we have $\gamma_t(x+y)=\gamma_t(x)\gamma_t(y)$. In particular $\gamma^0(x)=1$, $\gamma^1(x)=x$ and $\gamma^k(x+y)=\sum\gamma^i(x)\gamma^{k-i}(y)$. That is, the γ-operations satisfy the axioms for a λ-ring structure on K. An elementary calculation, left to the reader, yields the useful identity:

Formula 4.5. $\gamma^k(x)=\lambda^k(x+k-1)$. This implies that $\gamma^2(x)=\lambda^2(x)+x$ and

$$\gamma^k(x)=\lambda^k(x+k-1)=\lambda^k(x)+\binom{k-1}{1}\lambda^{k-1}(x)+\cdots+\binom{k-1}{k-2}\lambda^2(x)+x.$$

EXAMPLE 4.5.1. If H is a binomial ring then for all $x\in H$ we have

$$\gamma^k(x)=\binom{x+k-1}{k}=(-1)^k\binom{-x}{k}.$$

EXAMPLE 4.5.2. $\gamma^k(1)=1$ for all k, because $\lambda_s(1)=1+s=1/(1-t)$. More generally, if ℓ is a line element then $\gamma^k(\ell)=\ell$ for all $k\geq 1$.

LEMMA 4.5.3. *If $p\in P$ is a positive element with $\varepsilon(p)=n$, then $\gamma^k(p-n)=0$ for all $k>n$. In particular, if $\ell\in K$ is a line element then $\gamma^k(\ell-1)=0$ for every $k>1$.*

PROOF. If $k>n$ then $q=p+(k-n-1)$ is a positive element with $\varepsilon(q)=k-1$. Thus $\gamma^k(p-n)=\lambda^k(q)=0$. □

If $x\in K$, the *γ-dimension* $\dim_\gamma(x)$ of x is defined to be the largest integer n for which $\gamma^n(x-\varepsilon(x))\neq 0$, provided n exists. For example, $\dim_\gamma(h)=0$ for every $h\in H^0$ and $\dim_\gamma(\ell)=1$ for every line element ℓ (except $\ell=1$ of course). By the above remarks if $p\in P$ and $n=\varepsilon(p)$ then $\dim_\gamma(p)=\dim_\gamma(p-n)\leq n$. The supremum of the $\dim_\gamma(x)$ for $x\in K$ is called the *γ-dimension* of K.

EXAMPLES 4.5.4. If R is a commutative noetherian ring, the Serre Cancellation I.2.4 states that every element of $\widetilde{K}_0(R)$ is represented by $[P]-n$, where P is a finitely generated projective module of rank $<\dim(R)$. Hence $K_0(R)$ has γ-dimension at most $\dim(R)$.

Suppose that X is a CW complex with finite dimension d. The Real Cancellation Theorem I.4.3 allows us to use the same argument to deduce that $KO(X)$ has γ-dimension at most d; the Complex Cancellation Theorem I.4.4 shows that $KU(X)$ has γ-dimension at most $d/2$.

COROLLARY 4.5.5. *If K has a positive structure in which $\mathbb{N}$ is cofinal in P, then every element of $\widetilde{K}$ has finite γ-dimension.*

PROOF. Recall that "$\mathbb{N}$ is cofinal in P" means that for every p there is a p' so that $p+p'=n$ for some $n\in\mathbb{N}$. Therefore every $x\in\widetilde{K}$ can be written as $x=p-m$ for some $p\in P$ with $m=\varepsilon(p)$. By Lemma 4.5.3, $\dim_\gamma(x)\le m$. $\square$

THEOREM 4.6. *If every element of K has finite γ-dimension (e.g., K has a positive structure in which $\mathbb{N}$ is cofinal in P), then $\widetilde{K}$ is a nil ideal. That is, every element of $\widetilde{K}$ is nilpotent.*

PROOF. Fix $x\in\widetilde{K}$, and set $m=\dim_\gamma(x)$, $n=\dim_\gamma(-x)$. Then both $\gamma_t(x)=1+xt+\gamma^2(x)t^2+\cdots+\gamma^m(x)t^m$ and $\gamma_t(-x)=1-xt+\cdots+\gamma^n(-x)t^n$ are polynomials in t. Since $\gamma_t(x)\gamma_t(-x)=\gamma_t(0)=1$, the polynomials $\gamma_t(x)$ and $\gamma_t(-x)$ are units in the polynomial ring $K[t]$. By (I.3.12), the coefficients of these polynomials are nilpotent elements of K. $\square$

COROLLARY 4.6.1. *The ideal $\widetilde{K}_0(R)$ is the nilradical of $K_0(R)$ for every commutative ring R.*

If X is compact (or connected and paracompact) then $\widetilde{KO}(X)$ and $\widetilde{KU}(X)$ are the nilradicals of the rings $KO(X)$ and $KU(X)$, respectively.

EXAMPLE 4.6.2. The conclusion of Theorem 4.6 fails for the representation ring $R(G)$ of a cyclic group of order 2. If σ denotes the 1-dimensional sign representation, then $L=\{1,\sigma\}$ and $\widetilde{R}(G)\cong\mathbb{Z}$ is generated by $(\sigma-1)$. Since $(\sigma-1)^2=(\sigma^2-2\sigma+1)=(-2)(\sigma-1)$, we see that $(\sigma-1)$ is not nilpotent, and in fact that $\widetilde{R}(G)^n=(2^{n-1})\widetilde{R}(G)$ for every $n\ge 1$. The hypothesis of Corollary 4.5.5 fails here because σ cannot be a summand of a trivial representation. In fact $\dim_\gamma(1-\sigma)=\infty$, because $\gamma^n(1-\sigma)=(1-\sigma)^n=2^{n-1}(1-\sigma)$ for all $n\ge 1$.

The γ-Filtration

The *γ-filtration* on K is a descending sequence of ideals:

$$K=F^0_\gamma K\supset F^1_\gamma K\supset\cdots\supset F^n_\gamma K\supset\cdots.$$

It starts with $F^0_\gamma K=K$ and $F^1_\gamma K=\widetilde{K}$ (the kernel of ε). The first quotient F^0_γ/F^1_γ is clearly $H^0=K/\widetilde{K}$. For $n\ge 2$, $F^n_\gamma K$ is defined to be the ideal of K generated by the products $\gamma^{k_1}(x_1)\cdots\gamma^{k_m}(x_m)$ with $x_i\in\widetilde{K}$ and $\sum k_i\ge n$. In particular, $F^n_\gamma K$ contains $\gamma^k(x)$ for all $x\in\widetilde{K}$ and $k\ge n$.

It follows immediately from the definition that $F^i_\gamma F^j_\gamma\subseteq F^{i+j}_\gamma$. For $j=1$, this implies that the quotients $F^i_\gamma K/F^{i+1}_\gamma K$ are H^0-modules. We will prove that the quotient F^1_γ/F^2_γ is the group L of line elements in K:

THEOREM 4.7. *If K satisfies the Splitting Principle, then the map $\ell\mapsto\ell-1$ induces a group isomorphism, split by the map* det*:*

$$L\xrightarrow{\cong}F^1_\gamma K/F^2_\gamma K.$$

COROLLARY 4.7.1. *For every commutative ring R, the first two ideals in the γ-filtration of $K_0(R)$ are $F^1_\gamma = \widetilde{K}_0(R)$ and $F^2_\gamma = SK_0(R)$. (See 2.6.2.) In particular,*

$$F^0_\gamma/F^1_\gamma \cong H_0(R) \text{ and } \quad F^1_\gamma/F^2_\gamma \cong \mathrm{Pic}(R).$$

COROLLARY 4.7.2. *The first two quotients in the γ-filtration of $KO(X)$ are*

$$F^0_\gamma/F^1_\gamma \cong [X,\mathbb{Z}] \quad \text{ and } \quad F^1_\gamma/F^2_\gamma \cong H^1(X;\mathbb{Z}/2).$$

The first few quotients in the γ-filtration of $KU(X)$ are

$$F^0_\gamma/F^1_\gamma \cong [X,\mathbb{Z}] \text{ and } \quad F^1_\gamma/F^2_\gamma \cong H^2(X;\mathbb{Z}).$$

For the proof of Theorem 4.7, we shall need the following consequence of the Splitting Principle. A proof of this principle may be found in [FL, III.1].

FILTERED SPLITTING PRINCIPLE. *Let K be a λ-ring satisfying the Splitting Principle, and let x be an element of $F^n_\gamma K$. Then there exists a λ-ring extension $K \subset K'$ such that $F^n_\gamma K = K \cap F^n_\gamma K'$, and x is an H-linear combination of products $(\ell_1 - 1)\cdots(\ell_m - 1)$, where the ℓ_i are line elements of K' and $m \geq n$.*

PROOF OF THEOREM 4.7. Since $(\ell_1 - 1)(\ell_2 - 1) \in F^2_\gamma K$, the map $\ell \mapsto \ell - 1$ is a homomorphism. If ℓ_1, ℓ_2, ℓ_3 are line elements of K,

$$\det((\ell_1 - 1)(\ell_2 - 1)\ell_3) = \det(\ell_1\ell_2\ell_3)\det(\ell_3)/det(\ell_1\ell_3)\det(\ell_2\ell_3) = 1.$$

By Ex. 4.3, the Filtered Splitting Principle implies that every element of $F^2_\gamma K$ can be written as a sum of terms $(\ell_1 - 1)(\ell_2 - 1)\ell_3$ in some extension K' of K. This shows that $\det(F^2_\gamma) = 1$, so det induces a map $\widetilde{K}/F^2_\gamma K \to L$. Now det is the inverse of the map $\ell \mapsto \ell - 1$ because for $p \in P$ the Splitting Principle shows that $p - \varepsilon(p) \equiv \det(p) - 1$ modulo $F^2_\gamma K$. $\square$

PROPOSITION 4.8. *If the γ-filtration on K is finite then $\widetilde{K}$ is a nilpotent ideal. If $\widetilde{K}$ is a nilpotent ideal which is finitely generated as an abelian group, then the γ-filtration on K is finite. That is, $F^N_\gamma K = 0$ for some N.*

PROOF. The first assertion follows from the fact that $\widetilde{K}^n \subset F^n_\gamma K$ for all n. If $\widetilde{K}$ is additively generated by $\{x_1, ..., x_s\}$, then there is an upper bound on the k for which $\gamma^k(x_i) \neq 0$; using the sum formula there is an upper bound n on the k for which γ^k is nonzero on $\widetilde{K}$. If $\widetilde{K}^m = 0$ then clearly we must have $F^{mn}_\gamma K = 0$. $\square$

EXAMPLE 4.8.1. If X is a finite CW complex, both $KO(X)$ and $KU(X)$ are finitely generated abelian groups by Ex. 3.9. Therefore they have finite γ-filtrations.

EXAMPLE 4.8.2. If R is a commutative noetherian ring of Krull dimension d, then $F^{d+1}_\gamma K_0(R) = 0$ by [FL, V.3.10], even though $K_0(R)$ may not be a finitely generated abelian group.

EXAMPLE 4.8.3. For the representation ring $R(G)$, G cyclic of order 2, we saw in Example 4.6.2 that $\widetilde{R}$ is not nilpotent. In fact $F^n_\gamma R(G) = \widetilde{R}^n = 2^{n-1}\widetilde{R} \neq 0$. An even worse example is the λ-ring $R_\mathbb{Q} = R(G) \otimes \mathbb{Q}$, because $F^n_\gamma R_\mathbb{Q} = \widetilde{R}_\mathbb{Q} \cong \mathbb{Q}$ for all $n \geq 1$.

REMARK 4.8.4. Fix $x \in \widetilde{K}$. It follows from the nilpotence of the $\gamma^k(x)$ that there is an integer N such that $x^N = 0$, and for every $k_1, \dots, k_n$ with $\sum k_i \geq N$ we have

$$\gamma^{k_1}(x)\gamma^{k_2}(x)\cdots\gamma^{k_n}(x) = 0.$$

The best general bound for such an N is $N = mn = \dim_\gamma(x)\dim_\gamma(-x)$.

PROPOSITION 4.9. *Let k, $n \geq 1$ be integers. If $x \in F_\gamma^n K$ then modulo $F_\gamma^{n+1}K$:*

$$\psi^k(x) \equiv k^n x; \qquad \textit{and} \qquad \lambda^k(x) \equiv (-1)^k k^{n-1} x.$$

PROOF. If ℓ is a line element then modulo $(\ell-1)^2$ we have

$$\psi^k(\ell - 1) = (\ell^{k-1} + ... + \ell + 1)(\ell - 1) \equiv k(\ell - 1).$$

Therefore if $\ell_1, \cdots, \ell_m$ are line elements and $m \geq n$ we have

$$\psi^k((\ell_1 - 1)\cdots(\ell_n - 1)) \equiv k^n(\ell_1 - 1)\cdots(\ell_n - 1) \text{ modulo } F_\gamma^{n+1}K.$$

The Filtered Splitting Principle implies that $\psi^k(x) \equiv k^n x$ modulo $F_\gamma^{n+1}K$ for every $x \in F_\gamma^n K$. For λ^k, we use the inductive definition of ψ^k to see that $k^n x = (-1)^{k-1}k\lambda^k(x)$ for every $x \in F_\gamma^n K$. The Filtered Splitting Principle allows us to consider the universal case $W = W_s$ of Exercise 4.4. Since there is no torsion in $F_\gamma^n W/F_\gamma^{n+1}W$, we can divide by k to obtain the formula $k^{n-1}x = (-1)^{k-1}\lambda^k(x)$. □

THEOREM 4.10 (STRUCTURE OF $K \otimes \mathbb{Q}$). *Suppose that K has a positive structure in which every element has finite γ-dimension e.g., if $\mathbb{N}$ is cofinal in P). Then:*

(1) *The eigenvalues of ψ^k on $K_\mathbb{Q} = K \otimes \mathbb{Q}$ are a subset of $\{1, k, k^2, k^3, ...\}$ for each k;*
(2) *The subspace $K_\mathbb{Q}^{(n)} = K_\mathbb{Q}^{(n,k)}$ of eigenvectors for $\psi^k = k^n$ is independent of k;*
(3) *$K_\mathbb{Q}^{(n)}$ is isomorphic to $F_\gamma^n K_\mathbb{Q}/F_\gamma^{n+1}K_\mathbb{Q} \cong (F_\gamma^n K/F_\gamma^{n+1}K) \otimes \mathbb{Q}$;*
(4) *$K_\mathbb{Q}^{(0)} \cong H^0 \otimes \mathbb{Q}$ and $K_\mathbb{Q}^{(1)} \cong L \otimes \mathbb{Q}$;*
(5) *The ring $K \otimes \mathbb{Q}$ is isomorphic to the graded ring $K_\mathbb{Q}^{(0)} \oplus K_\mathbb{Q}^{(1)} \oplus \cdots \oplus K_\mathbb{Q}^{(n)} \oplus \cdots$.*

PROOF. For every positive p, consider the universal λ-ring $U_\mathbb{Q} = \mathbb{Q}[s_1, ...]$ of Example 4.3.2, and the map $U_\mathbb{Q} \to K_\mathbb{Q}$ sending s_1 to p and s_k to $\lambda^k(p)$. If $\varepsilon(p) = n$ then s_i maps to zero for $i > n$ and each $s_i - \binom{n}{i}$ maps to a nilpotent element by Theorem 4.6. The image A of this map is a λ-ring which is finite-dimensional over $\mathbb{Q}$, so A is an artinian ring. Clearly $F_\gamma^N A = 0$ for some large N. Consider the linear operation $\prod_{n=0}^N(\psi^k - k^n)$ on A; by Proposition 4.9 it is trivial on each $F_\gamma^n/F_\gamma^{n+1}$, so it must be zero. Therefore the characteristic polynomial of ψ^k on A divides $\Pi(t - k^n)$, and has distinct integer eigenvalues. This proves (1) and that $K_\mathbb{Q}$ is the direct sum of the eigenspaces $K_\mathbb{Q}^{(n,k)}$ for ψ^k. As ψ^k preserves products, Proposition 4.9 now implies (3) and (4). The rest is immediate from Theorem 4.7. □

Chern class homomorphisms

The formalism in §3 for the Chern classes $c_i: KU(X) \to H^{2i}(X;\mathbb{Z})$ extends to the current setting. Suppose we are given a λ-ring K with a positive structure and a commutative graded ring $A = A^0 \oplus A^1 \oplus \cdots$. *Chern classes* on K with values in A are set maps $c_n: K \to A^n$ for $n \geq 0$ with $c_0(x) = 1$, satisfying the following axioms:

(CC0) The c_n send H^0 to zero (for $n \geq 1$): $c_n(h) = 0$ for every $h \in H^0$.

(CC1) *Dimension.* $c_n(p) = 0$ whenever p is positive and $n \geq \varepsilon(p)$.

(CC2) *Sum Formula.* For every x, y in K and every n:

$$c_n(x+y) = \sum_{i=0}^{n} c_i(x)c_{n-i}(y).$$

(CC3) *Normalization.* $c_1: L \to A^1$ is a group homomorphism. That is, for ℓ, ℓ':

$$c_1(\ell\ell') = c_1(\ell) + c_1(\ell').$$

The *total Chern class* of x is the element $c(x) = \sum c_i(x)$ of the completion $\hat{A} = \prod A^i$ of A. In terms of the total Chern class, (CC2) becomes the product formula

$$c(x+y) = c(x)c(y).$$

Example 4.11.1. The Stiefel-Whitney classes $w_i: KO(X) \to A^i = H^i(X;\mathbb{Z}/2)$ and the Chern classes $c_i: KU(X) \to A^i = H^{2i}(X;\mathbb{Z})$ are Chern classes in this sense.

Example 4.11.2. Associated to the γ-filtration on K we have the associated graded ring $Gr^{\bullet}_{\gamma}K$ with $Gr^i_{\gamma}K = F^i_{\gamma}/F^{i+1}_{\gamma}$. For a positive element p in K, define $c_i(p)$ to be $\gamma^i(p - \varepsilon(p))$ modulo F^{i+1}_{γ}. The multiplicative formula for γ_t implies that $c_i(p+q) = c_i(p) + c_i(q)$, so that the c_i extend to classes $c_i: K \to Gr^{\bullet}_{\gamma}K$. The total Chern class $c: K \to Gr^{\bullet}_{\gamma}K$ is a group homomorphism with torsion kernel and cokernel, because by Theorem 4.10 and Ex. 4.10 the induced map $c_n: K^{(n)}_{\mathbb{Q}} \to Gr^n_{\gamma}K_{\mathbb{Q}} \cong K^{(n)}_{\mathbb{Q}}$ is multiplication by $(-1)^n(n-1)!$.

The Splitting Principle implies the following Splitting Principle (see [FL, I.3.1]).

Chern Splitting Principle. *Given a finite set $\{p_i\}$ of positive elements of K, there is a λ-ring extension $K \subset K'$ in which each p_i splits as a sum of line elements, and a graded extension $A \subset A'$ such that the c_i extend to maps $c_i: K' \to (A')^i$ satisfying (CC1) and (CC2).*

The existence of "Chern roots" is an important consequence of this Splitting Principle. Suppose that $p \in K$ is positive, and that in an extension K' of K we can write $p = \ell_1 + \cdots + \ell_n$, $n = \varepsilon(p)$. The *Chern roots of* p are the elements $a_i = c_1(\ell_i)$ in $(A')^1$; they determine the $c_k(p)$ in A^k. Indeed, because $c(p)$ is the product of the $c(\ell_i) = 1 + a_i$, we see that $c_k(p)$ is the k^{th} elementary symmetric polynomial $\sigma_k(a_1, ..., a_n)$ of the a_i in the larger ring A'. In particular, the first Chern class is $c_1(p) = \sum a_i$ and the "top" Chern class is $c_n(p) = \prod a_i$.

A famous theorem of Isaac Newton states that every symmetric polynomial in n variables $t_1, ..., t_n$ is in fact a polynomial in the symmetric polynomials $\sigma_k = \sigma_k(t_1, ..., t_n)$, $k = 1, 2, \cdots$. Therefore every symmetric polynomial in the Chern roots of p is also a polynomial in the Chern classes $c_k(p)$, and as such belongs to the subring A of A'. Here is an elementary application of these ideas.

PROPOSITION 4.11.3. *Suppose that K satisfies the Splitting Principle. Then $c_n(\psi^k x) = k^n c_n(x)$ for all $x \in K$. That is, the following diagram commutes:*

$$\begin{array}{ccc} K & \xrightarrow{c_n} & A^n \\ \downarrow{\scriptstyle \psi^k} & & \downarrow{\scriptstyle k^n} \\ K & \xrightarrow{c_n} & A^n. \end{array}$$

COROLLARY 4.11.4. *If $\mathbb{Q} \subset A$ then c_n vanishes on $K_{\mathbb{Q}}^{(i)}$, $i \neq n$.*

Chern character

As an application of the notion of Chern roots, suppose given Chern classes $c_i \colon K \to A^i$, where for simplicity A is an algebra over $\mathbb{Q}$. If $p \in K$ is a positive element, with Chern roots a_i, define $ch(p)$ to be the formal expansion

$$ch(p) = \sum_{i=0}^{n} \exp(a_i) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=0}^{n} a_i^k \right)$$

of terms in A'. The k^{th} term $\frac{1}{k!} \sum a_i^k$ is symmetric in the Chern roots, so it is a polynomial in the Chern classes $c_1(p), ..., c_k(p)$ and hence belongs to A^k. Therefore $ch(p)$ is a formal expansion of terms in A, *i.e.,* an element of $\hat{A} = \prod A^k$. For example, if ℓ is a line element of K then $ch(\ell)$ is just $\exp(c_1(\ell))$. From the definition, it is immediate that $ch(p+q) = ch(p) + ch(q)$, so ch extends to a map from $P^{-1}P$ to $\hat{A}$. Since $ch(1) = 1$, this is compatible with the given map $H^0 \to A^0$, and so it defines a map $ch \colon K \to \hat{A}$, called the *Chern character* on K. The first few terms in the expansion of the Chern character are

$$ch(x) = \varepsilon(x) + c_1(x) + \frac{1}{2}[c_1(x)^2 - c_2(x)] + \frac{1}{6}[c_1(x)^3 - 3c_1(x)c_2(x) + 3c_3(x)] + \cdots$$

An inductive formula for the term in $ch(x)$ is given in Exercise 4.14.

PROPOSITION 4.12. *If $\mathbb{Q} \subset A$ then the Chern character is a ring homomorphism*

$$ch \colon K \to \hat{A}.$$

PROOF. By the Splitting Principle, it suffices to verify that $ch(pq) = ch(p)ch(q)$ when p and q are sums of line elements. Suppose that $p = \sum \ell_i$ and $q = \sum m_j$ have Chern roots $a_i = c_1(\ell_i)$ and $b_j = c_1(m_j)$, respectively. Since $pq = \sum \ell_i m_j$, the Chern roots of pq are the $c_1(\ell_i m_j) = c_1(\ell_i) + c_1(m_j) = a_i + b_j$. Hence

$$ch(pq) = \sum ch(\ell_i m_j) = \sum exp(a_i + b_j) = \sum exp(a_i) \exp(b_j) = ch(p)ch(q). \qquad \square$$

COROLLARY 4.12.1. *Suppose that K has a positive structure in which every $x \in K$ has finite γ-dimension (e.g., $\mathbb{N}$ is cofinal in P). Then the Chern character lands in A, and the induced map from $K_{\mathbb{Q}} = \oplus K_{\mathbb{Q}}^{(n)}$ to A is a graded ring map. That is, the n^{th} term $ch_n \colon K_{\mathbb{Q}} \to A^n$ vanishes on $K_{\mathbb{Q}}^{(i)}$ for $i \neq n$.*

EXAMPLE 4.12.2. The universal Chern character $ch\colon K_{\mathbb{Q}} \to K_{\mathbb{Q}}$ is the identity map. Indeed, by Ex. 4.10(b) and Ex. 4.14 we see that ch_n is the identity map on each $K_{\mathbb{Q}}^{(n)}$.

The following result was proven by M. Karoubi in [Kar63]. (See Exercise 4.11 for the proof when X is a finite CW complex.)

THEOREM 4.13. *If X is a compact topological space and $\check{H}$ denotes Čech cohomology, then the Chern character is an isomorphism of graded rings.*

$$ch\colon KU(X) \otimes \mathbb{Q} \cong \bigoplus \check{H}^{2i}(X;\mathbb{Q})$$

EXAMPLE 4.13.1 (SPHERES). For each even sphere, we know by Example 3.7.1 that c_n maps $\widetilde{KU}(S^{2n})$ isomorphically onto $H^{2n}(S^{2n};\mathbb{Z}) = \mathbb{Z}$. The inductive formula for ch_n shows that in this case $ch(x) = \dim(x) + (-1)^n c_n(x)/(n-1)!$ for all $x \in KU(X)$. In this case it is easy to see directly that $ch\colon KU(S^{2n}) \otimes \mathbb{Q} \cong H^{2*}(S^{2n};\mathbb{Q})$

EXERCISES

4.1 Show that in $K_0(R)$ or $K^0(X)$ we have

$$\lambda^k([P]-n) = \sum (-1)^i \binom{n+i-1}{i} [\wedge^{k-i} P].$$

4.2 For every group G and every commutative ring A, let $R_A(G)$ denote the group $K_0(AG, A)$ of Ex. 2.14, *i.e.,* the group completion of the monoid $Rep(AG, A)$ of all AG-modules which are finitely generated projective as A-modules. Show that the $\wedge^k$ make $R_A(G)$ into a λ-ring with a positive structure given by $Rep(AG, A)$.

If $A = \mathbb{C}$, show that $R_{\mathbb{C}}(G)$ satisfies the Splitting Principle and hence is a special λ-ring (by 4.2.3); the line elements are the characters. Swan proved in [Swan70] that $R_A(G)$ satisfies the Splitting Principle for every A. (Another proof is in [SGA6], VI(3.3).) This proves that $R_A(G)$ is a special λ-ring for every A.

When $p = 0$ in A, show that $\psi^p = \Phi^*$ in $R_A(G)$, where $\Phi : A \to A$ is the Frobenius $\Phi(a) = a^p$. To do this, reduce to the case in which χ is a character and show that $\psi^k \chi(g) = \chi(g^p) = \chi(g)^p$.

4.3 Suppose that a λ-ring K is generated as an H-algebra by line elements. Show that $F_\gamma^n = \widetilde{K}^n$ for all n, so the γ-filtration is the adic filtration defined by the ideal $\widetilde{K}$. Then show that if K is any λ-ring satisfying the Splitting Principle every element x of $F_\gamma^n K$ can be written in an extension K' of K as a product

$$x = (\ell_1 - 1)\cdots(\ell_m - 1)$$

of line elements with $m \geq n$. In particular, show that every $x \in F_\gamma^2$ can be written as a sum of terms $(\ell_i - 1)(\ell_j - 1)\ell$ in K'.

4.4 *Universal special λ-ring.* Let W_s denote the Laurent polynomial ring $\mathbb{Z}[u_1, u_1^{-1}, ..., u_s, u_s^{-1}]$, and $\varepsilon\colon W_s \to \mathbb{Z}$ the augmentation defined by $\varepsilon(u_i) = 1$.

(a) Show that W_s is a λ-ring with a positive structure, the line elements being the monomials $u^\alpha = \prod u_i^{n_i}$. This implies that W_s is generated by the group $L \cong \mathbb{Z}^s$ of line elements, so by Exercise 4.3 the ideal $F_\gamma^n W_s$ is $\widetilde{W}^n$.

(b) Show that each $F^n_\gamma W/F^{n+1}_\gamma W$ is a torsionfree abelian group.

(c) If K is a special λ-ring show that any family $\{\ell_1, ..., \ell_s\}$ of line elements determines a λ-ring map $W_s \to K$ sending u_i to ℓ_i.

(d) (Splitting Principle for the free λ-ring) Let $U \to W_s$ be the λ-ring homomorphism sending s_1 to $\sum u_i$ (see 4.3.2). Show that U injects into $\varprojlim W_s$.

4.5 A line element ℓ is called *ample* for K if for every $x \in \widetilde{K}$ there is an integer $N = N(x)$ such that for every $n \geq N$ there is a positive element p_n so that $\ell^n x = p_n - \varepsilon(p_n)$. (The terminology comes from Algebraic Geometry; see 8.8.4 below.) If K has an ample line element, show that every element of $\widetilde{K}$ is nilpotent.

4.6 Verify that the inductive definition of ψ^k and the ψ_t definition of ψ^k agree.

4.7 If p is prime, use the Splitting Principle to verify that $\psi^p(x) \equiv x^p$ modulo p for every $x \in K$.

4.8 *Adams e-invariant.* Suppose given a map $f\colon S^{2m-1} \to S^{2n}$. The mapping cone $C(f)$ fits into a cofibration sequence $S^{2n} \overset{i}{\rightarrowtail} C(f) \overset{j}{\twoheadrightarrow} S^{2m}$. Associated to this is the exact sequence:

$$0 \to \widetilde{KU}(S^{2m}) \xrightarrow{j^*} \widetilde{KU}(C) \xrightarrow{i^*} \widetilde{KU}(S^{2n}) \to 0.$$

Choose $x, y \in \widetilde{KU}(C)$ so that $i^*(x)$ generates $\widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ and y is the image of a generator of $\widetilde{KU}(S^{2m}) \cong \mathbb{Z}$. Since j^* is a ring map, $y^2 = 0$.

(a) Show by applying ψ^k that $xy = 0$, and that if $m \neq 2n$ then $x^2 = 0$. (When $m = 2n$, x^2 defines the Hopf invariant of f; see the next exercise.)

(b) Show that $\psi^k(x) = k^n x + a_k y$ for appropriate integers a_k. Then show (for fixed x and y) that the rational number

$$e(f) = \frac{a_k}{k^m - k^n}$$

is independent of the choice of k.

(c) Show that a different choice of x only changes $e(f)$ by an integer, so that $e(f)$ is a well-defined element of $\mathbb{Q}/\mathbb{Z}$; $e(f)$ is called the *Adams e-invariant* of f.

(d) If f and f' are homotopic maps, it follows from the homotopy equivalence between $C(f)$ and $C(f')$ that $e(f) = e(f')$. By considering the mapping cone of $f \vee g$, show that the well-defined set map $e\colon \pi_{2m-1}(S^{2n}) \to \mathbb{Q}/\mathbb{Z}$ is a group homomorphism. J.F. Adams used this e-invariant to detect an important cyclic subgroup of $\pi_{2m-1}(S^{2n})$, namely the "image of J."

4.9 *Hopf Invariant One.* Given a continuous map $f\colon S^{4n-1} \to S^{2n}$, define an integer $H(f)$ as follows. Let $C(f)$ be the mapping cone of f. As in the previous exercise, we have an exact sequence:

$$0 \to \widetilde{KU}(S^{4n}) \xrightarrow{j^*} \widetilde{KU}(C(f)) \xrightarrow{i^*} \widetilde{KU}(S^{2n}) \to 0.$$

Choose $x, y \in \widetilde{KU}(C(f))$ so that $i^*(x)$ generates $\widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ and y is the image of a generator of $\widetilde{KU}(S^{4n}) \cong \mathbb{Z}$. Since $i^*(x^2) = 0$, we can write $x^2 = Hy$ for some integer H; this integer $H = H(f)$ is called the *Hopf invariant* of f.

(a) Show that $H(f)$ is well-defined, up to $\pm$ sign.

(b) If $H(f)$ is odd, show that n is 1, 2, or 4. *Hint:* Use Ex. 4.7 to show that the integer a_2 of the previous exercise is odd. Considering $e(f)$, show that 2^n divides $p^n - 1$ for every odd p.

It turns out that the classical "Hopf maps" $S^3 \to S^2$, $S^7 \to S^4$ and $S^{15} \to S^8$ all have Hopf invariant $H(f) = 1$. In contrast, for every even integer H there is a map $S^{4n-1} \to S^{2n}$ with Hopf invariant H.

4.10 *Operations.* A natural *operation* τ on λ-rings is a map $\tau\colon K \to K$ defined for every λ-ring K such that $f\tau = \tau f$ for every λ-ring map $f\colon K \to K'$. The operations λ_k, γ_k, and ψ_k are all natural operations on λ-rings.

(a) If K satisfies the Splitting Principle, generalize Proposition 4.9 to show that every natural operation τ preserves the γ-filtration of K and that there are integers $\omega_n = \omega_n(\tau)$, independent of K, such that for every $x \in F^n_\gamma K$

$$\tau(x) \equiv \omega_n x \text{ modulo } F^{n+1}_\gamma K.$$

(b) Show that for $\tau = \gamma^k$ and $x \in F^n_\gamma$ we have:

$$\gamma^k_{(x)} = \begin{cases} 0 \quad \text{if} \quad n < k \\ (-1)^{k-1}(k-1)! \text{ if } \quad n = k \\ \omega_n \neq 0 \quad \text{if} \quad n > k \end{cases}$$

(c) Show that $s_k \mapsto \lambda^k$ and $\tau \mapsto \tau(s_1)$ give λ-ring isomorphisms from the freeλ-ring $U = \mathbb{Z}[s_1, s_2, \cdots]$ of 4.3.2 to the ring of all natural operations on λ-rings. (See [Atiyah, 3.1.7].)

4.11 By Example 4.13.1, the Chern character $ch\colon KU(S^n) \otimes \mathbb{Q} \to H^{2*}(S^n; \mathbb{Q})$ is an isomorphism for every sphere S^n. Use this to show that $ch\colon KU(X) \otimes \mathbb{Q} \to H^{2*}(X; \mathbb{Q})$ is an isomorphism for every finite CW complex X.

4.12 Let K be a λ-ring. Given a K-module M, construct the ring $K \oplus M$ in which $M^2 = 0$. Given a sequence of K-linear endomorphisms φ_k of M with $\varphi_1(x) = x$, show that the formulae $\lambda^k(x) = \varphi_k(x)$ extend the λ-ring structure on K to a λ-ring structure on $K \oplus M$. Then show that $K \oplus M$ has a positive structure if K does, and that $K \oplus M$ satisfies the Splitting Principle whenever K does. (The elements in $1 + M$ are to be the new line elements.)

4.13 *Hirzebruch characters.* Suppose that A is an H^0-algebra and we fix a power series $\alpha(t) = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots$ in $A^0[[t]]$. Suppose given Chern classes $c_i\colon K \to A^i$. If $p \in K$ is a positive element, with Chern roots a_i, define $ch_\alpha(p)$ to be the formal expansion

$$ch_\alpha(p) = \sum_{i=0}^{n} \alpha(a_i) \sum_{k=0}^{\infty} \alpha_k \left(\sum_{i=0}^{n} a_i^k \right)$$

of terms in A'. Show that $ch_\alpha(p)$ belongs to the formal completion $\hat{A}$ of A, and that it defines a group homomorphism $ch_\alpha\colon K \to \hat{A}$. This map is called the *Hirzebruch character* for α.

4.14 Establish the following inductive formula for the n^{th} term ch_n in the Chern character:

$$ch_n - \frac{1}{n} c_1 ch_{n-1} + \cdots \pm \frac{1}{i!\binom{n}{i}} c_i ch_{n-i} + \cdots + \frac{(-1)^n}{(n-1)!} c_n = 0.$$

To do this, set $x = -t_i$ in the identity $\prod(x + a_i) = x^n + c_1 x^{n-1} + \cdots + c_n$.

§5. K_0 of a Symmetric Monoidal Category

The idea of group completion in §1 can be applied to more categories than just the categories $\mathbf{P}(R)$ in §2 and $\mathbf{VB}(X)$ in §3. It applies to any category with a "direct sum", or more generally any natural product $\Box$ making the isomorphism classes of objects into an abelian monoid. This leads us to the notion of a symmetric monoidal category.

DEFINITION 5.1. A *symmetric monoidal category* is a category S, equipped with a functor $\Box\colon S \times S \to S$, a distinguished object e and four basic natural isomorphisms:

$$e\Box s \cong s, \quad s\Box e \cong s, \quad s\Box(t\Box u) \cong (s\Box t)\Box u, \quad \text{and } s\Box t \cong t\Box s.$$

These basic isomorphisms must be "coherent" in the sense that two natural isomorphisms of products of $s_1, \dots, s_n$ built up from the four basic ones are the same whenever they have the same source and target. (We refer the reader to [Mac] for the technical details needed to make this definition of "coherent" precise.) Coherence permits us to write expressions without parentheses like $s_1\Box\cdots\Box s_n$ unambiguously (up to natural isomorphism).

EXAMPLE 5.1.1. Any category with a direct sum $\oplus$ is symmetric monoidal; this includes additive categories like $\mathbf{P}(R)$ and $\mathbf{VB}(X)$ as we have mentioned. More generally, a category with finite coproducts is symmetric monoidal with $s\Box t = s \amalg t$. Dually, any category with finite products is symmetric monoidal with $s\Box t = s \times t$.

DEFINITION 5.1.2 (K_0S). Suppose that the isomorphism classes of objects of S form a *set*, which we call S^{iso}. If S is symmetric monoidal, this set S^{iso} is an abelian monoid with product $\Box$ and identity e. The group completion of this abelian monoid is called the *Grothendieck group* of S, and is written as $K_0^{\Box}(S)$, or simply as $K_0(S)$ if $\Box$ is understood.

From §1 we see that $K_0^{\Box}(S)$ may be presented with one generator $[s]$ for each isomorphism class of objects, with relations that $[s\Box t] = [s] + [t]$ for each s and t. From Proposition 1.1 we see that every element of $K_0^{\Box}(S)$ may be written as a difference $[s] - [t]$ for some objects s and t.

EXAMPLES 5.2. (1) The category $\mathbf{P}(R)$ of finitely generated projective modules over a ring R is symmetric monoidal under direct sum. Since the above definition is identical to that in §2, we see that we have $K_0(R) = K_0^{\oplus}(\mathbf{P}(R))$.

(2) Similarly, the category $\mathbf{VB}(X)$ of (real or complex) vector bundles over a topological space X is symmetric monoidal, with $\Box$ being the Whitney sum $\oplus$. From the definition we see that we also have $K(X) = K_0^{\oplus}(\mathbf{VB}(X))$, or more explicitly:

$$KO(X) = K_0^{\oplus}(\mathbf{VB}_{\mathbb{R}}(X)), \qquad KU(X) = K_0^{\oplus}(\mathbf{VB}_{\mathbb{C}}(X)).$$

(3) If R is a commutative ring, let $\mathbf{Pic}(R)$ denote the category of algebraic line bundles L over R and their isomorphisms (§I.3). This is a symmetric monoidal category with $\Box = \otimes_R$, and the isomorphism classes of objects already form a group, so $K_0\mathbf{Pic}(R) = \operatorname{Pic}(R)$.

FINITE SETS 5.2.1. Let $\mathbf{Sets}_{\text{fin}}$ denote the category of finite sets. It has a coproduct, the disjoint sum $\amalg$, and it is not hard to see that $K_0^{\amalg}(\mathbf{Sets}_{\text{fin}}) = \mathbb{Z}$.

Another monoidal operation on $\mathbf{Sets}_{\text{fin}}$ is the product ($\times$). However, since the empty set satisfies $\emptyset = \emptyset \times X$ for all X we have $K_0^{\times}(\mathbf{Sets}_{\text{fin}}) = 0$.

The category $\mathbf{Sets}_{\text{fin}}^{\times}$ of nonempty finite sets has for its isomorphism classes the set $\mathbb{N}_{>0} = \{1, 2, \dots\}$ of positive integers, and the product of finite sets corresponds to multiplication. Since the group completion of $(\mathbb{N}_{>0}, \times)$ is the multiplicative monoid $\mathbb{Q}_{>0}^{\times}$ of positive rational numbers, we have $K_0^{\times}(\mathbf{Sets}_{\text{fin}}^{\times}) \cong \mathbb{Q}_{>0}^{\times}$.

BURNSIDE RING 5.2.2. Suppose that G is a finite group, and let G-$\mathbf{Sets}_{\text{fin}}$ denote the category of finite G-sets. It is a symmetric monoidal category under disjoint union. We saw in Example 1.5 that $K_0(G\text{-}\mathbf{Sets}_{\text{fin}})$ is the Burnside Ring $A(G) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of subgroups of G.

REPRESENTATION RING 5.2.3. Similarly, the finite-dimensional complex representations of a finite group G form a category $\mathbf{Rep}_{\mathbb{C}}(G)$. It is symmetric monoidal under $\oplus$. We saw in Example 1.6 that $K_0\mathbf{Rep}_{\mathbb{C}}(G)$ is the representation ring $R(G)$ of G, which is a free abelian group on the classes $[V_1], ..., [V_r]$ of the irreducible representations of G.

Cofinality

Let T be a full subcategory of a symmetric monoidal category S. If T contains e and is closed under finite products, then T is also symmetric monoidal. We say that T is *cofinal* in S if for every s in S there is an s' in S such that $s \Box s'$ is isomorphic to an element in T, *i.e.,* if the abelian monoid T^{iso} is cofinal in S^{iso} in the sense of §1. When this happens, we may restate Corollary 1.3 as follows.

COFINALITY THEOREM 5.3. *Let T be cofinal in a symmetric monoidal category S. Then (assuming S^{iso} is a set):*

(1) *$K_0(T)$ is a subgroup of $K_0(S)$;*
(2) *Every element of $K_0(S)$ is of the form $[s] - [t]$ for some s in S and t in T;*
(3) *If $[s] = [s']$ in $K_0(S)$ then $s \Box t \cong s' \Box t$ for some t in T.*

EXAMPLE 5.4.1 (FREE MODULES). Let R be a ring. The category $\mathbf{Free}(R)$ of finitely generated free R-modules is cofinal (for $\Box = \oplus$) in the category $\mathbf{P}(R)$ of finitely generated projective modules. Hence $K_0\mathbf{Free}(R)$ is a subgroup of $K_0(R)$. In fact $K_0\mathbf{Free}(R)$ is is a cyclic abelian group, equal to $\mathbb{Z}$ whenever R satisfies the Invariant Basis Property. Moreover, the subgroup $K_0\mathbf{Free}(R)$ of $K_0(R) = K_0\mathbf{P}(R)$ is the image of the map $\mathbb{Z} \to K_0(R)$ described in Lemma 2.1.

$\mathbf{Free}(R)$ is also cofinal in the smaller category $\mathbf{P}^{\text{st.free}}(R)$ of finitely generated stably free modules. Since every stably free module P satisfies $P \oplus R^m \cong R^n$ for some m and n, the Cofinality Theorem yields $K_0\mathbf{Free}(R) = K_0\mathbf{P}^{\text{st.free}}(R)$.

EXAMPLE 5.4.2. Let R be a commutative ring. A finitely generated projective R-module is called *faithfully projective* if its rank is never zero. The tensor product of faithfully projective modules is again faithfully projective by Ex. 2.7. Hence the category $\mathbf{FP}(R)$ of faithfully projective R-modules is a symmetric monoidal category under the tensor product $\otimes_R$. For example, if R is a field then the monoid $\mathbf{FP}^{\text{iso}}$ is the multiplicative monoid $(\mathbb{N}_{>0}, \times)$ of Example 5.2.1, so in this case we have $K_0^{\otimes}\mathbf{FP}(R) \cong \mathbb{Q}_{>0}^{\times}$. We will describe the group $K_0^{\otimes}\mathbf{FP}(R)$ in the exercises below.

EXAMPLE 5.4.3 (BRAUER GROUPS). Suppose first that F is a field, and let $\mathbf{Az}(F)$ denote the category of central simple F-algebras. This is a symmetric monoidal category with product $\otimes_F$, because if A and B are central simple then so is $A \otimes_F B$. The matrix rings $M_n(F)$ form a cofinal subcategory, with $M_m(F) \otimes_F M_n(F) \cong M_{mn}(F)$. From the previous example we see that the Grothendieck group of this subcategory is $\mathbb{Q}_{>0}^{\times}$. The classical *Brauer group* $Br(F)$ of the field F is the quotient of $K_0\mathbf{Az}(F)$ by this subgroup. That is, $Br(F)$ is generated by classes $[A]$ of central simple algebras with two families of relations: $[A \otimes_F B] = [A] + [B]$ and $[M_n(F)] = 0$.

More generally, suppose that R is a commutative ring. Recall (from [Milne, IV]) that an R-algebra A is is called an *Azumaya algebra* if there is another R-algebra B such that $A \otimes_R B \cong M_n(R)$ for some n. The category $\mathbf{Az}(R)$ of Azumaya R-algebras is thus symmetric monoidal with product $\otimes_R$. If P is a faithfully projective R-module, $\mathrm{End}_R(P)$ is an Azumaya algebra. Since $\mathrm{End}_R(P \otimes_R P') \cong \mathrm{End}_R(P) \otimes_R \mathrm{End}_R(P')$, there is a monoidal functor End_R from $\mathbf{FP}(R)$ to $\mathbf{Az}(R)$, and a group homomorphism $K_0\mathbf{FP}(R) \to K_0\mathbf{Az}(R)$. The cokernel $Br(R)$ of this map is called the *Brauer group* of R. That is, $Br(R)$ is generated by classes $[A]$ of Azumaya algebras with two families of relations: $[A \otimes_R B] = [A] + [B]$ and $[\mathrm{End}_R(P)] = 0$.

G-bundles and equivariant K-theory

The following discussion is taken from Atiyah's very readable book [Atiyah]. Suppose that G is a finite group and that X is a topological space on which G acts continuously. A (complex) vector bundle E over X is called a *G-vector bundle* if G acts continuously on E, the map $E \to X$ commutes with the action of G, and for each $g \in G$ and $x \in X$ the map $E_x \to E_{gx}$ is a vector space homomorphism. The category $\mathbf{VB}_G(X)$ of G-vector bundles over X is symmetric monoidal under the usual Whitney sum, and we write $K_G^0(X)$ for the Grothendieck group $K_0^{\oplus}\mathbf{VB}_G(X)$. For example, if X is a point then $\mathbf{VB}_G(X) = \mathbf{Rep}_{\mathbb{C}}(G)$, so we have $K_G^0(\text{point}) = R(G)$. More generally, if x is a fixed point of X, then $E \mapsto E_x$ defines a monoidal functor from $\mathbf{VB}_G(X)$ to $\mathbf{Rep}_{\mathbb{C}}(G)$, and hence a group map $K_G^0(X) \to R(G)$.

If G acts trivially on X, every vector bundle E on X can be considered as a G-bundle with trivial action, and the tensor product $E \otimes V$ with a representation V of G is another G-bundle. The following result is proven on p. 38 of [Atiyah].

PROPOSITION 5.5 (KRULL-SCHMIDT THEOREM). *Let $V_1, ..., V_r$ be a complete set of irreducible G-modules, and suppose that G acts trivially on X. Then for every G-bundle F over X there are unique vector bundles $E_i = \mathrm{Hom}_G(V_i, F)$ so that*

$$F \cong (E_1 \otimes V_1) \oplus \cdots \oplus (E_r \otimes V_r).$$

COROLLARY 5.5.1. *If G acts trivially on X then $K_G^0(X) \cong KU(X) \otimes_{\mathbb{Z}} R(G)$.*

The Witt ring $W(F)$ of a field

5.6. Symmetric bilinear forms over a field F provide another classical application of the K_0 construction. The following discussion is largely taken from Milnor's pretty book [M-SBF], and the reader is encouraged to look there for the connections with other branches of mathematics.

A *symmetric inner product space* (V, B) is a finite dimensional vector space V, equipped with a nondegenerate symmetric bilinear form $B\colon V \otimes V \to F$. The category $\mathbf{SBil}(F)$ of symmetric inner product spaces and form-preserving maps is symmetric monoidal, where the operation $\Box$ is the orthogonal sum $(V, B) \oplus (V', B')$, defined as the vector space $V \oplus V'$, equipped with the bilinear form $\beta(v \oplus v', w \oplus w') = B(v, w) + B'(v', w')$.

A crucial role is played by the *hyperbolic plane* H, which is $V = F^2$ equipped with the bilinear form B represented by the symmetric matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. An inner product space is called *hyperbolic* if it is isometric to an orthogonal sum of hyperbolic planes.

Let $(V, B) \otimes (V', B')$ denote the tensor product $V \otimes V'$, equipped with the bilinear form $\beta(v \otimes v', w \otimes w') = B(v, w)B'(v', w')$; this is also a symmetric inner product space, and the isometry classes of inner product spaces forms a semiring under $\oplus$ and $\otimes$ (see Ex. 5.10). Thus $K_0\mathbf{SBil}(F)$ is a commutative ring with unit $1 = \langle 1 \rangle$, and the forgetful functor $\mathbf{SBil}(F) \to \mathbf{P}(F)$ sending (V, B) to V induces a ring augmentation $\varepsilon\colon K_0\mathbf{SBil}(F) \to K_0(F) \cong \mathbb{Z}$. We write $\hat{I}$ for the augmentation ideal of $K_0\mathbf{SBil}(F)$.

Example 5.6.1. For each $a \in F^\times$, we write $\langle a \rangle$ for the inner product space with $V = F$ and $B(v, w) = avw$. Clearly $\langle a \rangle \otimes \langle b \rangle \cong \langle ab \rangle$. Note that a change of basis $1 \mapsto b$ of F induces an isometry $\langle a \rangle \cong \langle ab^2 \rangle$ for every unit b, so the inner product space only determines a up to a square.

If $\mathrm{char}(F) \neq 2$, it is well known that every symmetric bilinear form is diagonalizable. Thus every symmetric inner product space is isometric to an orthogonal sum $\langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$. For example, it is easy to see that $H \cong \langle 1 \rangle \oplus \langle -1 \rangle$. This also implies that $\hat{I}$ is additively generated by the elements $\langle a \rangle - 1$.

If $\mathrm{char}(F) = 2$, every symmetric inner product space is isomorphic to $\langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle \oplus N$, where N is hyperbolic; see [M-SBF, I.3]. In this case $\hat{I}$ has the extra generator $H - 2$.

If $\mathrm{char}(F) \neq 2$, there is a Cancellation Theorem due to Witt: if X, Y, Z are inner product spaces, then $X \oplus Y \cong X \oplus Z$ implies that $Y \cong Z$. For a proof, we refer the reader to [M-SBF]. We remark that cancellation fails if $\mathrm{char}(F) = 2$; see Ex. 5.11(d). The following definition is due to Knebusch.

Definition 5.6.2. Suppose that $\mathrm{char}(F) \neq 2$. The *Witt ring* $W(F)$ is defined to be the quotient of the ring $K_0\mathbf{SBil}(F)$ by the subgroup $\{nH\}$ generated by the hyperbolic plane H. This subgroup is an ideal by Ex. 5.11, so $W(F)$ is also a commutative ring.

Since the augmentation $K_0\mathbf{SBil}(F) \to \mathbb{Z}$ has $\varepsilon(H) = 2$, it induces an augmentation $\varepsilon\colon W(F) \to \mathbb{Z}/2$. We write I for the augmentation ideal $\ker(\varepsilon)$ of $W(F)$.

When $\mathrm{char}(F) = 2$, $W(F)$ is defined similarly, as the quotient of $K_0\mathbf{SBil}(F)$ by the subgroup of "split" spaces; see Ex. 5.11. In this case we have $2 = 0$ in the Witt ring $W(F)$, because the inner product space $\langle 1 \rangle \oplus \langle 1 \rangle$ is split (Ex. 5.11(d)).

When $\mathrm{char}(F) \neq 2$, the augmentation ideals of $K_0\mathbf{SBil}(F)$ and $W(F)$ are isomorphic: $\hat{I} \cong I$. This is because $\varepsilon(nH) = 2n$, so that $\{nH\} \cap \hat{I} = 0$ in $K_0\mathbf{SBil}(F)$.

Since $(V,B) + (V,-B) = 0$ in $W(F)$ by Ex. 5.11, every element of $W(F)$ is represented by an inner product space. In particular, I is additively generated by the classes $\langle a\rangle + \langle -1\rangle$, even if $\mathrm{char}(F) = 2$. The powers I^n of I form a decreasing chain of ideals $W(F) \supset I \supset I^2 \supset \cdots$. We shall describe I/I^2 now, and return to this topic in chapter III, §7.

The discriminant of an inner product space (V,B) is a classical invariant with values in $F^\times/F^{\times 2}$, where $F^{\times 2}$ denotes $\{a^2 | a \in F^\times\}$. For each basis of V, there is a matrix M representing B, and the determinant of M is a unit of F. A change of basis replaces M by A^tMA, and $\det(A^tMA) = \det(M)\det(A)^2$, so $w_1(V,B) = \det(M)$ is a well defined element in $F^\times/F^{\times 2}$, called the *first Stiefel-Whitney class* of (V,B). Since $w_1(H) = -1$, we have to modify the definition slightly in order to get an invariant on the Witt ring.

Definition 5.6.3. If $\dim(V) = r$, the *discriminant* of (V,B) is defined to be the element $d(V,B) = (-1)^{r(r-1)/2}\det(M)$ of $F^\times/F^{\times 2}$.

For example, we have $d(H) = d(1) = 1$ but $d(2) = -1$. It is easy to verify that the discriminant of $(V,B) \oplus (V',B')$ is $(-1)^{rr'}d(V,B)d(V',B')$, where $r = \dim(V)$ and $r' = \dim(V')$. In particular, (V,B) and $(V,B)\oplus H$ have the same discriminant. It follows that the discriminant is a well-defined map from $W(F)$ to $F^\times/F^{\times 2}$, and its restriction to I is additive.

Theorem 5.6.4. *(Pfister) The discriminant induces an isomorphism between I/I^2 and $F^\times/F^{\times 2}$.*

Proof. Since the discriminant of $\langle a\rangle \oplus \langle -1\rangle$ is a, the map $d\colon I \to F^\times/F^{\times 2}$ is onto. This homomorphism annihilates I^2 because I^2 is additively generated by products of the form

$$\big(\langle a\rangle - 1\big)\big(\langle b\rangle - 1\big) = \langle ab\rangle + \langle -a\rangle + \langle -b\rangle + 1,$$

and these have discriminant 1. Setting these products equal to zero, the identity $\langle a\rangle + \langle -a\rangle = 0$ yields the congruence

$$\text{(5.6.5)} \qquad \big(\langle a\rangle - 1\big) + \big(\langle b\rangle - 1\big) \equiv \langle ab\rangle - 1 \mod I^2.$$

Hence the formula $s(a) = \langle a\rangle - 1$ defines a surjective homomorphism $s\colon F^\times \to I/I^2$. Since $ds(a) = a$, it follows that s is an isomorphism with inverse induced by d. □

Corollary 5.6.6. *$W(F)$ contains $\mathbb{Z}/2$ as a subring (i.e., $2 = 0$) if and only if -1 is a square in F.*

Classical Examples 5.6.7. If F is an algebraically closed field, or more generally every element of F is a square, then $\langle a\rangle \cong \langle 1\rangle$ and $W(F) = \mathbb{Z}/2$.

If $F = \mathbb{R}$, every bilinear form is classified by its rank and signature. For example, $\langle 1\rangle$ has signature 1 but H has signature 0, with $H\otimes H \cong H\oplus H$. Thus $K_0\mathbf{SBil}(\mathbb{R}) \cong \mathbb{Z}[H]/(H^2 - 2H)$ and the signature induces a ring isomorphism $W(\mathbb{R}) \cong \mathbb{Z}$.

If $F = \mathbb{F}_q$ is a finite field with q odd, then $I/I^2 \cong \mathbb{Z}/2$, and an elementary argument due to Steinberg shows that the ideal I^2 is zero. The structure of the ring $W(F)$ now follows from 5.6.6: if $q \equiv 3 \pmod 4$ then $W(F) = \mathbb{Z}/4$; if $q \equiv 1 \pmod 4$, $W(\mathbb{F}_q) = \mathbb{Z}/2[\eta]/(\eta^2)$, where $\eta = \langle a \rangle - 1$ for some $a \in F$.

If F is a finite field extension of the p-adic rationals, then $I^3 = 0$ and I^2 is cyclic of order 2. If p is odd and the residue field is $\mathbb{F}_q$, then $W(F)$ contains $\mathbb{Z}/2$ as a subring if $q \equiv 1 \pmod 4$ and contains $\mathbb{Z}/4$ if $q \equiv 3 \pmod 4$. If $p = 2$ then $W(F)$ contains $\mathbb{Z}/2$ as a subring if and only if $\sqrt{-1} \in F$. Otherwise $W(F)$ contains $\mathbb{Z}/4$ or $\mathbb{Z}/8$, according to whether -1 is a sum of two squares, an issue which is somewhat subtle.

If $F = \mathbb{Q}$, the ring map $W(\mathbb{Q}) \to W(\mathbb{R}) = \mathbb{Z}$ is onto, with kernel N satisfying $N^3 = 0$. Since $I/I^2 = \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, the kernel is infinite but under control.

Quadratic Forms

The theory of symmetric bilinear forms is closely related to the theory of quadratic forms, which we now sketch.

Definition 5.7. Let V be a vector space over a field F. A quadratic form on V is a function $q\colon V \to F$ such that $q(av) = a^2\, q(v)$ for every $a \in F$ and $v \in V$, and such that the formula $B_q(v, w) = q(v + w) - q(v) - q(w)$ defines a symmetric bilinear form B_q on V. We call (V, q) a *quadratic space* if B_q is nondegenerate, and call (V, B_q) the underlying symmetric inner product space. We write $\mathbf{Quad}(F)$ for the category of quadratic spaces and form-preserving maps.

The orthogonal sum $(V, q) \oplus (V', q')$ of two quadratic spaces is defined to be $V \oplus V'$ equipped with the quadratic form $v \oplus v' \mapsto q(v) + q'(v')$. This is a quadratic space, whose underlying symmetric inner product space is the orthogonal sum $(V, B_q) \oplus (V', B_{q'})$. Thus $\mathbf{Quad}(F)$ is a symmetric monoidal category, and the underlying space functor $\mathbf{Quad}(F) \to \mathbf{SBil}(F)$ sending (V, q) to (V, B_q) is monoidal.

Here is one source of quadratic spaces. Suppose that β is a (possibly nonsymmetric) bilinear form on V. The function $q(v) = \beta(v, v)$ is visibly quadratic, with associated symmetric bilinear form $B_q(v, w) = \beta(v, w) + \beta(w, v)$. By choosing an ordered basis of V, it is easy to see that every quadratic form arises in this way. Note that when β is symmetric we have $B_q = 2\beta$; if $\mathrm{char}(F) \neq 2$ this shows that $\beta \mapsto \frac{1}{2}q$ defines a monoidal functor $\mathbf{SBil}(F) \to \mathbf{Quad}(F)$ inverse to the underlying functor, and proves the following result.

Lemma 5.7.1. *If* $\mathrm{char}(F) \neq 2$ *then the underlying space functor* $\mathbf{Quad}(F) \to \mathbf{SBil}(F)$ *is an equivalence of monoidal categories.*

A quadratic space (V, q) is said to be *split* if it contains a subspace N so that $q(N) = 0$ and $\dim(V) = 2\dim(N)$. For example, the quadratic forms $q(x, y) = xy + cy^2$ on $V = F^2$ are split.

Definition 5.7.2. The group $WQ(F)$ is defined to be the quotient of the group $K_0\mathbf{Quad}(F)$ by the subgroup of all split quadratic spaces.

It follows from Ex. 5.11 that the underlying space functor defines a homomorphism $WQ(F) \to W(F)$. By Lemma 5.7.1, this is an isomorphism when $\mathrm{char}(F) \neq 2$.

When $\mathrm{char}(F) = 2$, the underlying symmetric inner product space of a quadratic space (V, q) is always hyperbolic, and V is always even-dimensional; see Ex. 5.13.

In particular, $WQ(F) \to W(F)$ is the zero map when $\mathrm{char}(F) = 2$. By Ex. 5.13, $WQ(F)$ is a $W(F)$-module with $WQ(F)/I \cdot WQ(F)$ given by the Arf invariant. We will describe the rest of the filtration $I^n \cdot WQ(F)$ in III.7.10.4.

EXERCISES

5.1 Let R be a ring and let $\mathbf{P}^\infty(R)$ denote the category of all countably generated projective R-modules. Show that $K_0^\oplus \mathbf{P}^\infty(R) = 0$.

5.2 Suppose that the Krull-Schmidt Theorem holds in an additive category $\mathcal{C}$, *i.e.*, that every object of $\mathcal{C}$ can be written as a finite direct sum of indecomposable objects, in a way that is unique up to permutation. Show that $K_0^\oplus(\mathcal{C})$ is the free abelian group on the set of isomorphism classes of indecomposable objects.

5.3 Use Ex. 5.2 to prove Corollary 5.5.1.

5.4 Let R be a commutative ring, and let $H^0(\mathrm{Spec}\, R, \mathbb{Q}^\times_{>0})$ denote the free abelian group of all continuous maps $\mathrm{Spec}(R) \to \mathbb{Q}^\times_{>0}$. Show that $[P] \mapsto \mathrm{rank}(P)$ induces a split surjection from $K_0\mathbf{FP}(R)$ onto $H^0(\mathrm{Spec}\, R, \mathbb{Q}^\times_{>0})$. In the next two exercises, we shall show that the kernel of this map is isomorphic to $\widetilde{K}_0(R) \otimes \mathbb{Q}$.

5.5 Let R be a commutative ring, and let U_+ denote the subset of the ring $K_0(R) \otimes \mathbb{Q}$ consisting of all x such that $\mathrm{rank}(x)$ takes only positive values.

(a) Use the fact that the ideal $\widetilde{K}_0(R)$ is nilpotent to show that U_+ is an abelian group under multiplication, and that there is a split exact sequence

$$0 \to \widetilde{K}_0(R) \otimes \mathbb{Q} \xrightarrow{\exp} U_+ \xrightarrow{\mathrm{rank}} H^0(\mathrm{Spec}\, R, \mathbb{Q}^\times_{>0}) \to 0.$$

(b) Show that $P \mapsto [P] \otimes 1$ is an additive function from $\mathbf{FP}(R)$ to the multiplicative group U_+, and that it induces a map $K_0\mathbf{FP}(R) \to U_+$.

5.6 (Bass) Let R be a commutative ring. Show that the map $K_0\mathbf{FP}(R) \to U_+$ of the previous exercise is an isomorphism. *Hint:* The map is onto by Ex. 2.10. Conversely, if $[P] \otimes 1 = [Q] \otimes 1$ in U_+, show that $P \otimes R^n \cong Q \otimes R^n$ for some n.

5.7 Suppose that a finite group G acts freely on X, and let X/G denote the orbit space. Show that $\mathbf{VB}_G(X)$ is equivalent to the category $\mathbf{VB}(X/G)$, and conclude that $K_G^0(X) \cong KU(X/G)$.

5.8 Let R be a commutative ring. Show that the determinant of a projective module induces a monoidal functor $\det\colon \mathbf{P}(R) \to \mathbf{Pic}(R)$, and that the resulting map $K_0(\det)\colon K_0\mathbf{P}(R) \to K_0\mathbf{Pic}(R)$ is the determinant map $K_0(R) \to \mathrm{Pic}(R)$ of Proposition 2.6.

5.9 Let G be a finite group. Given a finite G-set X and a $\mathbb{Z}[G]$-module M, the abelian group $X \times M$ carries a $\mathbb{Z}[G]$-module structure by $g(x, m) = (gx, gm)$. Show that $X \times -$ induces an additive functor from $\mathbf{P}(\mathbb{Z}[G])$ to itself (2.8). Then show that the pairing $(X, M) \mapsto X \times M$ makes $K_0(\mathbb{Z}[G])$ into a module over the Burnside ring $A(G)$.

5.10 If $X = (V, B)$ and $X' = (V', B')$ are two inner product spaces, show that there is a nondegenerate bilinear form β on $V \otimes V'$ satisfying $\beta(v \otimes v', w \otimes w') = B(v, w)B'(v', w')$ for all $v, w \in V$ and $v', w' \in V'$. Writing $X \otimes X'$ for this inner product space, show that $X \otimes X' \cong X' \otimes X$ and $(X_1 \oplus X_2) \otimes X' \cong (X_1 \otimes X') \oplus (X_2 \otimes X')$. Then show that $X \otimes H \cong H \oplus \cdots \oplus H$.

5.11 A symmetric inner product space $S = (V, B)$ is called *split* if it has a basis so that B is represented by a matrix $\left(\begin{smallmatrix} 0 & I \\ I & A \end{smallmatrix}\right)$. Note that the sum of split spaces is also split, and that the hyperbolic plane is split. We define $W(F)$ to be the quotient of $K_0\mathbf{SBil}(F)$ by the subgroup of classes $[S]$ of split spaces.

(a) If $\mathrm{char}(F) \neq 2$, show that every split space S is hyperbolic. Conclude that this definition of $W(F)$ agrees with the definition given in 5.6.2.

(b) For any $a \in F^\times$, show that $\langle a \rangle \oplus \langle -a \rangle$ is split.

(c) If S is split, show that each $(V, B) \otimes S$ is split. In particular, $(V, B) \oplus (V, -B) = (V, B) \otimes \big(\langle 1 \rangle \oplus \langle -1 \rangle\big)$ is split. Conclude that $W(F)$ is also a ring when $\mathrm{char}(F) = 2$.

(d) If $\mathrm{char}(F) = 2$, show that the split space $S = \langle 1 \rangle \oplus \langle 1 \rangle$ is not hyperbolic, yet $\langle 1 \rangle \oplus S \cong \langle 1 \rangle \oplus H$. This shows that Witt Cancellation fails if $\mathrm{char}(F) = 2$. *Hint:* consider the associated quadratic forms. Then consider the basis $(1, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ of $\langle 1 \rangle \oplus S$.

5.12 If $a + b = 1$ in F, show that $\langle a \rangle \oplus \langle b \rangle \cong \langle ab \rangle \oplus \langle 1 \rangle$. Conclude that in both $K_0\mathbf{SBil}(F)$ and $W(F)$ we have the Steinberg identity $\big(\langle a \rangle - 1\big)\big(\langle b \rangle - 1\big) = 0$.

5.13 Suppose that $\mathrm{char}(F) = 2$ and that (V, q) is a quadratic form.

(a) Show that $B_q(v, v) = 0$ for every $v \in V$.

(b) Show that the underlying inner product space (V, B_q) is hyperbolic, hence split in the sense of Ex. 5.11. This shows that $\dim(V)$ is even, and that the map $WQ(F) \to W(F)$ is zero. *Hint:* Find two elements x, y in V so that $B_q(x, y) = 1$, and show that they span an orthogonal summand of V.

(c) If (W, β) is a symmetric inner product space, show that there is a unique quadratic form q' on $V' = V \otimes W$ satisfying $q'(v \oplus w) = q(v)\beta(w, w)$, such that the underlying bilinear form satisfies $B_{q'}(v \otimes w, v' \otimes w') = B_q(v, v')\beta(w, w')$. Show that this product makes $WQ(F)$ into a module over $W(F)$.

(d) (Arf invariant) Let $\wp\colon F \to F$ denote the additive map $\wp(a) = a^2 + a$. By (b), we may choose a basis $x_1, \dots, x_n, y_1 \dots, y_n$ of V so that each x_i, y_i span a hyperbolic plane. Show that the element $\Delta(V, q) = \sum q(x_i)q(y_i)$ of $F/\wp(F)$ is independent of the choice of basis, called the *Arf invariant* of the quadratic space (after C. Arf, who discovered it in 1941). Then show that Δ is an additive surjection. H. Sah showed that the Arf invariant and the module structure in (c) induces an isomorphism $WQ(F)/I \cdot WQ(F) \cong F/\wp(F)$.

(e) Consider the quadratic forms $q(a, b) = a^2 + ab + b^2$ and $q'(a, b) = ab$ on $V = F^2$. Show that they are isometric if and only if F contains the field $\mathbb{F}_4$.

5.14 (Kato) If $\mathrm{char}(F) = 2$, show that there is a ring homomorphism $W(F) \to F \otimes_{F^p} F$ sending $\langle a \rangle$ to $a^{-1} \otimes a$.

§6. K_0 of an Abelian Category

Another important situation in which we can define Grothendieck groups is when we have a (skeletally) small abelian category. This is due to the natural notion of exact sequence in an abelian category. We begin by quickly reminding the reader what an abelian category is, defining K_0 and then making a set-theoretic remark.

It helps to read the definitions below with some examples in mind. The reader should remember that the prototype abelian category is the category **mod**-R of right modules over a ring R, the morphisms being R-module homomorphisms. The full subcategory with objects the free R-modules $\{0, R, R^2, \dots\}$ is additive, and so is the slightly larger full subcategory $\mathbf{P}(R)$ of finitely generated projective R-modules (this observation was already made in chapter I). For more information on abelian categories, see textbooks like [Mac] or [WHomo].

Definitions 6.1. (1) An *additive category* is a category containing a zero object '0' (an object which is both initial and terminal), having all products $A \times B$, and such that every set $\mathrm{Hom}(A, B)$ is given the structure of an abelian group in such a way that composition is bilinear. In an additive category the product $A \times B$ is also the coproduct $A \amalg B$ of A and B; we call it the *direct sum* and write it as $A \oplus B$ to remind ourselves of this fact.

(2) An *abelian category* $\mathcal{A}$ is an additive category in which (i) every morphism $f\colon B \to C$ has a kernel and a cokernel, and (ii) every monic arrow is a kernel, and every epi is a cokernel. (Recall that $f\colon B \to C$ is called *monic* if $fe_1 \neq fe_2$ for every $e_1 \neq e_2\colon A \to B$; it is called *epi* if $g_1 f \neq g_2 f$ for every $g_1 \neq g_2\colon C \to D$.)

(3) In an abelian category, we call a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ *exact* if $\ker(g)$ equals $\mathrm{im}(f) \equiv \ker\{B \to \mathrm{coker}(f)\}$. A longer sequence is *exact* if it is exact at all places. By the phrase *short exact sequence* in an abelian category $\mathcal{A}$ we mean an exact sequence of the form:

$$0 \to A' \to A \to A'' \to 0. \qquad (*)$$

Definition 6.1.1 ($K_0\mathcal{A}$). Let $\mathcal{A}$ be an abelian category. Its *Grothendieck group* $K_0(\mathcal{A})$ is the abelian group presented as having one generator $[A]$ for each object A of $\mathcal{A}$, with one relation $[A] = [A'] + [A'']$ for every short exact sequence $(*)$ in $\mathcal{A}$.

Here are some useful identities which hold in $K_0(\mathcal{A})$.

a) $[0] = 0$ (take $A = A'$).

b) if $A \cong A'$ then $[A] = [A']$ (take $A'' = 0$).

c) $[A' \oplus A''] = [A'] + [A'']$ (take $A = A' \oplus A''$).

If two abelian categories are equivalent, their Grothendieck groups are naturally isomorphic, as *b*) implies they have the same presentation. By *c*), the group $K_0(\mathcal{A})$ is a quotient of the group $K_0^{\oplus}(\mathcal{A})$ defined in §5 by considering $\mathcal{A}$ as a symmetric monoidal category.

Universal Property 6.1.2. An *additive function* from $\mathcal{A}$ to an abelian group Γ is a function f from the objects of $\mathcal{A}$ to Γ such that $f(A) = f(A') + f(A'')$ for every short exact sequence $(*)$ in $\mathcal{A}$. By construction, the function $A \mapsto [A]$ defines an additive function from $\mathcal{A}$ to $K_0(\mathcal{A})$. This has the following universal property: any additive function f from $\mathcal{A}$ to Γ induces a unique group homomorphism $f\colon K_0(\mathcal{A}) \to \Gamma$, with $f([A]) = f(A)$ for every A.

For example, the direct sum $\mathcal{A}_1 \oplus \mathcal{A}_2$ of two abelian categories is also abelian. Using the universal property of K_0 it is clear that $K_0(\mathcal{A}_1 \oplus \mathcal{A}_2) \cong K_0(\mathcal{A}_1) \oplus K_0(\mathcal{A}_2)$. More generally, an arbitrary direct sum $\bigoplus \mathcal{A}_i$ of abelian categories is abelian, and we have $K_0(\bigoplus \mathcal{A}_i) \cong \bigoplus K_0(\mathcal{A}_i)$.

Set-theoretic Considerations 6.1.3. There is an obvious set-theoretic difficulty in defining $K_0\mathcal{A}$ when $\mathcal{A}$ is not small; recall that a category $\mathcal{A}$ is called *small* if the class of objects of $\mathcal{A}$ forms a set.

We will always implicitly assume that our abelian category $\mathcal{A}$ is *skeletally small*, *i.e.*, it is equivalent to a small abelian category $\mathcal{A}'$. In this case we define $K_0(\mathcal{A})$ to be $K_0(\mathcal{A}')$. Since any other small abelian category equivalent to $\mathcal{A}$ will also be equivalent to $\mathcal{A}'$, the definition of $K_0(\mathcal{A})$ is independent of this choice.

Example 6.1.4 (All R-modules). We cannot take the Grothendieck group of the abelian category $\mathbf{mod}$-R because it is not skeletally small. To finesse this difficulty, fix an infinite cardinal number κ and let $\mathbf{mod}_\kappa$-R denote the full subcategory of $\mathbf{mod}$-R consisting of all R-modules of cardinality $< \kappa$. As long as $\kappa \geq |R|$, $\mathbf{mod}_\kappa$-R is an abelian subcategory of $\mathbf{mod}$-R having a set of isomorphism classes of objects. The Eilenberg Swindle I.2.8 applies to give $K_0(\mathbf{mod}_\kappa\text{-}R) = 0$. In effect, the formula $M \oplus M^\infty \cong M^\infty$ implies that $[M] = 0$ for every module M.

6.1.5. The natural type of functor $F: \mathcal{A} \to \mathcal{B}$ between two abelian categories is an *additive* functor; this is a functor for which all the maps $\operatorname{Hom}(A, A') \to \operatorname{Hom}(FA, FA')$ are group homomorphisms. However, not all additive functors induce homomorphisms $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

We say that an additive functor F is *exact* if it preserves exact sequences—that is, for every exact sequence $(*)$ in $\mathcal{A}$, the sequence $0 \to F(A') \to F(A) \to F(A'') \to 0$ is exact in $\mathcal{B}$. The presentation of K_0 implies that any exact functor F defines a group homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{B})$ by the formula $[A] \mapsto [F(A)]$.

Suppose given an inclusion $\mathcal{A} \subset \mathcal{B}$ of abelian categories, with $\mathcal{A}$ a full subcategory of $\mathcal{B}$. If the inclusion is an exact functor, we say that $\mathcal{A}$ is an *exact abelian subcategory* of $\mathcal{B}$. As with all exact functors, the inclusion induces a natural map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$.

Definition 6.2 (G_0R). If R is a (right) noetherian ring, let $\mathbf{M}(R)$ denote the subcategory of $\mathbf{mod}$-R consisting of all finitely generated R-modules. The noetherian hypothesis implies that $\mathbf{M}(R)$ is an abelian category, and we write $G_0(R)$ for $K_0\mathbf{M}(R)$. (We will give a definition of $\mathbf{M}(R)$ and $G_0(R)$ for non-noetherian rings in Example 7.1.4 below.)

The presentation of $K_0(R)$ in §2 shows that there is a natural map $K_0(R) \to G_0(R)$, which is called the *Cartan homomorphism* (send $[P]$ to $[P]$).

Associated to a ring homomorphism $f: R \to S$ are two possible maps on G_0: the contravariant transfer map and the covariant base change map.

When S is finitely generated as an R-module (*e.g.*, $S = R/I$), there is a "transfer" homomorphism $f_*: G_0(S) \to G_0(R)$. It is induced from the forgetful functor $f_*: \mathbf{M}(S) \to \mathbf{M}(R)$, which is exact.

Whenever S is flat as an R-module, there is a "base change" homomorphism $f^*: G_0(R) \to G_0(S)$. Indeed, the base change functor $f^*: \mathbf{M}(R) \to \mathbf{M}(S)$, $f^*(M) = M \otimes_R S$, is exact if and only if S is flat over R. We will extend the definition of f^*

in §7 to the case in which S has a finite resolution by flat R-modules using Serre's Formula (7.9.3): $f^*([M]) = \sum(-1)^i[\mathrm{Tor}_i^R(M,S)]$.

If F is a field then every exact sequence in $\mathbf{M}(F)$ splits, and it is easy to see that $G_0(F) \cong K_0(F) \cong \mathbb{Z}$. In particular, if R is an integral domain with field of fractions F, then there is a natural map $G_0(R) \to G_0(F) = \mathbb{Z}$, sending $[M]$ to the integer $\dim_F(M \otimes_R F)$.

EXAMPLE 6.2.1 (ABELIAN GROUPS). When $R = \mathbb{Z}$ the Cartan homomorphism is an isomorphism: $K_0(\mathbb{Z}) \cong G_0(\mathbb{Z}) \cong \mathbb{Z}$. To see this, first observe that the sequences

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

imply that $[\mathbb{Z}/n\mathbb{Z}] = [\mathbb{Z}] - [\mathbb{Z}] = 0$ in $G_0(\mathbb{Z})$ for every n. By the Fundamental Theorem of finitely generated Abelian Groups, every finitely generated abelian group M is a finite sum of copies of the groups $\mathbb{Z}$ and $\mathbb{Z}/n$, $n \geq 2$. Hence $G_0(\mathbb{Z})$ is generated by $[\mathbb{Z}]$. To see that $G_0(\mathbb{Z}) \cong \mathbb{Z}$, observe that since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module there is a homomorphism from $G_0(\mathbb{Z})$ to $G_0(\mathbb{Q}) \cong \mathbb{Z}$ sending $[M]$ to $r(M) = \dim_{\mathbb{Q}}(M \otimes \mathbb{Q})$. In effect, $r(M)$ is an additive function; as such it induces a homomorphism $r: G_0(\mathbb{Z}) \to \mathbb{Z}$. As $r(\mathbb{Z}) = 1$, r is an isomorphism.

More generally, the Cartan homomorphism is an isomorphism whenever R is a principal ideal domain, and $K_0(R) \cong G_0(R) \cong \mathbb{Z}$. The proof is identical.

EXAMPLE 6.2.2 (p-GROUPS). Let $\mathbf{Ab}_p$ denote the abelian category of all finite abelian p-groups for some prime p. Then $K_0(\mathbf{Ab}_p) \cong \mathbb{Z}$ on generator $[\mathbb{Z}/p]$. To see this, we observe that the length $\ell(M)$ of a composition series for a finite p-group M is well-defined by the Jordan-Hölder Theorem. Moreover ℓ is an additive function, and defines a homomorphism $K_0(\mathbf{Ab}_p) \to \mathbb{Z}$ with $\ell(\mathbb{Z}/p) = 1$. To finish we need only observe that $\mathbb{Z}/p$ generates $K_0(\mathbf{Ab}_p)$; this follows by induction on the length of a p-group, once we observe that any $L \subset M$ yields $[M] = [L] + [M/L]$ in $K_0(\mathbf{Ab}_p)$.

EXAMPLE 6.2.3. The category $\mathbf{Ab}_{fin}$ of all finite abelian groups is the direct sum of the categories $\mathbf{Ab}_p$ of Example 6.2.2. Therefore $K_0(\mathbf{Ab}_{fin}) = \bigoplus K_0(\mathbf{Ab}_p)$ is the free abelian group on the set $\{[\mathbb{Z}/p],\ p \text{ prime}\}$.

EXAMPLE 6.2.4. The category $\mathbf{M}(\mathbb{Z}/p^n)$ of all finite $\mathbb{Z}/p^n$-modules is an exact abelian subcategory of $\mathbf{Ab}_p$, and the argument above applies verbatim to prove that the simple module $[\mathbb{Z}/p]$ generates the group $G_0(\mathbb{Z}/p^n) \cong \mathbb{Z}$. In particular, the canonical maps from $G_0(\mathbb{Z}/p^n) = K_0\mathbf{M}(\mathbb{Z}/p^n)$ to $K_0(\mathbf{Ab}_p)$ are all isomorphisms.

Recall from Lemma 2.2 that $K_0(\mathbb{Z}/p^n) \cong \mathbb{Z}$ on $[\mathbb{Z}/p^n]$. The Cartan homomorphism from $K_0 \cong \mathbb{Z}$ to $G_0 \cong \mathbb{Z}$ is not an isomorphism; it sends $[\mathbb{Z}/p^n]$ to $n[\mathbb{Z}/p]$.

DEFINITION 6.2.5 ($G_0(X)$). Let X be a noetherian scheme. The category $\mathbf{M}(X)$ of all coherent $\mathcal{O}_X$-modules is an abelian category. (See [Hart, II.5.7].) We write $G_0(X)$ for $K_0\mathbf{M}(X)$. When $X = \mathrm{Spec}(R)$ this agrees with Definition 6.2: $G_0(X) \cong G_0(R)$, because of the equivalence of $\mathbf{M}(X)$ and $\mathbf{M}(R)$.

If $f: X \to Y$ is a morphism of schemes, there is a *base change functor* $f^*: \mathbf{M}(Y) \to \mathbf{M}(X)$ sending $\mathcal{F}$ to $f^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_X$; see I.5.2. When f is flat, the base change f^* is exact and therefore the formula $f^*([\mathcal{F}]) = [f^*\mathcal{F}]$ defines a homomorphism $f^*: G_0(Y) \to G_0(X)$. Thus G_0 is contravariant for flat maps.

If $f: X \to Y$ is a finite morphism, the direct image $f_*\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ is coherent, and $f_*: \mathbf{M}(X) \to \mathbf{M}(Y)$ is an exact functor [EGA, I(1.7.8)]. In this case the formula $f_*([\mathcal{F}]) = [f_*\mathcal{F}]$ defines a "transfer" map $f_*: G_0(X) \to G_0(Y)$.

If $f: X \to Y$ is a proper morphism, the direct image $f_*\mathcal{F}$ of a coherent sheaf $\mathcal{F}$ is coherent, and so are its higher direct images $R^i f_*\mathcal{F}$. (This is Serre's "Theorem B"; see I.5.2 or [EGA, III(3.2.1)].) The functor $f_*: \mathbf{M}(X) \to \mathbf{M}(Y)$ is not usually exact (unless f is finite). Instead we have:

LEMMA 6.2.6. *If $f: X \to Y$ is a proper morphism of noetherian schemes, there is a "transfer" homomorphism $f_*: G_0(X) \to G_0(Y)$. It is defined by the formula $f_*([\mathcal{F}]) = \sum(-1)^i[R^i f_*\mathcal{F}]$. The transfer homomorphism makes G_0 functorial for proper maps.*

PROOF. For each coherent $\mathcal{F}$ the $R^i f_*\mathcal{F}$ vanish for large i, so the sum is finite. By 6.2.1 it suffices to show that the formula gives an additive function. But if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence in $\mathbf{M}(X)$ there is a finite long exact sequence in $\mathbf{M}(Y)$:

$$0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to R^1 f_*\mathcal{F}' \to R^1 f_*\mathcal{F} \to R^1 f_*\mathcal{F}'' \to R^2 f_*\mathcal{F}' \to \cdots$$

and the alternating sum of the terms is $f_*[\mathcal{F}'] - f_*[\mathcal{F}] + f_*[\mathcal{F}'']$. This alternating sum must be zero by Proposition 6.6 below, so f_* is additive as desired. (Functoriality is relegated to Ex. 6.15.) $\square$

The next lemma follows by inspection of the definition of the direct limit (or filtered colimit) $\mathcal{A} = \varinjlim \mathcal{A}_i$ of a filtered system of small categories; the objects and morphisms of $\mathcal{A}$ are the direct limits of the object and morphisms of the $\mathcal{A}_i$.

LEMMA 6.2.7 (FILTERED COLIMITS). *Suppose that $\{\mathcal{A}_i\}_{i\in I}$ is a filtered family of small abelian categories and exact functors. Then the direct limit $\mathcal{A} = \varinjlim \mathcal{A}_i$ is also an abelian category, and*

$$K_0(\mathcal{A}) = \varinjlim K_0(\mathcal{A}_i).$$

EXAMPLE 6.2.8 (S-TORSION MODULES). Suppose that S is a multiplicatively closed set of elements in a noetherian ring R. Let $\mathbf{M}_S(R)$ be the subcategory of $\mathbf{M}(R)$ consisting of all finitely generated R-modules M such that $Ms = 0$ for some $s \in S$. For example, if $S = \{p^n\}$ then $\mathbf{M}_S(\mathbb{Z}) = \mathbf{Ab}_p$ was discussed in Example 6.2.2. In general $\mathbf{M}_S(R)$ is not only the union of the $\mathbf{M}(R/RsR)$, but is also the union of the $\mathbf{M}(R/I)$ as I ranges over the ideals of R with $I \cap S \neq \phi$. By 6.2.7,

$$K_0\mathbf{M}_S(R) = \varinjlim_{I\cap S\neq\phi} G_0(R/I) = \varinjlim_{s\in S} G_0(R/RsR).$$

Devissage

The method behind the computation in Example 6.2.4 that $G_0(\mathbb{Z}/p^n) \cong K_0\mathbf{Ab}_p$ is called Devissage, a French word referring to the "unscrewing" of the composition series. Here is a formal statement of the process, due to Alex Heller.

Devissage Theorem 6.3. *Let $\mathcal{B} \subset \mathcal{A}$ be small abelian categories. Suppose that*

(a) *$\mathcal{B}$ is an exact abelian subcategory of $\mathcal{A}$, closed in $\mathcal{A}$ under subobjects and quotient objects; and*

(b) *Every object A of $\mathcal{A}$ has a finite filtration $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ with all quotients A_i/A_{i+1} in $\mathcal{B}$.*

Then the inclusion functor $\mathcal{B} \subset \mathcal{A}$ is exact and induces an isomorphism

$$K_0(\mathcal{B}) \cong K_0(\mathcal{A}).$$

Proof. Let $i_*: K_0(\mathcal{B}) \to K_0(\mathcal{A})$ denote the canonical homomorphism. To see that i_* is onto, observe that every filtration $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ yields $[A] = \sum [A_i/A_{i+1}]$ in $K_0(\mathcal{A})$. This follows by induction on n, using the observation that $[A_i] = [A_{i+1}] + [A_i/A_{i+1}]$. Since by (b) such a filtration exists with the A_i/A_{i+1} in $\mathcal{B}$, this shows that the canonical i_* is onto.

For each A in $\mathcal{A}$, fix a filtration $A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$ with each A_i/A_{i+1} in $\mathcal{B}$, and define $f(A)$ to be the element $\sum [A_i/A_{i+1}]$ of $K_0(\mathcal{B})$. We claim that $f(A)$ is independent of the choice of filtration. Because any two filtrations have equivalent refinements (Ex. 6.2), we only need check refinements of our given filtration. By induction we need only check for one insertion, say changing $A_i \supset A_{i+1}$ to $A_i \supset A' \supset A_{i+1}$. Appealing to the exact sequence

$$0 \to A'/A_{i+1} \to A_i/A_{i+1} \to A_i/A' \to 0,$$

we see that $[A_i/A_{i+1}] = [A_i/A'] + [A'/A_{i+1}]$ in $K_0(\mathcal{B})$, as claimed.

Given a short exact sequence $0 \to A' \to A \to A'' \to 0$, we may construct a filtration $\{A_i\}$ on A by combining our chosen filtration for A' with the inverse image in A of our chosen filtration for A''. For this filtration we have $\sum [A_i/A_{i+1}] = f(A') + f(A'')$. Therefore f is an additive function, and defines a map $K_0(\mathcal{A}) \to K_0(\mathcal{B})$. By inspection, f is the inverse of the canonical map i_*. □

Corollary 6.3.1. *Let I be a nilpotent ideal of a noetherian ring R. Then the inclusion* **mod**-$(R/I) \subset$ **mod**-R *induces an isomorphism*

$$G_0(R/I) \cong G_0(R).$$

Proof. To apply Devissage, we need to observe that if M is a finitely generated R-module, the filtration $M \supseteq MI \supseteq MI^2 \supseteq \cdots \supseteq MI^n = 0$ is finite, and all the quotients MI^n/MI^{n+1} are finitely generated R/I-modules. □

Notice that this also proves the scheme version:

Corollary 6.3.2. *Let X be a noetherian scheme, and X_{red} the associated reduced scheme. Then $G_0(X) \cong G_0(X_{\text{red}})$.*

Application 6.3.3 (R-modules with support). Example 6.2.2 can be generalized as follows. Given a central element s in a ring R, let $\mathbf{M}_s(R)$ denote the abelian subcategory of $\mathbf{M}(R)$ consisting of all finitely generated R-modules M such that $Ms^n = 0$ for some n. That is, modules such that $M \supset Ms \supset Ms^2 \supset \cdots$ is a finite filtration. By Devissage,

$$K_0\mathbf{M}_s(R) \cong G_0(R/sR).$$

More generally, suppose we are given an ideal I of R. Let $\mathbf{M}_I(R)$ be the (exact) abelian subcategory of $\mathbf{M}(R)$ consisting of all finitely generated R-modules M such that the filtration $M \supset MI \supset MI^2 \supset \cdots$ is finite, *i.e.,* such that $MI^n = 0$ for some n. By Devissage,

$$K_0\mathbf{M}_I(R) \cong K_0\mathbf{M}(R/I) = G_0(R/I).$$

EXAMPLE 6.3.4. Let X be a noetherian scheme, and $i: Z \subset X$ the inclusion of a closed subscheme. Let $\mathbf{M}_Z(X)$ denote the abelian category of coherent $\mathcal{O}_X$-modules Z supported on Z, and $\mathcal{I}$ the ideal sheaf in $\mathcal{O}_X$ such that $\mathcal{O}_X/\mathcal{I} \cong \mathcal{O}_Z$. Via the direct image $i_*: \mathbf{M}(Z) \subset \mathbf{M}(X)$, we can consider $\mathbf{M}(Z)$ as the subcategory of all modules M in $\mathbf{M}_Z(X)$ such that $\mathcal{I}M = 0$. Every M in $\mathbf{M}_Z(X)$ has a finite filtration $M \supset M\mathcal{I} \supset M\mathcal{I}^2 \supset \cdots$ with quotients in $\mathbf{M}(Z)$, so by Devissage:

$$K_0\mathbf{M}_Z(X) \cong K_0\mathbf{M}(Z) = G_0(Z).$$

The Localization Theorem

Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* of $\mathcal{A}$ is an abelian subcategory $\mathcal{B}$ which is closed under subobjects, quotients and extensions. That is, if $0 \to B \to C \to D \to 0$ is exact in $\mathcal{A}$ then

$$C \in \mathcal{B} \Leftrightarrow B, D \in \mathcal{B}.$$

Now assume for simplicity that $\mathcal{A}$ is small. If $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$, we can form a quotient abelian category $\mathcal{A}/\mathcal{B}$ as follows. Call a morphism f in $\mathcal{A}$ a $\mathcal{B}$*-iso* if $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{B}$. The objects of $\mathcal{A}/\mathcal{B}$ are the objects of $\mathcal{A}$, and morphisms $A_1 \to A_2$ are equivalence classes of diagrams in $\mathcal{A}$:

$$A_1 \xleftarrow{f} A' \xrightarrow{g} A_2, \quad f \text{ a } \mathcal{B}\text{-iso}.$$

Such a morphism is equivalent to $A_1 \leftarrow A'' \to A_2$ if and only if there is a commutative diagram:

$$\begin{array}{ccccc} & & A' & & \\ & \swarrow & \uparrow & \searrow & \\ A_1 & \leftarrow & A & \rightarrow & A_2 \\ & \nwarrow & \downarrow & \nearrow & \\ & & A'' & & \end{array} \qquad \text{where } A' \leftarrow A \to A'' \text{ are } \mathcal{B}\text{-isos.}$$

The composition with $A_2 \xleftarrow{f'} A'' \xrightarrow{h} A_3$ is $A_1 \xleftarrow{f} A' \leftarrow A \to A'' \xrightarrow{h} A_3$, where A is the pullback of A' and A'' over A_2. The proof that $\mathcal{A}/\mathcal{B}$ is abelian, and that the quotient functor $\operatorname{loc}: \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is exact, may be found in [Swan, p.44ff] or [Gabriel]. (See the appendix to this chapter.)

It is immediate from the construction of $\mathcal{A}/\mathcal{B}$ that $\operatorname{loc}(A) \cong 0$ if and only if A is an object of $\mathcal{B}$, and that for a morphism $f: A \to A'$ in $\mathcal{A}$, $\operatorname{loc}(f)$ is an isomorphism if and only if f is a $\mathcal{B}$-iso. In fact $\mathcal{A}/\mathcal{B}$ solves a universal problem (see *op. cit.*): if $T: \mathcal{A} \to \mathcal{C}$ is an exact functor such that $T(B) \cong 0$ for all B in $\mathcal{B}$, then there is a unique exact functor $T': \mathcal{A}/\mathcal{B} \to \mathcal{C}$ so that $T = T' \circ \operatorname{loc}$.

Localization Theorem 6.4. *(Heller) Let $\mathcal{A}$ be a small abelian category, and $\mathcal{B}$ a Serre subcategory of $\mathcal{A}$. Then the following sequence is exact:*

$$K_0(\mathcal{B}) \to K_0(\mathcal{A}) \xrightarrow{loc} K_0(\mathcal{A}/\mathcal{B}) \to 0.$$

Proof. By the construction of $\mathcal{A}/\mathcal{B}$, $K_0(\mathcal{A})$ maps onto $K_0(\mathcal{A}/\mathcal{B})$ and the composition $K_0(\mathcal{B}) \to K_0(\mathcal{A}/\mathcal{B})$ is zero. Hence if Γ denotes the cokernel of the map $K_0(\mathcal{B}) \to K_0(\mathcal{A})$ there is a natural surjection $\Gamma \to K_0(\mathcal{A}/\mathcal{B})$; to prove the theorem it suffices to give an inverse. For this it suffices to show that $\gamma(\text{loc}(A)) = [A]$ defines an additive function from $\mathcal{A}/\mathcal{B}$ to Γ, because the induced map $\gamma\colon K_0(\mathcal{A}/\mathcal{B}) \to \Gamma$ will be inverse to the natural surjection $\Gamma \to K_0(\mathcal{A}/\mathcal{B})$.

Since $\text{loc}\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a bijection on objects, γ is well-defined. We claim that if $\text{loc}(A_1) \cong \text{loc}(A_2)$ in $\mathcal{A}/\mathcal{B}$ then $[A_1] = [A_2]$ in Γ. To do this, represent the isomorphism by a diagram $A_1 \xleftarrow{f} A \xrightarrow{g} A_2$ with f a $\mathcal{B}$-iso. As $\text{loc}(g)$ is an isomorphism in $\mathcal{A}/\mathcal{B}$, g is also a $\mathcal{B}$-iso. In $K_0(\mathcal{A})$ we have

$$[A] = [A_1] + [\ker(f)] - [\text{coker}(f)] = [A_2] + [\ker(g)] - [\text{coker}(g)].$$

Hence $[A] = [A_1] = [A_2]$ in Γ, as claimed.

To see that γ is additive, suppose given an exact sequence in $\mathcal{A}/\mathcal{B}$ of the form:

$$0 \to \text{loc}(A_0) \xrightarrow{i} \text{loc}(A_1) \xrightarrow{j} \text{loc}(A_2) \to 0;$$

we have to show that $[A_1] = [A_0] + [A_2]$ in Γ. Represent j by a diagram $A_1 \xleftarrow{f} A \xrightarrow{g} A_2$ with f a $\mathcal{B}$-iso. Since $[A] = [A_1] + [\ker(f)] - [\text{coker}(f)]$ in $K_0(\mathcal{A})$, $[A] = [A_1]$ in Γ. Applying the exact functor loc to

$$0 \to \ker(g) \to A \xrightarrow{g} A_2 \to \text{coker}(g) \to 0,$$

we see that $\text{coker}(g)$ is in $\mathcal{B}$ and that $\text{loc}(\ker(g)) \cong \text{loc}(A_0)$ in $\mathcal{A}/\mathcal{B}$. Hence $[\ker(g)] \equiv [A_0]$ in Γ, and in Γ we have

$$[A_1] = [A] = [A_2] + [\ker(g)] - [\text{coker}(g)] \equiv [A_0] + [A_2]$$

proving that γ is additive, and finishing the proof of the Localization Theorem. □

Application 6.4.1. Let S be a central multiplicative set in a ring R, and let $\mathbf{mod}_S(R)$ denote the Serre subcategory of $\mathbf{mod}\text{-}R$ consisting of S-torsion modules, *i.e.*, those R-modules M such that every $m \in M$ has $ms = 0$ for some $s \in S$. Then there is a natural equivalence between $\mathbf{mod}\text{-}(S^{-1}R)$ and the quotient category $\mathbf{mod}\text{-}R/\mathbf{mod}_S(R)$. If R is noetherian and $\mathbf{M}_S(R)$ denotes the Serre subcategory of $\mathbf{M}(R)$ consisting of finitely generated S-torsion modules, then $\mathbf{M}(S^{-1}R)$ is equivalent to $\mathbf{M}(R)/\mathbf{M}_S(R)$. The Localization exact sequence becomes:

$$K_0\mathbf{M}_S(R) \to G_0(R) \to G_0(S^{-1}R) \to 0.$$

In particular, if $S = \{s^n\}$ for some s then by Application 6.3.3 we have an exact sequence

$$G_0(R/sR) \to G_0(R) \to G_0(R[\frac{1}{s}]) \to 0.$$

More generally, if I is an ideal of a noetherian ring R, we can consider the Serre subcategory $\mathbf{M}_I(R)$ of modules with some $MI^n = 0$ discussed in Application 6.3.3. The quotient category $\mathbf{M}(R)/\mathbf{M}_I(R)$ is known to be isomorphic to the category $\mathbf{M}(U)$ of coherent $\mathcal{O}_U$-modules, where U is the open subset of Spec(R) defined by I. The composition of the isomorphism $K_0\mathbf{M}(R/I) \cong K_0\mathbf{M}_I(R)$ of 6.3.3 with $K_0\mathbf{M}_I(R) \to K_0\mathbf{M}(R)$ is evidently the transfer map $i_*: G_0(R/I) \to G_0(R)$. Hence the Localization Sequence becomes the exact sequence

$$G_0(R/I) \xrightarrow{i_*} G_0(R) \to G_0(U) \to 0$$

Application 6.4.2. Let X be a scheme, and $i: Z \subset X$ a closed subscheme with complement $j: U \subset X$. Let $\mathbf{mod}_Z(X)$ denote the Serre subcategory of $\mathcal{O}_X$-**mod** consisting of all $\mathcal{O}_X$-modules $\mathcal{F}$ with support in Z, *i.e.,* such that $\mathcal{F}|_U = 0$. Gabriel proved [Gabriel] that j^* induces an equivalence: $\mathcal{O}_U$-**mod** $\cong \mathcal{O}_X$-**mod**/$\mathbf{mod}_Z(X)$.

Morover, if X is noetherian and $\mathbf{M}_Z(X)$ denotes the category of coherent sheaves supported in Z, then $\mathbf{M}(X)/\mathbf{M}_Z(X) \cong \mathbf{M}(U)$. The inclusion $i: Z \subset X$ induces an exact functor $i_*: \mathbf{M}(Z) \subset \mathbf{M}(X)$, and $G_0(Z) \cong K_0\mathbf{M}_Z(X)$ by Example 6.3.4. Therefore the Localization sequence becomes:

$$G_0(Z) \xrightarrow{i_*} G_0(X) \xrightarrow{j^*} G_0(U) \to 0.$$

For example, if $X = \text{Spec}(R)$ and $Z = \text{Spec}(R/I)$, we recover the exact sequence in the previous application.

Application 6.4.3 (Higher Divisor Class Groups). Given a commutative noetherian ring R, let $D^i(R)$ denote the free abelian group on the set of prime ideals of height exactly i; this is generalizes the group of Weil divisors in Ch.I, §3. Let $\mathbf{M}^i(R)$ denote the category of finitely generated R-modules M whose associated prime ideals all have height $\geq i$. Each $\mathbf{M}^i(R)$ is a Serre subcategory of $\mathbf{M}(R)$; see Ex. 6.9. Let $F^iG_0(R)$ denote the image of $K_0\mathbf{M}^i(R)$ in $G_0(R) = K_0\mathbf{M}(R)$. These subgroups form a filtration $\cdots \subset F^2 \subset F^1 \subset F^0 = G_0(R)$, called the *coniveau filtration* of $G_0(R)$.

It turns out that there is an equivalence $\mathbf{M}^i/\mathbf{M}^{i+1}(R) \cong \bigoplus \mathbf{M}_\mathfrak{p}(R_\mathfrak{p})$, $ht(\mathfrak{p}) = i$. By Application 6.3.3 of Devissage, $K_0\mathbf{M}_\mathfrak{p}(R_\mathfrak{p}) \cong G_0(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \cong \mathbb{Z}$, so there is an isomorphism $D^i(R) \xrightarrow{\cong} K_0\mathbf{M}^i/\mathbf{M}^{i+1}(R)$, $[\mathfrak{p}] \mapsto [R/\mathfrak{p}]$. By the Localization Theorem, we have an exact sequence

$$K_0\mathbf{M}^{i+1}(R) \to K_0\mathbf{M}^i(R) \to D^i(R) \to 0.$$

Thus $G_0(R)/F^1 \cong D^0(R)$, and each subquotient F^i/F^{i+1} is a quotient of $D^i(R)$.

For $i \geq 1$, the *generalized Weil divisor class group* $CH^i(R)$ is defined to be the subgroup of $K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}(R)$ generated by the classes $[R/\mathfrak{p}]$, $ht(\mathfrak{p}) \geq i$. This definition is due to L. Claborn and R. Fossum; the notation reflects a theorem (in V.9 below) that the kernel of $D^i(R) \to CH^i(R)$ is generated by rational equivalence. For example, we will see in Ex. 6.9 that if R is a Krull domain then $CH^1(R)$ is the usual divisor class group $Cl(R)$, and $G_0(R)/F^2 \cong \mathbb{Z} \oplus Cl(R)$.

Similarly, if X is a noetherian scheme, there is a coniveau filtration on $G_0(X)$. Let $\mathbf{M}^i(X)$ denote the subcategory of $\mathbf{M}(X)$ consisting of coherent modules whose support has codimension $\geq i$, and let $D^i(X)$ denote the free abelian group on the set of points of X having codimension i. Then each $\mathbf{M}^i(X)$ is a Serre subcategory and $\mathbf{M}^i/\mathbf{M}^{i+1}(X) \cong \bigoplus \mathbf{M}_x(\mathcal{O}_{X,x})$, where x runs over all points of codimension i in X. Again by Devissage, there is an isomorphism $K_0\mathbf{M}^i/\mathbf{M}^{i+1}(X) \cong D^i(X)$ and hence $G_0(X)/F^1 \cong D^0(X)$. For $i \geq 1$, the *generalized Weil divisor class group* $CH^i(X)$ is defined to be the subgroup of $K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}(X)$ generated by the classes $[\mathcal{O}_Z]$, $\mathrm{codim}_X(Z) = i$. We will see later on (in V.9.4.1) that $CH^i(X)$ is the usual Chow group of codimension i cycles on X modulo rational equivalence, as defined in [Fulton]. The verification that $CH^1(X) = Cl(X)$ is left to Ex. 6.10.

We now turn to a clasical application of the Localization Theorem: the Fundamental Theorem for G_0 of a noetherian ring R. Via the ring map $\pi\colon R[t] \to R$ sending t to zero, we have an inclusion $\mathbf{M}(R) \subset \mathbf{M}(R[t])$ and hence a transfer map $\pi_*\colon G_0(R) \to G_0(R[t])$. By 6.4.1 there is an exact localization sequence

$$G_0(R) \xrightarrow{\pi_*} G_0(R[t]) \xrightarrow{j^*} G_0(R[t,t^{-1}]) \to 0. \tag{6.4.4}$$

Given an R-module M, the exact sequence of $R[t]$-modules

$$0 \to M[t] \xrightarrow{t} M[t] \to M \to 0$$

shows that in $G_0(R[t])$ we have

$$\pi_*[M] = [M] = [M[t]] - [M[t]] = 0.$$

Thus $\pi_* = 0$, because every generator $[M]$ of $G_0(R)$ becomes zero in $G_0(R[t])$. From the Localization sequence (6.4.4) it follows that j^* is an isomorphism. This proves the easy part of the following result.

Fundamental Theorem for G_0-theory of Rings 6.5. *For every noetherian ring R, the inclusions $R \overset{i}{\hookrightarrow} R[t] \overset{j}{\hookrightarrow} R[t,t^{-1}]$ induce isomorphisms*

$$G_0(R) \cong G_0(R[t]) \cong G_0(R[t,t^{-1}]).$$

Proof. The ring inclusions are flat, so they induce maps $i^*\colon G_0(R) \to G_0(R[t])$ and $j^*\colon G_0(R[t]) \to G_0(R[t,t^{-1}])$. We have already seen that j^* is an isomorphism; it remains to show that i^* is an isomorphism.

Because $R = R[t]/tR[t]$, Serre's formula defines a map $\pi^*\colon G_0(R[t]) \to G_0(R)$ by the formula: $\pi^*[M] = [M/Mt] - [\mathrm{ann}_M(t)]$, where $\mathrm{ann}_M(t) = \{x \in M : xt = 0\}$. (See Ex. 6.6 or 7.9.3 below.) Since $\pi^* i^*[M] = \pi^*[M[t]] = [M]$, i^* is an injection split by π^*.

We shall present Grothendieck's proof that $i^*\colon G_0(R) \to G_0(R[t])$ is onto, which assumes that R is a commutative ring. A proof in the non-commutative case (due to Serre) will be sketched in Ex. 6.13.

If $G_0(R) \neq G_0(R[t])$, we proceed by noetherian induction to a contradiction. Among all ideals J for which $G_0(R/J) \neq G_0(R/J[t])$, there is a maximal one. Replacing R by R/J, we may assume that $G_0(R/I) = G_0(R/I[t])$ for each $I \neq 0$ in R. Such a ring R must be reduced by Corollary 6.3.1. Let S be the set of non-zero divisors in R; by elementary ring theory $S^{-1}R$ is a finite product $\prod F_i$ of fields F_i, so $G_0(S^{-1}R) \cong \oplus G_0(F_i)$. Similarly $S^{-1}R[t] = \prod F_i[t]$ and $G_0(S^{-1}R[t]) \cong \oplus G_0(F_i[t])$. By Application 6.4.1 and Example 6.2.8 we have a diagram with exact rows:

$$\begin{array}{ccccccc} \varinjlim G_0(R/sR) & \longrightarrow & G_0(R) & \longrightarrow & \oplus G_0(F_i) & \longrightarrow & 0 \\ \cong \downarrow i^* & & \downarrow i^* & & \downarrow & & \\ \varinjlim G_0(R/sR[t]) & \longrightarrow & G_0(R[t]) & \longrightarrow & \oplus G_0(F_i[t]) & \longrightarrow & 0. \end{array}$$

Since the direct limits are taken over all $s \in S$, the left vertical arrow is an isomorphism by induction. Because each $F_i[t]$ is a principal ideal domain, (2.6.3) and Example 6.2.1 imply that the right vertical arrow is the sum of the isomorphisms

$$G_0(F_i) \cong K_0(F_i) \cong \mathbb{Z} \cong K_0(F_i[t]) \cong G_0(F_i[t]).$$

By the 5-lemma, the middle vertical arrow is onto, hence an isomorphism. □

We can generalize the Fundamental Theorem from rings to schemes by a slight modification of the proof. For every scheme X, let $X[t]$ and $X[t,t^{-1}]$ denote the schemes $X \times \mathrm{Spec}(\mathbb{Z}[t])$ and $X \times \mathrm{Spec}(\mathbb{Z}[t,t^{-1}])$ respectively. Thus if $X = \mathrm{Spec}(R)$ we have $X[t] = \mathrm{Spec}(R[t])$ and $X[t,t^{-1}] = \mathrm{Spec}(R[t,t^{-1}])$. Now suppose that X is noetherian. Via the map $\pi: X \to X[t]$ defined by $t = 0$, we have an inclusion $\mathbf{M}(X) \subset \mathbf{M}(X[t])$ and hence a transfer map $\pi_*: G_0(X) \to G_0(X[t])$ as before. The argument we gave after (6.4.4) above goes through to show that $\pi_* = 0$ here too, because any generator $[\mathcal{F}]$ of $G_0(X)$ becomes zero in $G_0(X[t])$. By 6.4.2 we have an exact sequence

$$G_0(X) \xrightarrow{\pi_*} G_0(X[t]) \to G_0(X[t,t^{-1}]) \to 0$$

and therefore $G_0(X[t]) \cong G_0(X[t,t^{-1}])$.

Fundamental Theorem for G_0-theory of Schemes 6.5.1. *If X is a noetherian scheme then the flat maps $X[t,t^{-1}] \overset{j}{\hookrightarrow} X[t] \overset{i}{\to} X$ induce isomorphisms:*

$$G_0(X) \cong G_0(X[t]) \cong G_0(X[t,t^{-1}]).$$

Proof. We have already seen that j^* is an isomorphism. By Ex. 6.7 there is a map $\pi^*: G_0(X[t]) \to G_0(X)$ sending $[\mathcal{F}]$ to $[\mathcal{F}/t\mathcal{F}] - [\mathrm{ann}_{\mathcal{F}}(t)]$. Since $\pi^* i^*[\mathcal{F}] = (i\pi)^*[\mathcal{F}] = [\mathcal{F}]$, we again see that i^* is an injection, split by π^*.

It suffices to show that i^* is a surjection for all X. By noetherian induction, we may suppose that the result is true for all proper closed subschemes Z of X. In particular, if Z is the complement of an affine open subscheme $U = \mathrm{Spec}(R)$ of X, we have a commutative diagram whose rows are exact by Application 6.4.2.

$$\begin{array}{ccccccc} G_0(Z) & \longrightarrow & G_0(X) & \longrightarrow & G_0(R) & \longrightarrow & 0 \\ \cong \downarrow i^* & & \downarrow i^* & & \cong \downarrow i^* & & \\ G_0(Z[t]) & \longrightarrow & G_0(X[t]) & \longrightarrow & G_0(R[t]) & \longrightarrow & 0 \end{array}$$

The outside vertical arrows are isomorphisms, by induction and Theorem 6.5. By the 5-lemma, $G_0(X) \xrightarrow{i^*} G_0(X[t])$ is onto, and hence an isomorphism. □

Euler Characteristics

Suppose that $C_\bullet$: $0 \to C_m \to \cdots \to C_n \to 0$ is a bounded chain complex of objects in an abelian category $\mathcal{A}$. We define the *Euler characteristic* $\chi(C_\bullet)$ of $C_\bullet$ to be the following element of $K_0(\mathcal{A})$:

$$\chi(C_\bullet) = \sum (-1)^i [C_i].$$

PROPOSITION 6.6. *If $C_\bullet$ is a bounded complex of objects in $\mathcal{A}$, the element $\chi(C_\bullet)$ depends only upon the homology of $C_\bullet$:*

$$\chi(C_\bullet) = \sum (-1)^i [H_i(C_\bullet)].$$

In particular, if $C_\bullet$ is acyclic (exact as a sequence) then $\chi(C_\bullet) = 0$.

PROOF. Write Z_i and B_{i-1} for the kernel and image of the map $C_i \to C_{i-1}$, respectively. Since $B_{i-1} = C_i/Z_i$ and $H_i(C_\bullet) = Z_i/B_i$, we compute in $K_0(\mathcal{A})$:

$$\begin{aligned} \sum (-1)^i [H_i(C_\bullet)] &= \sum (-1)^i [Z_i] - \sum (-1)^i [B_i] \\ &= \sum (-1)^i [Z_i] + \sum (-1)^i [B_{i-1}] \\ &= \sum (-1)^i [C_i] = \chi(C_\bullet). \qquad \square \end{aligned}$$

Let $\mathbf{Ch}^{hb}(\mathcal{A})$ denote the abelian category of (possibly unbounded) chain complexes of objects in $\mathcal{A}$ having only finitely many nonzero homology groups. We call such complexes *homologically bounded.*

COROLLARY 6.6.1. *There is a natural surjection $\chi_H \colon K_0(\mathbf{Ch}^{hb}) \to K_0(\mathcal{A})$ sending $C_\bullet$ to $\sum(-1)^i[H_i(C_\bullet)]$. In particular, if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a exact sequence of homologically bounded complexes then:*

$$\chi_H(B_\bullet) = \chi_H(A_\bullet) + \chi_H(C_\bullet).$$

EXERCISES

6.1 Let R be a ring and $\mathbf{mod}_{\mathrm{fl}}(R)$ the abelian category of R-modules with finite length. Show that $K_0\mathbf{mod}_{\mathrm{fl}}(R)$ is the free abelian group $\bigoplus_{\mathfrak{m}} \mathbb{Z}$, a basis being $\{[R/\mathfrak{m}], \mathfrak{m}$ a maximal right ideal of $R\}$. *Hint:* Use the Jordan-Hölder Theorem for modules of finite length.

6.2 *Schreier Refinement Theorem.* Let $A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_r = 0$ and $A = A'_0 \supseteq A'_1 \supseteq \cdots \supseteq A'_s = 0$ be two filtrations of an object A in an abelian category $\mathcal{A}$. Show that the subobjects $A_{i,j} = (A_i \cap A'_j) + A_{i+1}$, ordered lexicographically, form a filtration of A which refines the filtration $\{A_i\}$. By symmetry, there is also a filtration by the $A'_{j,i} = (A_i \cap A'_j) + A'_{j+1}$ which refines the filtration $\{A'_j\}$.

Prove *Zassenhaus' Lemma*, that $A_{i,j}/A_{i,j+1} \cong A'_{j,i}/A'_{j,i+1}$. This shows that the factors in the two refined filtrations are isomorphic up to a permutation; the slogan is that "any two filtrations have equivalent refinements."

6.3 *Jordan-Hölder Theorem in $\mathcal{A}$.* An object A in an abelian category $\mathcal{A}$ is called *simple* if it has no proper subobjects. We say that an object A has *finite length* if it has a composition series $A = A_0 \supset \cdots \supset A_s = 0$ in which all the quotients A_i/A_{i+1} are simple. By Ex. 6.2, the Jordan-Hölder Theorem holds in $\mathcal{A}_{\text{fl}}$: the simple factors in any composition series of A are unique up to permutation and isomorphism. Let $\mathcal{A}_{\text{fl}}$ denote the subcategory of objects in $\mathcal{A}$ of finite length. Show that $\mathcal{A}_{\text{fl}}$ is a Serre subcategory of $\mathcal{A}$, and that $K_0(\mathcal{A}_{\text{fl}})$ is the free abelian group on the set of isomorphism classes of simple objects.

6.4 Let $\mathcal{A}$ be a small abelian category. If $[A_1] = [A_2]$ in $K_0(\mathcal{A})$, show that there are short exact sequences in $\mathcal{A}$

$$0 \to C' \to C_1 \to C'' \to 0, \quad 0 \to C' \to C_2 \to C'' \to 0$$

such that $A_1 \oplus C_1 \cong A_2 \oplus C_2$. *Hint:* First find sequences $0 \to D_i' \to D_i \to D_i'' \to 0$ such that $A_1 \oplus D_1' \oplus D_1'' \oplus D_2 \cong A_2 \oplus D_2' \oplus D_2'' \oplus D_1$, and set $C_i = D_i' \oplus D_i'' \oplus D_j$.

6.5 *Resolution.* Suppose that R is a regular noetherian ring, *i.e.,* that every R-module has a finite projective resolution. Show that the Cartan homomorphism $K_0(R) \to G_0(R)$ is onto. (We will see in Theorem 7.8 that it is an isomorphism.)

6.6 *Serre's Formula.* (Cf. 7.9.3) If s is a central element of a ring R, show that there is a map $\pi^*\colon G_0(R) \to G_0(R/sR)$ sending $[M]$ to $[M/Ms] - [\text{ann}_M(s)]$, where $\text{ann}_M(s) = \{x \in M : xs = 0\}$. Theorem 6.5 gives an example where π^* is onto, and if s is nilpotent the map is zero by Devissage 6.3.1. *Hint:* Use the map $M \xrightarrow{s} M$.

6.7 Let Y be a noetherian scheme over the ring $\mathbb{Z}[t]$, and let $X \overset{\pi}{\hookrightarrow} Y$ be the closed subscheme defined by $t = 0$. If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, let $\text{ann}_{\mathcal{F}}(t)$ denote the submodule of $\mathcal{F}$ annihilated by t. Show that there is a map $\pi^*\colon G_0(Y) \to G_0(X)$ sending $[\mathcal{F}]$ to $[\mathcal{F}/t\mathcal{F}] - [\text{ann}_{\mathcal{F}}(t)]$.

6.8 (Heller-Reiner) Let R be a commutative domain with field of fractions F. If $S = R - \{0\}$, show that there is a well-defined map $\Delta : F^\times \to K_0\mathbf{M}_S(R)$ sending the fraction $r/s \in F^\times$ to $[R/rR] - [R/sR]$. Then use Ex. 6.4 to show that the localization sequence extends to the exact sequence

$$1 \to R^\times \to F^\times \xrightarrow{\Delta} K_0\mathbf{M}_S(R) \to G_0(R) \to \mathbb{Z} \to 0.$$

6.9 *Weil Divisor Class groups.* Let R be a commutative noetherian ring.

(a) Show that each $\mathbf{M}^i(R)$ is a Serre subcategory of $\mathbf{M}(R)$.

(b) Show that $K_0\mathbf{M}^{i-1}/\mathbf{M}^{i+1}(R) \cong CH^i(R) \oplus D^{i-1}(R)$. In particular, if R is a 1-dimensional domain then $G_0(R) = \mathbb{Z} \oplus CH^1(R)$.

(c) Show that each $F^iG_0(R)/F^{i+1}G_0(R)$ is a quotient of the group $CH^i(R)$.

(d) Suppose that R is a domain with field of fractions F. As in Ex. 6.8, show that there is an exact sequence generalizing Proposition I.3.6:

$$0 \to R^\times \to F^\times \xrightarrow{\Delta} D^1(R) \to CH^1(R) \to 0.$$

In particular, if R is a Krull domain, conclude that $CH^1(R) \cong Cl(R)$ and $G_0(R)/F^2 \cong \mathbb{Z} \oplus Cl(R)$.

(e) If $(R, \mathfrak{m})$ is a 1-dimensional local domain and $k_1, \dots, k_n$ are the residue fields of the normalization of R over $k = R/\mathfrak{m}$, show that $CH^1(R) \cong \mathbb{Z}/\gcd\{[k_i : k]\}$.

6.10 Generalize the preceding exercise to a noetherian scheme X, as indicated in Application 6.4.3. *Hint:* F becomes the function field of X, and (d) becomes I.5.12.

6.11 If S is a multiplicatively closed set of central elements in a noetherian ring R, show that

$$K_0\mathbf{M}_S(R) \cong K_0\mathbf{M}_S(R)[t]) \cong K_0\mathbf{M}_S(R[t, t^{-1}]).$$

6.12 *Graded modules.* When $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ is a noetherian graded ring, let $\mathbf{M}_{gr}(S)$ denote the abelian category of finitely generated graded S-modules. Write σ for the shift automorphism $M \mapsto M(-1)$ of the category $\mathbf{M}_{gr}(S)$. Show that:

(a) $K_0\mathbf{M}_{gr}(S)$ is a module over the ring $\mathbb{Z}[\sigma, \sigma^{-1}]$

(b) If S is flat over R, there is a map from the direct sum $G_0(R)[\sigma, \sigma^{-1}] = \bigoplus_{n\in\mathbb{Z}} G_0(R)\sigma^n$ to $K_0\mathbf{M}_{gr}(S)$ sending $[M]\sigma^n$ to $[\sigma^n(M \otimes S)]$.

(c) If $S = R$, the map in (b) is an isomorphism: $K_0\mathbf{M}_{gr}(R) \cong G_0(R)[\sigma, \sigma^{-1}]$.

(d) If $S = R[x_1, \cdots, x_m]$ with $x_1, \cdots, x_m$ in S_1, the map is surjective, *i.e.,* $K_0\mathbf{M}_{gr}(S)$ is generated by the classes $[\sigma^n M[x_1, ..., x_m]]$. We will see in Ex. 7.14 that the map in (b) is an isomorphism for $S = R[x_1, \cdots, x_m]$.

(e) Let $\mathcal{B}$ be the subcategory of $\mathbf{M}_{gr}(R[x, y])$ of modules on which y is nilpotent. Show that $\mathcal{B}$ is a Serre subcategory, and that

$$K_0\mathcal{B} \cong K_0\mathbf{M}_{gr}(R) \cong G_0(R)[\sigma, \sigma^{-1}]$$

6.13 In this exercise we sketch Serre's proof of the Fundamental Theorem 6.5 when R is a non-commutative ring. We assume the results of the previous exercise. Show that the formula $j(M) = M/(y - 1)M$ defines an exact functor $j: \mathbf{M}_{gr}(R[x, y]) \to \mathbf{M}(R[x])$, sending $\mathcal{B}$ to zero. In fact, j induces an equivalence

$$\mathbf{M}_{gr}(R[x, y])/\mathcal{B} \cong \mathbf{M}(R[x]).$$

Then use this equivalence to show that the map $i^*: G_0(R) \to G_0(R[x])$ is onto.

6.14 G_0 *of projective space.* Let k be a field and set $S = k[x_0, \ldots, x_m]$, with $X = \mathbb{P}^m_k$. Using the notation of Exercises 6.3 and 6.12, let $\mathbf{M}^b_{gr}(S)$ denote the Serre subcategory of $\mathbf{M}_{gr}(S)$ consisting of graded modules of finite length. It is well-known (see [Hart, II.5.15]) that every coherent $\mathcal{O}_X$-module is of the form $\tilde{M}$ for some M in $\mathbf{M}_{gr}(S)$, *i.e.,* that the associated sheaf functor $\mathbf{M}_{gr}(S) \to \mathbf{M}(X)$ is onto, and that if M has finite length then $\tilde{M} = 0$. In fact, there is an equivalence

$$\mathbf{M}_{gr}(S)/\mathbf{M}^b_{gr}(S) \cong \mathbf{M}(\mathbb{P}^m_k).$$

(See [Hart, Ex. II.5.9(c)].) Under this equivalence $\sigma^i(S)$ represents $\mathcal{O}_X(-i)$.

(a) Let F denote the graded S-module S^{m+1}, whose basis lies in degree 0. Use the Koszul exact sequence of (I.5.4):

$$0 \to \sigma^{m+1}(\bigwedge^{m+1} F) \to \cdots \to \sigma^2(\bigwedge^2 F) \to \sigma F \xrightarrow{x_0,\ldots} S \to k \to 0$$

to show that in $K_0\mathbf{M}_{gr}(S)$ every finitely generated k-module M satisfies

$$[M] = \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} \sigma^i [M \otimes_k S] = (1 - \sigma)^{m+1}[M \otimes_k S].$$

(b) Show that in $G_0(\mathbb{P}^m_k)$ every $[\mathcal{O}_X(n)]$ is a linear combination of the classes $[\mathcal{O}_X], [\mathcal{O}_X(-1)], \cdots, [\mathcal{O}_X(-m)]$, and that

$$\sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} [\mathcal{O}(-i)] = 0 \quad \text{in } G_0\, \mathbb{P}^m_k\,.$$

(c) We will see in Ex. 7.14 that the map in Ex. 6.12(b) is an isomorphism:

$$K_0\mathbf{M}_{gr}(S) \cong G_0(R)[\sigma, \sigma^{-1}].$$

Assume this calculation, and show that

$$G_0(\mathbb{P}^m_k) \cong \mathbb{Z}^m \quad \text{on generators } \ [\mathcal{O}_X], [\mathcal{O}_X(-1)], \cdots, [\mathcal{O}_X(-m)].$$

6.15 *Naturality of f_*.* Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper morphisms between noetherian schemes. Show that $(gf)_* = g_* f_*$ as maps $G_0(X) \to G_0(Z)$.

6.16 Let R be a noetherian ring, and $r \in R$. If r is a nonzerodivisor on modules M_j whose associated primes all have height i, and $0 = \sum \pm[M_j]$ in $K_0\mathbf{M}^i(R)$, show that $0 = \sum \pm[M_j/rM_j]$ in $K_0\mathbf{M}^{i+1}(R)$. *Hint:* By Devissage, the formula holds in $K_0(R/I)$ for some product I of height i primes. Modify r to be a nonzerodivisor on R/I without changing the M_j/rM_j and use $f_* : G_0(R/I) \to G_0(R/(I + rR))$.

§7. K_0 of an Exact Category

If $\mathcal{C}$ is an additive subcategory of an abelian category $\mathcal{A}$, we may still talk about exact sequences: an *exact sequence* in $\mathcal{C}$ is a sequence of objects (and maps) in $\mathcal{C}$ which is exact as a sequence in $\mathcal{A}$. With hindsight, we know that it helps to require $\mathcal{C}$ to be closed under extensions. Thus we formulate the following definitions.

Definition 7.0 (Exact categories). An *exact category* is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is an additive category and $\mathcal{E}$ is a family of sequences in $\mathcal{C}$ of the form

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0, \qquad (\dagger)$$

satisfying the following condition: there is an embedding of $\mathcal{C}$ as a full subcategory of an abelian category $\mathcal{A}$ so that

(1) $\mathcal{E}$ is the class of all sequences $(\dagger)$ in $\mathcal{C}$ which are exact in $\mathcal{A}$;
(2) $\mathcal{C}$ is *closed under extensions* in $\mathcal{A}$ in the sense that if $(\dagger)$ is an exact sequence in $\mathcal{A}$ with $B, D \in \mathcal{C}$ then C is isomorphic to an object in $\mathcal{C}$.

The sequences in $\mathcal{E}$ are called the *short exact sequences* of $\mathcal{C}$. We will often abuse notation and just say that $\mathcal{C}$ is an exact category when the class $\mathcal{E}$ is clear. We call a map in $\mathcal{C}$ an *admissible monomorphism* (resp. an *admissible epimorphism*) if it occurs as the monomorphism i (resp. as the epi j) in some sequence $(\dagger)$ in $\mathcal{E}$.

The following hypothesis is commonly satisfied in applications, and is needed for Euler characteristics and the Resolution Theorem 7.6 below.

(7.0.1) We say that $\mathcal{C}$ is *closed under kernels of surjections* in $\mathcal{A}$ provided that whenever a map $f: B \to C$ in $\mathcal{C}$ is a surjection in $\mathcal{A}$ then $\ker(f) \in \mathcal{C}$. The well-read reader will observe that the definition of exact category in [Bass] is what we call an exact category closed under kernels of surjections.

An *exact functor* $F: \mathcal{B} \to \mathcal{C}$ between exact categories is an additive functor F carrying short exact sequences in $\mathcal{B}$ to exact sequences in $\mathcal{C}$. If $\mathcal{B}$ is a full subcategory of $\mathcal{C}$, and the exact sequences in $\mathcal{B}$ are precisely the sequences $(\dagger)$ in $\mathcal{B}$ which are exact in $\mathcal{C}$, we call $\mathcal{B}$ an *exact subcategory* of $\mathcal{C}$. This is consistent with the notion of an exact abelian subcategory in 6.1.5.

Definition 7.1 (K_0). Let $\mathcal{C}$ be a small exact category. $K_0(\mathcal{C})$ is the abelian group having generators $[C]$, one for each object C of $\mathcal{C}$, and relations $[C] = [B]+[D]$ for every short exact sequence $0 \to B \to C \to D \to 0$ in $\mathcal{C}$.

As in 6.1.1, we have $[0] = 0$, $[B \oplus D] = [B] + [D]$ and $[B] = [C]$ if B and C are isomorphic. As before, we could actually define $K_0(\mathcal{C})$ when $\mathcal{C}$ is only skeletally small, but we shall not dwell on these set-theoretic intricacies. Clearly, $K_0(\mathcal{C})$ satisfies the universal property 6.1.2 for additive functions from $\mathcal{C}$ to abelian groups.

Example 7.1.1. The category $\mathbf{P}(R)$ of finitely generated projective R-modules is exact by virtue of its embedding in $\mathbf{mod}$-R. As every exact sequence of projective modules splits, we have $K_0\mathbf{P}(R) = K_0(R)$.

Any additive category is a symmetric monoidal category under $\oplus$, and the above remarks show that $K_0(\mathcal{C})$ is a quotient of the group $K_0^{\oplus}(\mathcal{C})$ of §5. Since abelian categories are exact, Examples 6.2.1–4 show that these groups are not identical.

Example 7.1.2 (Split exact categories). A *split exact* category $\mathcal{C}$ is an exact category in which every short exact sequence in $\mathcal{E}$ is split (*i.e.,* isomorphic to $0 \to B \to B \oplus D \to D \to 0$). In this case we have $K_0(\mathcal{C}) = K_0^{\oplus}(\mathcal{C})$ by definition. For example, the category $\mathbf{P}(R)$ is split exact.

If X is a topological space, embedding $\mathbf{VB}(X)$ in the abelian category of families of vector spaces over X makes $\mathbf{VB}(X)$ into an exact category. By the Subbundle Theorem I.4.1, $\mathbf{VB}(X)$ is a split exact category, so that $K^0(X) = K_0(\mathbf{VB}(X))$.

We will see in Exercise 7.7 that any additive category $\mathcal{C}$ may be made into a split exact category by equipping it with the class $\mathcal{E}_{split}$ of sequences isomorphic to $0 \to B \to B \oplus D \to D \to 0$

Warning. Every abelian category $\mathcal{A}$ has a natural exact category structure, but it also has the split exact structure. These will yield different K_0 groups in general, unless something like a Krull-Schmidt Theorem holds in $\mathcal{A}$. We will always use the natural exact structure unless otherwise indicated.

Example 7.1.3 (K_0 of a scheme). Let X be a scheme (or more generally a ringed space). The category $\mathbf{VB}(X)$ of algebraic vector bundles on X, introduced in (I.5), is an exact category by virtue of its being an additive subcategory of the abelian category $\mathcal{O}_X$-**mod** of all $\mathcal{O}_X$-modules. If X is quasi-projective over a commutative ring, we write $K_0(X)$ for $K_0\mathbf{VB}(X)$. If X is noetherian, the inclusion $\mathbf{VB}(X) \subset \mathbf{M}(X)$ yields a *Cartan homomorphism* $K_0(X) \to G_0(X)$. We saw in (I.5.3) that exact sequences in $\mathbf{VB}(X)$ do not always split, so $\mathbf{VB}(X)$ is not always a split exact category.

Example 7.1.4 (G_0 of non-noetherian rings). If R is a non-noetherian ring, the category $\mathbf{mod}_{fg}(R)$ of all finitely generated R-modules will not be abelian, because $R \to R/I$ has no kernel inside this category. However, it is still an exact subcategory of **mod**-R, so once again we might try to consider the group $K_0\mathbf{mod}_{fg}(R)$. However, it turns out that this definition does not have good properties (see Ex. 7.3 and 7.4).

Here is a more suitable definition, based upon [SGA6, I.2.9]. An R-module M is called *pseudo-coherent* if it has an infinite resolution $\cdots \to P_1 \to P_0 \to M \to 0$ by finitely generated projective R-modules. Pseudo-coherent modules are clearly finitely presented, and if R is right noetherian then every finitely generated module is pseudo-coherent. Let $\mathbf{M}(R)$ denote the category of all pseudo-coherent R-modules. The "Horseshoe Lemma" [WHomo, 2.2.8] shows that $\mathbf{M}(R)$ is closed under extensions in **mod**-R, so it is an exact category. (It is also closed under kernels of surjections, and cokernels of injections in **mod**-R, as can be seen using the mapping cone.)

Now we define $G_0(R) = K_0\mathbf{M}(R)$. Note that if R is right noetherian then $\mathbf{M}(R)$ is the usual category of §6, and we have recovered the definition of $G_0(R)$ in 6.2.

Example 7.1.5. The opposite category $\mathcal{C}^{op}$ has an obvious notion of exact sequence: turn the arrows around in the exact sequences of $\mathcal{C}$. Formally, this arises from the inclusion of $\mathcal{C}^{op}$ in $\mathcal{A}^{op}$. Clearly $K_0(\mathcal{C}) \cong K_0(\mathcal{C}^{op})$.

Example 7.1.6. The direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ of two exact categories is also exact, the ambient abelian category being $\mathcal{A}_1 \oplus \mathcal{A}_2$. Clearly $K_0(\mathcal{C}_1 \oplus \mathcal{C}_2) \cong K_0(\mathcal{C}_1) \oplus K_0(\mathcal{C}_2)$.

More generally, the direct sum $\bigoplus \mathcal{C}_i$ of exact categories is an exact category (inside the abelian category $\oplus \mathcal{A}_i$), and as in 6.1.2 this yields $K_0(\oplus \mathcal{C}_i) \cong \oplus K_0(\mathcal{C}_i)$.

EXAMPLE 7.1.7 (FILTERED COLIMITS). Suppose that $\{\mathcal{C}_i\}$ is a filtered family of exact subcategories of a fixed abelian category $\mathcal{A}$. Then $\mathcal{C} = \cup \mathcal{C}_i$ is also an exact subcategory of $\mathcal{A}$, and by inspection of the definition we see that

$$K_0(\bigcup \mathcal{C}_i) = \varinjlim K_0(\mathcal{C}_i).$$

The ambient $\mathcal{A}$ is unnecessary: if $\{\mathcal{C}_i\}$ is a filtered family of exact categories and exact functors, then $K_0(\varinjlim \mathcal{C}_i) = \varinjlim K_0(\mathcal{C}_i)$; see Ex. 7.9. As a case in point, if a ring R is the union of subrings R_α then $\mathbf{P}(R)$ is the direct limit of the $\mathbf{P}(R_\alpha)$, and we have $K_0(R) = \varinjlim K_0(R_\alpha)$, as in 2.1.6.

COFINALITY LEMMA 7.2. *Let $\mathcal{B}$ be an exact subcategory of $\mathcal{C}$ which is closed under extensions in $\mathcal{C}$, and which is cofinal in the sense that for every C in $\mathcal{C}$ there is a C' in $\mathcal{C}$ so that $C \oplus C'$ is in $\mathcal{B}$. Then $K_0\mathcal{B}$ is a subgroup of $K_0\mathcal{C}$.*

PROOF. By (1.3) we know that $K_0^{\oplus}\mathcal{B}$ is a subgroup of $K_0^{\oplus}\mathcal{C}$. Given a short exact sequence $0 \to C_0 \to C_1 \to C_2 \to 0$ in $\mathcal{C}$, choose C_0' and C_2' in $\mathcal{C}$ so that $B_0 = C_0 \oplus C_0'$ and $B_2 = C_2 \oplus C_2'$ are in $\mathcal{B}$. Setting $B_1 = C_1 \oplus C_0' \oplus C_2'$, we have the short exact sequence $0 \to B_0 \to B_1 \to B_2 \to 0$ in $\mathcal{C}$. As $\mathcal{B}$ is closed under extensions in $\mathcal{C}$, $B_1 \in \mathcal{B}$. Therefore in $K_0^{\oplus}\mathcal{C}$:

$$[C_1] - [C_0] - [C_2] = [B_1] - [B_0] - [B_2].$$

Thus the kernel of $K_0^{\oplus}\mathcal{C} \to K_0\mathcal{C}$ equals the kernel of $K_0^{\oplus}\mathcal{B} \to K_0\mathcal{B}$, which implies that $K_0\mathcal{B} \to K_0\mathcal{C}$ is an injection. $\square$

REMARK 7.2.1. The proof shows that $K_0(\mathcal{C})/K_0(\mathcal{B}) \cong K_0^{\oplus}\mathcal{C}/K_0^{\oplus}\mathcal{B}$, and that every element of $K_0(\mathcal{C})$ has the form $[C] - [B]$ for some B in $\mathcal{B}$ and C in $\mathcal{C}$.

Idempotent completion.

7.3. A category $\mathcal{C}$ is called *idempotent complete* if every idempotent endomorphism e of an object C factors as $C \to B \to C$ with the composite $B \to C \to B$ being the identity. Given $\mathcal{C}$, we can form a new category $\widehat{\mathcal{C}}$ whose objects are pairs (C, e) with e an idempotent endomorphism of an object C of $\mathcal{C}$; a morphism from (C, e) to (C', e') is a map $f\colon C \to C'$ in $\mathcal{C}$ such that $f = e'fe$. The category $\widehat{\mathcal{C}}$ is idempotent complete, since an idempotent endomorphism f of (C, e) factors through the object (C, efe).

$\widehat{\mathcal{C}}$ is called the *idempotent completion* of $\mathcal{C}$. To see why, consider the natural embedding of $\mathcal{C}$ into $\widehat{\mathcal{C}}$ sending C to (C, id). It is easy to see that any functor from $\mathcal{C}$ to an idempotent complete category $\mathcal{D}$ must factor through a functor $\widehat{\mathcal{C}} \to \mathcal{D}$ that is unique up to natural equivalence. In particular, if $\mathcal{C}$ is idempotent then $\mathcal{C} \cong \widehat{\mathcal{C}}$.

If $\mathcal{C}$ is an additive subcategory of an abelian category $\mathcal{A}$, then $\widehat{\mathcal{C}}$ is equivalent to a larger additive subcategory $\mathcal{C}'$ of $\mathcal{A}$ (see Ex. 7.6). Moreover, $\mathcal{C}$ is cofinal in $\widehat{\mathcal{C}}$, because (C, e) is a summand of C in $\mathcal{A}$. By the Cofinality Lemma 7.2, we see that $K_0(\mathcal{C})$ is a subgroup of $K_0(\widehat{\mathcal{C}})$.

EXAMPLE 7.3.1. Consider the subcategory $\mathbf{Free}(R)$ of $\mathbf{M}(R)$ consisting of finitely generated free R-modules. The idempotent completion of $\mathbf{Free}(R)$ is the category $\mathbf{P}(R)$ of finitely generated projective modules. Thus the cyclic group $K_0\mathbf{Free}(R)$ is a subgroup of $K_0(R)$. If R satisfies the Invariant Basis Property (IBP), then $K_0\mathbf{Free}(R) \cong \mathbb{Z}$ and we have recovered the conclusion of Lemma 2.1.

EXAMPLE 7.3.2. Let $R \to S$ be a ring homomorphism, and let $\mathcal{B}$ denote the full subcategory of $\mathbf{P}(S)$ on the modules of the form $P \otimes_R S$ for P in $\mathbf{P}(R)$. Since it contains all the free modules S^n, $\mathcal{B}$ is cofinal in $\mathbf{P}(S)$, so $K_0\mathcal{B}$ is a subgroup of $K_0(S)$. Indeed, $K_0\mathcal{B}$ is the image of the natural map $K_0(R) \to K_0(S)$.

Products

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be exact categories. A functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called *biexact* if $F(A, -)$ and $F(-, B)$ are exact functors for every A in $\mathcal{A}$ and B in $\mathcal{B}$, and $F(0, -) = F(-, 0) = 0$. (The last condition, not needed in this chapter, can always be arranged by replacing $\mathcal{C}$ by an equivalent category.) The following result is completely elementary.

LEMMA 7.4. *A biexact functor $F: \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces a bilinear map*

$$K_0\mathcal{A} \otimes K_0\mathcal{B} \to K_0\mathcal{C}.$$
$$[A] \otimes [B] \mapsto [F(A, B)]$$

APPLICATION 7.4.1. Let R be a commutative ring. The tensor product $\otimes_A$ defines a biexact functor $\mathbf{P}(R) \times \mathbf{P}(R) \to \mathbf{P}(R)$, as well as a biexact functor $\mathbf{P}(R) \times \mathbf{M}(R) \to \mathbf{M}(R)$. The former defines the product $[P][Q] = [P \otimes Q]$ in the commutative ring $K_0(R)$, as we saw in §2. The latter defines an action of $K_0(R)$ on $G_0(R)$, making $G_0(R)$ into a $K_0(R)$-module.

APPLICATION 7.4.2. Let X be a scheme (or more generally a locally ringed space) The tensor product of vector bundles defines a biexact functor $\mathbf{VB}(X) \times \mathbf{VB}(X) \to \mathbf{VB}(X)$ (see I.5.3). This defines a product on $K_0(X)$ satisfying $[\mathcal{E}][\mathcal{F}] = [\mathcal{E} \otimes \mathcal{F}]$. This product is clearly commutative and associative, so it makes $K_0(X)$ into a commutative ring. We will discuss this ring further in the next section.

If X is noetherian, recall from 6.2.5 that $G_0(X)$ denotes $K_0\mathbf{M}(X)$. Since the tensor product of a vector bundle and a coherent module is coherent, we have a functor $\mathbf{VB}(X) \times \mathbf{M}(X) \to \mathbf{M}(X)$. It is biexact (why?), so it defines an action of $K_0(X)$ on $G_0(X)$, making $G_0(X)$ into a $K_0(X)$-module.

APPLICATION 7.4.3 (ALMKVIST). If R is a ring, let $\mathbf{End}(R)$ denote the exact category whose objects (P, α) are pairs, where P is a finitely generated projective R-module and α is an endomorphism of P. A morphism $(P, \alpha) \to (Q, \beta)$ in $\mathbf{End}(R)$ is a morphism $f: P \to Q$ in $\mathbf{P}(R)$ such that $f\alpha = \beta f$, and exactness in $\mathbf{End}(R)$ is determined by exactness in $\mathbf{P}(R)$.

If R is commutative, the tensor product of modules gives a biexact functor

$$\otimes_R : \ \mathbf{End}(R) \times \mathbf{End}(R) \to \mathbf{End}(R),$$
$$((P, \alpha), (Q, \beta)) \mapsto (P \otimes_R Q, \alpha \otimes_R \beta).$$

As $\otimes_R$ is associative and symmetric up to isomorphism, the induced product makes $K_0\mathbf{End}(R)$ into a commutative ring with unit $[(R,1)]$. The inclusion of $\mathbf{P}(R)$ in $\mathbf{End}(R)$ by $\alpha = 0$ is split by the forgetful functor, and the kernel $\mathrm{End}_0(R)$ of $K_0\mathbf{End}(R) \to K_0(R)$ is not only an ideal but a commutative ring with unit $1 = [(R,1)] - [(R,0)]$. Almkvist proved that $(P,\alpha) \mapsto \det(1-\alpha t)$ defines an isomorphism of $\mathrm{End}_0(R)$ with the subgroup of the multiplicative group $W(R) = 1+tR[[t]]$ consisting of all quotients $f(t)/g(t)$ of polynomials in $1+tR[t]$ (see Ex. 7.18). Almkvist also proved that $\mathrm{End}_0(R)$ is a subring of $W(R)$ under the ring structure of 4.3.

If A is an R-algebra, $\otimes_R$ is also a pairing $\mathbf{End}(R)\times\mathbf{End}(A) \to \mathbf{End}(A)$, making $\mathrm{End}_0(A)$ into an $\mathrm{End}_0(R)$-module. We leave the routine details to the reader.

Example 7.4.4. If R is a ring, let $\mathbf{Nil}(R)$ denote the category whose objects (P,ν) are pairs, where P is a finitely generated projective R-module and ν is a nilpotent endomorphism of P. This is an exact subcategory of $\mathbf{End}(R)$. The forgetful functor $\mathbf{Nil}(R) \to \mathbf{P}(R)$ sending (P,ν) to P is exact, and is split by the exact functor $\mathbf{P}(R) \to \mathbf{Nil}(R)$ sending P to $(P,0)$. Therefore $K_0(R) = K_0\mathbf{P}(R)$ is a direct summand of $K_0\mathbf{Nil}(R)$. We write $\mathrm{Nil}_0(R)$ for the kernel of $K_0\mathbf{Nil}(R) \to \mathbf{P}(R)$, so that there is a direct sum decomposition $K_0\mathbf{Nil}(R) = K_0(R) \oplus \mathrm{Nil}_0(R)$. Since $[P,\nu] = [P\oplus Q, \nu\oplus 0] - [Q,0]$ in $K_0\mathbf{Nil}(R)$, we see that $\mathrm{Nil}_0(R)$ is generated by elements of the form $\big[(R^n,\nu)\big] - n\big[(R,0)\big]$ for some n and some nilpotent matrix ν.

If A is an R-algebra, then the tensor product pairing on $\mathbf{End}$ restricts to a biexact functor $F\colon \mathbf{End}(R)\times\mathbf{Nil}(A) \to \mathbf{Nil}(A)$. The resulting bilinear map $K_0\mathbf{End}(R) \times K_0\mathbf{Nil}(A) \to K_0\mathbf{Nil}(A)$ is associative, and makes $\mathrm{Nil}_0(A)$ into a module over the ring $\mathrm{End}_0(R)$, and makes $\mathrm{Nil}_0(A) \to \mathrm{End}_0(A)$ an $\mathrm{End}_0(R)$-module map.

Any additive functor $T : \mathbf{P}(A) \to \mathbf{P}(B)$ induces an exact functor $\mathbf{Nil}(A) \to \mathbf{Nil}(B)$ and a homomorphism $\mathrm{Nil}_0(A) \to \mathrm{Nil}_0(B)$. If A and B are R-algebras and T is R-linear, $\mathrm{Nil}_0(A) \to \mathrm{Nil}_0(B)$ is an $\mathrm{End}_0(R)$-module homomorphism. (Exercise!)

Example 7.4.5. If R is a commutative regular ring, and $A = R[x]/(x^N)$, we will see in III.3.8.1 that $\mathrm{Nil}_0(A) \to \mathrm{End}_0(A)$ is an injection, identifying $\mathrm{Nil}_0(A)$ with the ideal $(1+xtA[t])^\times$ of $\mathrm{End}_0(A)$, and identifying $[(A,x)]$ with $1-xt$.

This isomorphism $\mathrm{End}_0(A) \cong (1+xtA[t])^\times$ is universal in the following sense. If B is an R-algebra and (P,ν) is in $\mathbf{Nil}(B)$, with $\nu^N = 0$, we may regard P as an A–B bimodule. By 2.8, this yields an R-linear functor $\mathbf{Nil}_0(A) \to \mathbf{Nil}_0(B)$ sending (A,x) to (P,ν). By 7.4.4, there is an $\mathrm{End}_0(R)$-module homomorphism $(1+xtA[t])^\times \to \mathrm{Nil}_0(B)$ sending $1-xt$ to $[(P,\nu)]$.

The following result shows that Euler characteristics can be useful in exact categories as well as in abelian categories, and is the analogue of Proposition 6.6.

Proposition 7.5. *Suppose that $\mathcal{C}$ is closed under kernels of surjections in an abelian category $\mathcal{A}$. If $C_\bullet$ is a bounded chain complex in $\mathcal{C}$ whose homology $H_i(C_\bullet)$ is also in $\mathcal{C}$ then in $K_0(\mathcal{C})$:*

$$\chi(C_\bullet) = \sum (-1)^i [C_i] \quad \textit{equals} \quad \sum (-1)^i [H_i(C_\bullet)].$$

In particular, if $C_\bullet$ is any exact sequence in $\mathcal{C}$ then $\chi(C_\bullet) = 0$.

PROOF. The proof we gave in 6.6 for abelian categories will go through, provided that the Z_i and B_i are objects of $\mathcal{C}$. Consider the exact sequences:

$$0 \to Z_i \to C_i \to B_i \to 0$$
$$0 \to B_i \to Z_i \to H_i(C_\bullet) \to 0.$$

Since $B_i = 0$ for $i << 0$, the following inductive argument shows that all the B_i and Z_i belong to $\mathcal{C}$. If $B_{i-1} \in \mathcal{C}$ then the first sequence shows that $Z_i \in \mathcal{C}$; since $H_i(C_\bullet)$ is in $\mathcal{C}$, the second sequence shows that $B_i \in \mathcal{C}$. □

COROLLARY 7.5.1. *Suppose $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$. If $f: C'_\bullet \to C_\bullet$ is a morphism of bounded complexes in $\mathcal{C}$, inducing an isomorphism on homology, then*

$$\chi(C'_\bullet) = \chi(C_\bullet).$$

PROOF. Form cone(f), the mapping cone of f, which has $C_n \oplus C'_{n-1}$ in degree n. By inspection, $\chi(\text{cone}(f)) = \chi(C_\bullet) - \chi(C'_\bullet)$. But cone$(f)$ is an exact complex because f is a homology isomorphism, so $\chi(\text{cone}(f)) = 0$. □

The Resolution Theorem

We need a definition in order to state our next result. Suppose that $\mathcal{P}$ is an additive subcategory of an abelian category $\mathcal{A}$. A $\mathcal{P}$*-resolution* $P_\bullet \to C$ of an object C of $\mathcal{A}$ is an exact sequence in $\mathcal{A}$

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to C \to 0$$

in which all the P_i are in $\mathcal{P}$. The $\mathcal{P}$*-dimension* of C is the minimum n (if it exists) such that there is a resolution $P_\bullet \to C$ with $P_i = 0$ for $i > n$.

RESOLUTION THEOREM 7.6. *Let $\mathcal{P} \subset \mathcal{C} \subset \mathcal{A}$ be an inclusion of additive categories with $\mathcal{A}$ abelian ($\mathcal{A}$ gives the notion of exact sequence to $\mathcal{P}$ and $\mathcal{C}$). Assume that:*
(a) Every object C of $\mathcal{C}$ has finite $\mathcal{P}$-dimension; and
(b) $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$.
Then the inclusion $\mathcal{P} \subset \mathcal{C}$ induces an isomorphism $K_0(\mathcal{P}) \cong K_0(\mathcal{C})$.

PROOF. To see that $K_0(\mathcal{P})$ maps onto $K_0(\mathcal{C})$, observe that if $P_\bullet \to C$ is a finite $\mathcal{P}$-resolution, then the exact sequence

$$0 \to P_n \to \cdots \to P_0 \to C \to 0$$

has $\chi = 0$ by 7.5, so $[C] = \sum(-1)^i[P] = \chi(P_\bullet)$ in $K_0(\mathcal{C})$. To see that $K_0(\mathcal{P}) \cong K_0(\mathcal{C})$, we will show that the formula $\chi(C) = \chi(P_\bullet)$ defines an additive function from $\mathcal{C}$ to $K_0(\mathcal{P})$. For this, we need the following lemma, due to Grothendieck.

LEMMA 7.6.1. *Given a map $f: C' \to C$ in $\mathcal{C}$ and a finite $\mathcal{P}$-resolution $P_\bullet \to C$, there is a finite $\mathcal{P}$-resolution $P'_\bullet \to C'$ and a commutative diagram*

$$\begin{array}{ccccccccccccccc}
0 & \to & P'_m & \to & \cdots & \to & P'_n & \to & \cdots & \to & P'_1 & \to & P'_0 & \to & C' & \to & 0 \\
 & & & & & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow f & & \\
 & & & & & 0 & \to P_n & \to & \cdots & \to & P_1 & \to & P_0 & \to & C & \to & 0
\end{array}$$

We will prove this lemma in a moment. First we shall use it to finish the proof of Theorem 7.6. Suppose given two finite $\mathcal{P}$-resolutions $P_\bullet \to C$ and $P'_\bullet \to C$ of an object C. Applying the lemma to the diagonal map $C \to C \oplus C$ and $P_\bullet \oplus P'_\bullet \to C \oplus C$, we get a $\mathcal{P}$-resolution $P''_\bullet \to C$ and a map $P''_\bullet \to P_\bullet \oplus P'_\bullet$ of complexes. Since the maps $P_\bullet \leftarrow P''_\bullet \to P'_\bullet$ are quasi-isomorphisms, Corollary 7.5.1 implies that $\chi(P_\bullet) = \chi(P''_\bullet) = \chi(P'_\bullet)$. Hence $\chi(C) = \chi(P_\bullet)$ is independent of the choice of $\mathcal{P}$-resolution.

Given a short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$ and a $\mathcal{P}$-resolution $P_\bullet \to C$, the lemma provides a $\mathcal{P}$-resolution $P'_\bullet \to C'$ and a map $f\colon P'_\bullet \to P_\bullet$. Form the mapping cone complex cone(f), which has $P_n \oplus P'_n[-1]$ in degree n, and observe that $\chi(\text{cone}(f)) = \chi(P_\bullet) - \chi(P'_\bullet)$. The homology exact sequence

$$H_i(P') \to H_i(P) \to H_i(\text{cone}(f)) \to H_{i-1}(P') \to H_{i-1}(P)$$

shows that $H_i\text{cone}(f) = 0$ for $i \neq 0$, and $H_0(\text{cone}(f)) = C''$. Thus $\text{cone}(f) \to C''$ is a finite $\mathcal{P}$-resolution, and so

$$\chi(C'') = \chi(\text{cone}(f)) = \chi(P_\bullet) - \chi(P'_\bullet) = \chi(C) - \chi(C').$$

This proves that χ is an additive function, so it induces a map $\chi\colon K_0\mathcal{C} \to K_0(\mathcal{P})$. If P is in $\mathcal{P}$ then evidently $\chi(P) = [P]$, so χ is the inverse isomorphism to the map $K_0(\mathcal{P}) \to K_0(\mathcal{C})$. This finishes the proof of the Resolution Theorem 7.6. $\square$

Proof of Lemma 7.6.1. We proceed by induction on the length n of $P_\bullet$. If $n = 0$, we may choose any $\mathcal{P}$-resolution of C'; the only nonzero map $P'_n \to P_n$ is $P'_0 \to C' \to C \cong P_0$. If $n \geq 1$, let Z denote the kernel (in $\mathcal{A}$) of $\varepsilon\colon P_0 \to C$ and let B denote the kernel (in $\mathcal{A}$) of $(\varepsilon, -f)\colon P_0 \oplus C' \to C$. As $\mathcal{C}$ is closed under kernels, both Z and B are in $\mathcal{C}$. Moreover, the sequence

$$0 \to Z \to B \to C' \to 0$$

is exact in $\mathcal{C}$ (because it is exact in $\mathcal{A}$). Choose a surjection $P'_0 \to B$ with P'_0 in $\mathcal{P}$, let f_0 be the composition $P'_0 \to B \to P_0$ and let Y denote the kernel of the surjection $P'_0 \to B \to C'$. By induction applied to the induced map $Y \to Z$, we can find a $\mathcal{P}$-resolution $P'_\bullet[+1]$ of Y and maps $f_i\colon P'_i \to P_i$ making the following diagram commute (the rows are not exact at Y and Z):

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P'_2 & \longrightarrow & P'_1 & \longrightarrow & Y & \xrightarrow{\text{monic}} & P'_0 & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & Z & \xrightarrow{\text{monic}} & P_0 & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Splicing the rows by deleting Y and Z yields the desired $\mathcal{P}$-resolution of C'. $\square$

Definition 7.7 ($\mathbf{H}(R)$). Given a ring R, let $\mathbf{H}(R)$ denote the category of all R-modules M having a finite resolution by finitely generated projective modules, and let $\mathbf{H}_n(R)$ denote the subcategory in which the resolutions have length $\leq n$.

By the Horseshoe Lemma [WHomo, 2.2.8], both $\mathbf{H}(R)$ and $\mathbf{H}_n(R)$ are exact subcategories of **mod**-R. The following Lemma shows that they are also closed under kernels of surjections in **mod**-R.

Lemma 7.7.1. *If* $0 \to L \to M \xrightarrow{f} N \to 0$ *is a short exact sequence of modules, with* M *in* $\mathbf{H}_m(R)$ *and* N *in* $\mathbf{H}_n(R)$*, then* L *is in* $\mathbf{H}_\ell(R)$*, where* $\ell = \min\{m, n-1\}$*.*

Proof. If $P_\bullet \to M$ and $Q_\bullet \to N$ are projective resolutions, and $P_\bullet \to Q_\bullet$ lifts f, then the kernel P_0' of the surjection $P_0 \oplus Q_1 \to Q_0$ is finitely generated projective, and the truncated mapping cone $\cdots \to P_1 \oplus Q_2 \to P_0'$ is a resolution of L. □

Corollary 7.7.2. $K_0(R) \cong K_0\mathbf{H}(R) \cong K_0\mathbf{H}_n(R)$ *for all* $n \geq 1$*.*

Proof. Apply the Resolution Theorem to $\mathbf{P}(R) \subset \mathbf{H}(R)$. □

Here is a useful variant of the above construction. Let S be a multiplicatively closed set of central nonzerodivisors in a ring R. We say a module M is *S-torsion* if $Ms = 0$ for some $s \in S$ (*cf.* Example 6.2.8), and write $\mathbf{H}_S(R)$ for the exact subcategory $\mathbf{H}(R) \cap \mathbf{M}_S(R)$ of S-torsion modules M in $\mathbf{H}(R)$. Similarly, we write $\mathbf{H}_{n,S}(R)$ for the S-torsion modules in $\mathbf{H}_n(R)$. Note that $\mathbf{H}_{0,S}(R) = 0$, and that the modules R/sR belong to $\mathbf{H}_{1,S}(R)$.

Corollary 7.7.3. $K_0\mathbf{H}_S(R) \cong K_0\mathbf{H}_{n,S}(R) \cong K_0\mathbf{H}_{1,S}(R)$ *for all* $n \geq 1$*.*

Proof. We apply the Resolution Theorem with $\mathcal{P} = \mathbf{H}_{1,S}(R)$. By Lemma 7.7.1, each $\mathbf{H}_{n,S}(R)$ is closed under kernels of surjections. Every N in $\mathbf{H}_{n,S}(R)$ is finitely generated, so if $Ns = 0$ there is an exact sequence $0 \to L \to (R/sR)^m \to N \to 0$. If $n \geq 2$ then L is in $\mathbf{H}_{n-1,S}(R)$ by Lemma 7.7.1. By induction, L and hence N has a $\mathcal{P}$-resolution. □

Corollary 7.7.4. *If* S *is a multiplicatively closed set of central nonzerodivisors in a ring* R*, the sequence* $K_0\mathbf{H}_S(R) \to K_0(R) \to K_0(S^{-1}R)$ *is exact.*

Proof. If $[P] - [R^n] \in K_0(R)$ vanishes in $K_0(S^{-1}R)$, $S^{-1}P$ is stably free (Cor. 1.3). Hence there is an isomorphism $(S^{-1}R)^{m+n} \to S^{-1}P \oplus (S^{-1}R)^m$. Clearing denominators yields a map $f\colon R^{m+n} \to P \oplus R^m$ whose kernel and cokernel are S-torsion. But $\ker(f) = 0$ because S consists of nonzerodivisors, and therefore $M = \operatorname{coker}(f)$ is in $\mathbf{H}_{1,S}(R)$. But the map $K_0\mathbf{H}_S(R) \to K_0\mathbf{H}(R) = K_0(R)$ sends $[M]$ to $[M] = [P] - [R^n]$. □

Let R be a regular noetherian ring. Since every module has finite projective dimension, $\mathbf{H}(R)$ is the abelian category $\mathbf{M}(R)$ discussed in §6. Combining Corollary 7.7.2 with the Fundamental Theorem for G_0 6.5, we have:

Fundamental Theorem for K_0 of regular rings 7.8. *If* R *is a regular noetherian ring, then* $K_0(R) \cong G_0(R)$*. Moreover,*

$$K_0(R) \cong K_0(R[t]) \cong K_0(R[t, t^{-1}]).$$

If R is not regular, we can still use the localization sequence 7.7.4 to get a partial result, which will be considerably strengthened by the Fundamental Theorem for K_0 in chapter III.

Proposition 7.8.1. $K_0(R[t]) \to K_0(R[t, t^{-1}])$ *is injective for every ring* R*.*

To prove this, we need the following lemma. Recall from Example 7.4.4 that $\mathbf{Nil}(R)$ is the category of pairs (P, ν) with ν a nilpotent endomorphism of $P \in \mathbf{P}(R)$.

LEMMA 7.8.2. *Let S be the multiplicative set $\{t^n\}$ in the polynomial ring $R[t]$. Then $\mathbf{Nil}(R)$ is equivalent to the category $\mathbf{H}_{1,S}(R[t])$ of t-torsion $R[t]$-modules M in $\mathbf{H}_1(R[t])$.*

PROOF. If (P,ν) is in $\mathbf{Nil}(R)$, let P_ν denote the $R[t]$-module P on which t acts as ν. It is a t-torsion module because $t^nP_\nu = \nu^nP = 0$ for large n. A projective resolution of P_ν is given by the "characteristic sequence" of ν:

$$0 \to P[t] \xrightarrow{t-\nu} P[t] \to P_\nu \to 0, \tag{7.8.3}$$

Thus P_ν is an object of $\mathbf{H}_{1,S}(R[t])$. Conversely, each M in $\mathbf{H}_{1,S}(R[t])$ has a projective resolution $0 \to P \to Q \to M \to 0$ by finitely generated projective $R[t]$-modules, and M is killed by some power t^n of t. From the exact sequence

$$0 \to \mathrm{Tor}_1^{R[t]}\big(M, R[t]/(t^n)\big) \to P/t^nP \to Q/t^nQ \to M \to 0$$

and the identification of the first term with M we obtain the exact sequence $0 \to M \xrightarrow{t^n} P/t^nP \to P/t^nQ \to 0$. Since P/t^nP is a projective R-module and $pd_R(P/t^nQ) \le 1$, we see that M must be a projective R-module. Thus (M,t) is an object of $\mathbf{Nil}(R)$. □

Combining Lemma 7.8.2 with Corollary 7.7.3 yields:

COROLLARY 7.8.4. *$K_0\mathbf{Nil}(R) \cong K_0\mathbf{H}_S(R[t])$.*

PROOF OF PROPOSITION 7.8.1. By Corollaries 7.7.4 and 7.8.4, we have an exact sequence

$$K_0\mathbf{Nil}(R) \to K_0(R[t]) \to K_0(R[t,t^{-1}]).$$

The result will follow once we calculate that the left map is zero. This map is induced by the forgetful functor $\mathbf{Nil}(R) \to \mathbf{H}(R[t])$ sending (P,ν) to P. Since the characteristic sequence (7.8.3) of ν shows that $[P] = 0$ in $K_0(R[t])$, we are done. □

Base change and Transfer Maps for Rings

7.9. Let $f\colon R \to S$ be a ring homomorphism. We have already seen that the base change $\otimes_R S\colon \mathbf{P}(R) \to \mathbf{P}(S)$ is an exact functor, inducing $f^*\colon K_0(R) \to K_0(S)$. If $S \in \mathbf{P}(R)$, we observed in (2.8.1) that the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ is exact, inducing the transfer map $f_*\colon K_0(S) \to K_0(R)$.

Using the Resolution Theorem, we can also define a transfer map f_* if $S \in \mathbf{H}(R)$. In this case every finitely generated projective S-module is in $\mathbf{H}(R)$, because if $P \oplus Q = S^n$ then $pd(P) \le pd(S^n) = pd(S) < \infty$. Hence there is an (exact) forgetful functor $\mathbf{P}(S) \to \mathbf{H}(R)$, and we define the transfer map to be the induced map

$$f_*\colon K_0(S) = K_0\mathbf{P}(S) \to K_0\mathbf{H}(R) \cong K_0(R). \tag{7.9.1}$$

A similar trick works to construct base change maps for the groups G_0. We saw in 6.2 that if S is flat as an R-module then $\otimes_R S$ is an exact functor $\mathbf{M}(R) \to \mathbf{M}(S)$ and we obtained a map $f^*\colon G_0(R) \to G_0(S)$. More generally, suppose that S has finite flat dimension $fd_R(S) = n$ as a left R-module, *i.e.*, that there is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to S \to 0$$

of R-modules, with the F_i flat. Let $\mathcal{F}$ denote the full subcategory of $\mathbf{M}(R)$ consisting of all finitely generated R-modules M with $\mathrm{Tor}_i^R(M,S)=0$ for $i\neq 0$; $\mathcal{F}$ is an exact category concocted so that $\otimes_R S$ defines an exact functor from $\mathcal{F}$ to $\mathbf{M}(S)$. Not only does $\mathcal{F}$ contain $\mathbf{P}(R)$, but from homological algebra one knows that (if R is noetherian) every finitely generated R-module has a finite resolution by objects in $\mathcal{F}$: for any projective resolution $P_\bullet \to M$ the kernel of $P_n \to P_{n-1}$ (the n^{th} *syzygy*) of any projective resolution will be in $\mathcal{F}$. The long exact Tor sequence shows that $\mathcal{F}$ is closed under kernels, so the Resolution Theorem applies to yield $K_0(\mathcal{F})\cong K_0(\mathbf{M}(R))=G_0(R)$. Therefore if R is noetherian and $fd_R(S)<\infty$ we can define the base change map $f^*\colon G_0(R)\to G_0(S)$ as the composite

$$G_0(R)\cong K_0(\mathcal{F}) \xrightarrow{\otimes} K_0\mathbf{M}(S)=G_0(S). \tag{7.9.2}$$

The following formula for f^* was used in §6 to show that $G_0(R)\cong G_0(R[x])$.

Serre's Formula 7.9.3. *Let $f\colon R\to S$ be a map between noetherian rings with $fd_R(S)<\infty$. Then the base change map $f^*\colon G_0(R)\to G_0(S)$ of (7.9.2) satisfies:*

$$f^*([M])=\sum(-1)^i\left[\mathrm{Tor}_i^R(M,S)\right].$$

Proof. Choose an $\mathcal{F}$-resolution $L_\bullet\to M$ (by R-modules L_i in $\mathcal{F}$):

$$0\to L_n\to\cdots\to L_1\to L_0\to M\to 0.$$

From homological algebra, we know that $\mathrm{Tor}_i^R(M,S)$ is the i^{th} homology of the chain complex $L_\bullet\otimes_R S$. By Prop. 7.5, the right-hand side of (7.9.3) equals

$$\chi(L_\bullet\otimes_R S)=\sum(-1)^i[L_i\otimes_R S]=f^*(\sum(-1)^i[L_i])=f^*([M]).$$

EXERCISES

7.1 Suppose that $\mathbf{P}$ is an exact subcategory of an abelian category $\mathcal{A}$, closed under kernels of surjections in $\mathcal{A}$. Suppose further that every object of $\mathcal{A}$ is a quotient of an object of $\mathbf{P}$ (as in Corollary 7.7.2). Let $\mathbf{P}_n\subset\mathcal{A}$ be the full subcategory of objects having $\mathbf{P}$-dimension $\le n$. Show that each $\mathbf{P}_n$ is an exact category closed under kernels of surjections, so that by the Resolution Theorem $K_0(\mathbf{P})\cong K_0(\mathbf{P}_n)$. *Hint.* If $0\to L\to P\to M\to 0$ is exact with $P\in\mathbf{P}$ and $M\in\mathbf{P}_1$, show that $L\in\mathbf{P}$.

7.2 Let $\mathcal{A}$ be a small exact category. If $[A_1]=[A_2]$ in $K_0(\mathcal{A})$, show that there are short exact sequences in $\mathcal{A}$

$$0\to C'\to C_1\to C''\to 0,\quad 0\to C'\to C_2\to C''\to 0$$

such that $A_1\oplus C_1\cong A_2\oplus C_2$. (Cf. Ex. 6.4.)

7.3 This exercise shows why the noetherian hypothesis was needed for G_0 in Corollary 6.3.1, and motivates the definition of $G_0(R)$ in 7.1.4. Let R be the ring $k\oplus I$, where I is an infinite-dimensional vector space over a field k, with multiplication given by $I^2=0$.

(a) (Swan) Show that $K_0\mathbf{mod}_{fg}(R) = 0$ but $K_0\mathbf{mod}_{fg}(R/I) = G_0(R/I) = \mathbb{Z}$.
(b) Show that every pseudo-coherent R-module is isomorphic to R^n for some n. Conclude that $G_0(R) = \mathbb{Z}$.

7.4 The groups $G_0(\mathbb{Z}[G])$ and $K_0\mathbf{mod}_{fg}(\mathbb{Z}[G])$ are very different for the free group G on two generators x and y. Let I be the two-sided ideal of $\mathbb{Z}[G]$ generated by y, so that $\mathbb{Z}[G]/I = \mathbb{Z}[x, x^{-1}]$. As a right module, $\mathbb{Z}[G]/I$ is not finitely presented.
(a) (Lück) Construct resolutions $0 \to \mathbb{Z}[G]^2 \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z}[G]/I \to \mathbb{Z}[G]/I \to \mathbb{Z} \to 0$, and conclude that $K_0\mathbf{mod}_{fg}(\mathbb{Z}[G]) = 0$
(b) Gersten proved in [Ger74] that $K_0(\mathbb{Z}[G]) = \mathbb{Z}$ by showing that every finitely presented $\mathbb{Z}[G]$-module is in $\mathbf{H}(\mathbb{Z}[G])$, i.e., has a finite resolution by finitely generated projective modules. Show that $G_0(\mathbb{Z}[G]) \cong K_0(\mathbb{Z}[G]) \cong \mathbb{Z}$.

7.5 *Naturality of base change.* Let $R \xrightarrow{f} S \xrightarrow{g} T$ be maps between noetherian rings, with $\mathrm{fd}_R(S)$ and $\mathrm{fd}_S(T)$ finite. Show that $g^*f^* = (gf)^*$ as maps $G_0(R) \to G_0(T)$.

7.6 *Idempotent completion.* Suppose that $(\mathcal{C}, \mathcal{E})$ is an exact category. Show that there is a natural way to make the idempotent completion $\widehat{\mathcal{C}}$ of $\mathcal{C}$ into an exact category, with $\mathcal{C}$ an exact subcategory. As noted in 7.3, this proves that $K_0(\mathcal{C})$ is a subgroup of $K_0(\widehat{\mathcal{C}})$.

7.7 Let $\mathcal{C}$ be a small additive category, and $\mathcal{A} = \mathbf{Ab}^{\mathcal{C}^{op}}$ the (abelian) category of all additive contravariant functors from $\mathcal{C}$ to $\mathbf{Ab}$. The Yoneda embedding $h: \mathcal{C} \to \mathcal{A}$, defined by $h(C) = \mathrm{Hom}_{\mathcal{C}}(-, C)$, embeds $\mathcal{C}$ as a full subcategory of $\mathcal{A}$. Show that every object of $\mathcal{C}$ is a projective object in $\mathcal{A}$. Then conclude that this embedding makes $\mathcal{C}$ into a split exact category (see 7.1.2).

7.8 (Quillen). Let $\mathcal{C}$ be an exact category, with the family $\mathcal{E}$ of short exact sequences (and admissible monics i and admissible epis j)

$$0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0 \qquad (\dagger)$$

as in Definition 7.0. Show that the following three conditions hold:

(1) Any sequence in $\mathcal{C}$ isomorphic to a sequence in $\mathcal{E}$ is in $\mathcal{E}$. If (†) is a sequence in $\mathcal{E}$ then i is a kernel for j (resp. j is a cokernel for i) in $\mathcal{C}$. The class $\mathcal{E}$ contains all of the sequences

$$0 \to B \xrightarrow{(1,0)} B \oplus D \xrightarrow{(0,1)} D \to 0.$$

(2) The class of admissible epimorphisms (resp. monomorphisms) is closed under composition. If (†) is in $\mathcal{E}$ and $B \to B''$, $D' \to D$ are maps in $\mathcal{C}$ then the base change sequence $0 \to B \to (C \times_D D') \to D' \to 0$ and the cobase change sequence $0 \to B'' \to (B'' \amalg_B C) \to D \to 0$ are in $\mathcal{E}$.
(3) If $C \to D$ is a map in $\mathcal{C}$ possessing a kernel, and there is a map $C' \to C$ in $\mathcal{C}$ so that $C' \to D$ is an admissible epimorphism, then $C \to D$ is an admissible epimorphism. Dually, if $B \to C$ has a cokernel and some $B \to C \to C''$ is admissible monomorphism, then so is $B \to C$.

Keller [Ke90, App. A] has proven that (1) and (2) imply (3).

Quillen observed that a converse is true: let $\mathcal{C}$ be an additive category, equipped with a family $\mathcal{E}$ of sequences of the form (†). If conditions (1) and (2) hold, then

$\mathcal{C}$ is an exact category in the sense of definition 7.0. The ambient abelian category used in 7.0 is the category $\mathcal{L}$ of contravariant *left exact* functors: additive functors $F: \mathcal{C} \to \mathbf{Ab}$ which carry each (†) to a "left" exact sequence

$$0 \to F(D) \to F(C) \to F(B),$$

and the embedding $\mathcal{C} \subset \mathcal{L}$ is the Yoneda embedding.

We refer the reader to Appendix A of [TT] for a detailed proof that $\mathcal{E}$ is the class of sequences in $\mathcal{C}$ which are exact in $\mathcal{L}$, as well as the following useful result: If $\mathcal{C}$ is idempotent complete then it is closed under kernels of surjections in $\mathcal{L}$.

7.9 Let $\{\mathcal{C}_i\}$ be a filtered system of exact categories and exact functors. Use Ex. 7.8 to generalize Example 7.1.7, showing that $\mathcal{C} = \varinjlim \mathcal{C}_i$ is an exact category and that $K_0(\mathcal{C}) = \varinjlim K_0(\mathcal{C}_i)$.

7.10 *Projection Formula for rings.* Suppose that R is a commutative ring, and A is an R-algebra which as an R-module is in $\mathbf{H}(R)$. By Ex. 2.1, $\otimes_R$ makes $K_0(A)$ into a $K_0(R)$-module. Generalize Ex. 2.2 to show that the transfer map $f_*: K_0(A) \to K_0(R)$ is a $K_0(R)$-module map, *i.e.,* that the *projection formula* holds:

$$f_*(x \cdot f^*y) = f_*(x) \cdot y \quad \text{for every} \quad x \in K_0(A),\ y \in K_0(R).$$

7.11 For a localization $f : R \to S^{-1}R$ at a central set of nonzerodivisors, every $\alpha : S^{-1}P \to S^{-1}Q$ has the form $\alpha = \gamma/s$ for some $\gamma \in \mathrm{Hom}_R(P,Q)$ and $s \in S$. Show that $[(P, \gamma/s, Q)] \mapsto [Q/\gamma(P)] - [Q/sQ]$ defines an isomorphism $K_0(f) \to K_0\mathbf{H}_S(R)$ identifying the sequences (2.10.2) and 7.7.4.

7.12 This exercise generalizes the Localization Theorem 6.4. Let $\mathcal{C}$ be an exact subcategory of an abelian category $\mathcal{A}$, closed under extensions and kernels of surjections, and suppose that $\mathcal{C}$ contains a Serre subcategory $\mathcal{B}$ of $\mathcal{A}$. Let $\mathcal{C}/\mathcal{B}$ denote the full subcategory of $\mathcal{A}/\mathcal{B}$ on the objects of $\mathcal{C}$. Considering $\mathcal{B}$-isos $A \to C$ with C in $\mathcal{C}$, show that the following sequence is exact:

$$K_0(\mathcal{B}) \to K_0(\mathcal{C}) \xrightarrow{\text{loc}} K_0(\mathcal{C}/\mathcal{B}) \to 0.$$

7.13 *δ-functors.* Let $T = \{T_i: \mathcal{C} \to \mathcal{A}, i \geq 0\}$ be a homological δ-functor from an exact category $\mathcal{C}$ to an abelian category $\mathcal{A}$, *i.e.,* for every exact sequence (†) in $\mathcal{C}$ we have a long exact sequence in $\mathcal{A}$:

$$\cdots \to T_1(D) \xrightarrow{\delta} T_0(B) \to T_0(C) \to T_0(D) \to 0.$$

Let $\mathcal{F}$ denote the category of all C in $\mathcal{C}$ such that $T_i(C) = 0$ for all $i > 0$, and assume that every C in $\mathcal{C}$ is a quotient of some object of $\mathcal{F}$.

(a) Show that $K_0(\mathcal{F}) \cong K_0(\mathcal{C})$, and that T defines a map $K_0(\mathcal{C}) \to K_0(\mathcal{A})$ sending $[C]$ to $\sum(-1)^i[T_iC]$. (Cf. Ex. 6.6.)

(b) Suppose that $f: X \to Y$ is a map of noetherian schemes, and that $\mathcal{O}_X$ has finite flat dimension over $f^{-1}\mathcal{O}_Y$. Show that there is a base change map $f^*: G_0(Y) \to G_0(X)$ satisfying $f^*g^* = (gf)^*$, generalizing (7.9.2) and Ex. 7.5.

7.14 This exercise is a refined version of Ex. 6.12. Consider $S = R[x_0, \cdots, x_m]$ as a graded ring with $x_1, \cdots, x_n$ in S_1, and let $\mathbf{M}_{gr}(S)$ denote the exact category of finitely generated graded S-modules.

(a) Use Ex. 7.13 with $T_i = \mathrm{Tor}_i^S(-, R)$ to show that $K_0\mathbf{M}_{gr}(S) \cong G_0(R)[\sigma, \sigma^{-1}]$.

(b) Use (a) and Ex. 6.12(e) to obtain an exact sequence

$$G_0(R)[\sigma, \sigma^{-1}] \xrightarrow{i} G_0(R)[\sigma, \sigma^{-1}] \to G_0(R[x]) \to 0.$$

Then show that the map i sends α to $\alpha - \sigma\alpha$.

(c) Conclude that $G_0(R) \cong G_0(R[x])$.

7.15 Let R be a noetherian ring. Show that the groups $K_0\mathbf{M}_i(R)$ of Application 6.4.3 are all $K_0(R)$-modules, and that the subgroups F^i in the coniveau filtration of $G_0(R)$ are $K_0(R)$-submodules. Conclude that if R is regular then the F^i are ideals in the ring $K_0(R)$.

7.16 (Grayson) Show that the operations $\lambda^n(P, \alpha) = (\wedge^n P, \wedge^n \alpha)$ make $K_0\mathbf{End}(R)$ and $\mathrm{End}_0(R)$ into λ-rings. Then show that the ring map $\mathrm{End}_0(R) \to W(R)$ (of 7.4.3) is a λ-ring injection, where $W(R)$ is the ring of big Witt vectors of R (see Example 4.3). Conclude that $\mathrm{End}_0(R)$ is a special λ-ring (4.3.1).

The exact endofunctors $F_m : (P, \alpha) \mapsto (P, \alpha^m)$ and $V_m : (P, \alpha) \mapsto (P[t]/t^m - \alpha, t)$ on $\mathbf{End}(R)$ induce operators F_m and V_m on $\mathrm{End}_0(R)$. Show that they agree with the classical Frobenius and Verschiebung operators, respectively.

7.17 This exercise is a refinement of 7.4.4. Let $F_n\mathbf{Nil}(R)$ denote the full subcategory of $\mathbf{Nil}(R)$ on the (P, ν) with $\nu^n = 0$. Show that $F_n\mathbf{Nil}(R)$ is an exact subcategory of $\mathbf{Nil}(R)$. If R is an algebra over a commutative ring k, show that the kernel $F_n\,\mathrm{Nil}_0(R)$ of $K_0F_n\mathbf{Nil}(R) \to K_0\mathbf{P}(R)$ is an $\mathrm{End}_0(k)$-module, and $F_n\,\mathrm{Nil}_0(R) \to \mathrm{Nil}_0(R)$ is a module map.

The exact endofunctor $F_m : (P, \nu) \mapsto (P, \nu^m)$ on $\mathbf{Nil}(R)$ is zero on $F_n\mathbf{Nil}(R)$. For $\alpha \in \mathrm{End}_0(k)$ and $(P, \nu) \in \mathrm{Nil}_0(R)$, show that $(V_m\alpha) \cdot (P, \nu) = V_m(\alpha \cdot F_m(P, \nu))$, and conclude that $V_m\,\mathrm{End}_0(k)$ acts trivially on the image of $F_m\,\mathrm{Nil}_0(A)$ in $\mathrm{Nil}_0(A)$.

7.18 Let $\alpha_n = \alpha_n(a_1, ..., a_n)$ denote the $n \times n$ matrix over a commutative ring R:

$$\alpha_n(a_1, ..., a_n) = \begin{pmatrix} 0 & & & -a_n \\ 1 & 0 & & -a_{n-1} \\ & \ddots & \ddots & \vdots \\ & & 1 & -a_1 \end{pmatrix}.$$

(a) Show that $[(R^n, \alpha_n)] = 1 + a_1t + \cdots + a_nt^n$ in $W(R)$. Conclude that the image of the map $\mathrm{End}_0(R) \to W(R)$ in 7.4.3 is indeed the subgroup of all quotients $f(t)/g(t)$ of polynomials in $1 + tR[t]$.

(b) Let A be an R-algebra. Recall that $(R^n, \alpha_n) * [(P, \nu)] = [(P^n, \alpha_n\nu)]$ in the $\mathrm{End}_0(R)$-module $\mathrm{Nil}_0(A)$ (see 7.4.4). Show that $(R^{n+1}, \alpha_{n+1}(a_1, \ldots, a_n, 0)) * [(P, \nu)] = (R^n, \alpha_n) * [(P, \nu)]$.

(c) Use 7.4.5 with $R = \mathbb{Z}[a_1, ..., a_n]$ to show that $(R^n, \alpha_n) * [(P, \nu)] = [(P^n, \beta)]$, $\beta = \alpha_n(a_1\nu, ..., a_n\nu^n)$. If $\nu^N = 0$, this is clearly independent of the a_i for $i \geq N$.

(d) Conclude that the $\mathrm{End}_0(R)$-module structure on $\mathrm{Nil}_0(A)$ extends to a $W(R)$-module structure by the formula

$$\left(1 + \sum a_it^i\right) * [(P, \nu)] = (R^n, \alpha_n(a_1, ..., a_n)) * [(P, \nu)], \quad n \gg 0.$$

7.19 (Lam) If R is a commutative ring, and Λ is an R-algebra, we write $G_0^R(\Lambda)$ for $K_0\mathbf{Rep}_R(\Lambda)$, where $\mathbf{Rep}_R(\Lambda)$ denotes the full subcategory of **mod**-Λ consisting of modules M which are finitely generated and projective as R-modules. If $\Lambda = R[G]$ is the group ring of a group G, the tensor product $M \otimes_R N$ of two $R[G]$-modules is again an $R[G]$-module where $g \in G$ acts by $(m \otimes n)g = mg \otimes ng$. Show that:

(a) $\otimes_R$ makes $G_0^R(R[G])$ an associative, commutative ring with identity $[R]$.

(b) $G_0^R(R[G])$ is an algebra over the ring $K_0(R)$, and $K_0(R[G])$ is a $G_0^R(R[G])$-module.

(c) If R is a regular ring and Λ is finitely generated projective as an R-module, $G_0^R(\Lambda) \cong G_0(\Lambda)$.

(d) If R is regular and G is finite, then $G_0(R[G])$ is a commutative $K_0(R)$-algebra, and that $K_0(R[G])$ is a module over $G_0(R[G])$.

7.20 (Deligne) A *filtered object* in an abelian category $\mathcal{A}$ is an object A together with a finite filtration $\cdots \subseteq W_nA \subseteq W_{n+1}A \subseteq \cdots$; if A and B are filtered, a morphism $f : A \to B$ in $\mathcal{A}$ is filtered if $F(W_nA) \subseteq W_nB$ for all n. The category $\mathcal{A}_{\text{filt}}$ of filtered objects in $\mathcal{A}$ is additive but not abelian (because images and coimages can differ). Let $\mathcal{E}$ denote the family of all sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}_{\text{filt}}$ such that each sequence $0 \to gr_n^W A \to gr_n^W B \to gr_n^W C \to 0$ is exact in $\mathcal{A}$.

(a) Show that $(\mathcal{A}_{\text{filt}}, \mathcal{E})$ is an exact category. (See [BBD, 1.1.4].)

(b) Show that $K_0(\mathcal{A}_{\text{filt}}) \cong \mathbb{Z} \times K_0(\mathcal{A})$.

7.21 *Replete exact categories.* A sequence $0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0$ in an additive category $\mathcal{C}$ is called *replete* if i is the categorical kernel of j, and j is the categorical cokernel of i. Let $\mathcal{E}_{rep}$ denote the class of all replete sequences, and show that $(\mathcal{C}, \mathcal{E}_{rep})$ is an exact category.

7.22 Fix a prime p, let $\mathbf{Ab}_p$ be the category of all finite abelian p-groups (6.2.2), and let $\mathcal{C}$ denote the full subcategory of all groups in $\mathbf{Ab}_p$ whose cyclic summands have even length (e.g., $\mathbb{Z}/p^{2i}$). Show that $\mathcal{C}$ is an additive category, but not an exact subcategory of $\mathbf{Ab}_p$. Let $\mathcal{E}$ be the sequences in $\mathcal{C}$ which are exact in $\mathbf{Ab}_p$; is $(\mathcal{C}, \mathcal{E})$ an exact category?

7.23 Give an example of a cofinal exact subcategory $\mathcal{B}$ of an exact category $\mathcal{C}$, such that the map $K_0\mathcal{B} \to K_0\mathcal{C}$ is not an injection (see 7.2).

7.24 Suppose that $\mathcal{C}_i$ are exact categories. Show that the product category $\prod \mathcal{C}_i$ is an exact category. Need $K_0(\prod \mathcal{C}_i) \to \prod K_0(\mathcal{C}_i)$ be an isomorphism?

7.25 (Claborn-Fossum). Set $R_n = \mathbb{C}[x_0, \cdots, x_n]/(\sum x_i^2 = 1)$. This is the complex coordinate ring of the n-sphere; it is a regular ring for every n, and $R_1 \cong \mathbb{C}[z, z^{-1}]$. In this exercise, we show that

$$\tilde{K}_0(R_n) \cong \widetilde{KU}(S^n) \cong \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbb{Z} & \text{if } n \text{ is even, } (n \neq 0) \end{cases}.$$

(a) Set $z = x_0 + ix_1$ and $\bar{z} = x_0 - ix_1$, so that $z\bar{z} = x_0^2 + x_1^2$. Show that

$$R_n[z^{-1}] \cong \mathbb{C}[z, z^{-1}, x_2, \ldots, x_n]$$

$$R_n/zR_n \cong R_{n-2}[\bar{z}], \quad n \geq 2.$$

(b) Use (a) to show that $\tilde{K}_0(R_n) = 0$ for n odd, and that if n is even there is a surjection $\beta\colon K_0(R_{n-2}) \to \tilde{K}_0(R_n)$.

(c) If n is even, show that β sends $[R_{n-2}]$ to zero, and conclude that there is a surjection $\mathbb{Z} \to \tilde{K}_0(R_n)$.

Fossum produced a finitely generated projective R_{2n}-module P_n such that the map $\tilde{K}_0(R_{2n}) \to \widetilde{KU}(S^{2n}) \cong \mathbb{Z}$ sends $[P_n]$ to the generator. (See [Foss].)

(d) Use the existence of P_n to finish the calculation of $K_0(R_n)$.

7.26 (Keller) Recall that any exact category $\mathcal{C}$ embeds into the abelian category $\mathcal{L}$ of left exact functors $\mathcal{C} \to \mathbf{Ab}$, and is closed under extensions (see Ex. 7.8). The *countable envelope* $\mathcal{C}^e$ of $\mathcal{C}$ is the full subcategory of $\mathcal{L}$ consisting of all colimits of sequences $A_0 \rightarrowtail A_1 \rightarrowtail \cdots$ of admissible monics in $\mathcal{C}$. Show that countable direct sums exist in $\mathcal{C}^e$. Then use the Eilenberg Swindle (I.2.8) to show that $K_0(\mathcal{C}^e) = 0$.

§8. K_0 of Schemes and Varieties

We have already introduced the Grothendieck group $K_0(X)$ of a scheme X in Example 7.1.3. By definition, it is $K_0\mathbf{VB}(X)$, where $\mathbf{VB}(X)$ denotes the (exact) category of vector bundles on X. The tensor product of vector bundles makes $K_0(X)$ into a commutative ring, as we saw in 7.4.2. This ring structure is natural in X: K_0 is a contravariant functor from schemes to commutative rings. Indeed, we saw in I.5.2 that a morphism of schemes $f: X \to Y$ induces an exact base change functor $f^*: \mathbf{VB}(Y) \to \mathbf{VB}(X)$, preserving tensor products, and such an exact functor induces a (ring) homomorphism $f^*: K_0(Y) \to K_0(X)$.

In this section we shall study $K_0(X)$ in more depth. Such a study requires that the reader has somewhat more familiarity with algebraic geometry than we assumed in the previous section, which is why this study has been isolated in its own section. We begin with two general invariants: the rank and determinant of a vector bundle.

The ring of continuous functions $X \to \mathbb{Z}$ is isomorphic to the global sections of the constant sheaf $\mathbb{Z}$, i.e., to the cohomology group $H^0(X;\mathbb{Z})$; see [Hart, I.1.0.3]. We saw in I.5.1 that the rank of a vector bundle $\mathcal{F}$ is a continuous function, so $\text{rank}(\mathcal{F}) \in H^0(X;\mathbb{Z})$. Similarly, we saw in I.5.3 that the determinant of $\mathcal{F}$ is a line bundle on X, *i.e.,* $\det(\mathcal{F}) \in \text{Pic}(X)$.

THEOREM 8.1. *Let X be a scheme. Then $H^0(X;\mathbb{Z})$ is isomorphic to a subring of $K_0(X)$, and the rank of a vector bundle induces a split surjection of rings*

$$\text{rank}: K_0(X) \to H^0(X;\mathbb{Z}).$$

Similarly, the determinant of a vector bundle induces a surjection of abelian groups

$$\det: K_0(X) \to \text{Pic}(X).$$

Their sum $\text{rank} \oplus \det: K_0(X) \to H^0(X;\mathbb{Z}) \oplus \text{Pic}(X)$ *is a surjective ring map.*

The ring structure on $H^0(X;\mathbb{Z}) \oplus \text{Pic}(X)$ is $(a_1, \mathcal{L}_1)\cdot(a_2, \mathcal{L}_2) = (a_1a_2, \mathcal{L}_1^{a_2} \otimes \mathcal{L}_2^{a_1})$.

PROOF. Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of vector bundles on X. At any point x of X we have an isomorphism of free $\mathcal{O}_x$-modules $\mathcal{F}_x \cong \mathcal{E}_x \oplus \mathcal{G}_x$, so $\text{rank}_x(\mathcal{F}) = \text{rank}_x(\mathcal{E}) + \text{rank}_x(\mathcal{G})$. Hence each rank_x is an additive function on $\mathbf{VB}(X)$. As x varies rank becomes an additive function with values in $H^0(X;\mathbb{Z})$, so by 6.1.2 it induces a map $\text{rank}: K_0(X) \to H^0(X;\mathbb{Z})$. This is a ring map, since the formula $\text{rank}(\mathcal{E} \otimes \mathcal{F}) = \text{rank}(\mathcal{E}) \cdot \text{rank}(\mathcal{F})$ may be checked at each point x. If $f: X \to \mathbb{N}$ is continuous, the componentwise free module $\mathcal{O}_X^f$ has rank f. It follows that rank is onto. Since the class of componentwise free $\mathcal{O}_X$-modules are closed under $\oplus$ and $\otimes$, the elements $[\mathcal{O}_X^f] - [\mathcal{O}_X^g]$ in $K_0(X)$ form a subring isomorphic to $H^0(X;\mathbb{Z})$.

Similarly, det is an additive function, because we have $\det(\mathcal{F}) \cong \det(\mathcal{E}) \otimes \det(\mathcal{G})$ by Ex. I.5.4. Hence det induces a map $K_0(X) \to \text{Pic}(X)$ by 6.1.2. If $\mathcal{L}$ is a line bundle on X, then the element $[\mathcal{L}] - [\mathcal{O}_X]$ of $K_0(X)$ has rank zero and determinant $\mathcal{L}$. Hence $\text{rank} \oplus \det$ is onto; the proof that it is a ring map is given in Ex. 8.5. $\square$

Definition 8.1.1. As in 2.3 and 2.6.1, the ideal $\widetilde{K}_0(X)$ of $K_0(X)$ is defined to be the kernel of the rank map, so that $K_0(X) = H^0(X;\mathbb{Z}) \oplus \widetilde{K}_0(X)$ as an abelian group. In addition, we let $SK_0(X)$ denote the kernel of rank $\oplus$ det. By Theorem 8.1, these are both ideals of the ring $K_0(X)$. In fact, they form the beginning of the γ-filtration.

Regular Noetherian Schemes and the Cartan Map

Historically, the group $K_0(X)$ first arose when X is a smooth projective variety, in Grothendieck's proof of the Riemann-Roch Theorem (see [BoSe]). The following theorem was central to that proof.

Recall from §6 that $G_0(X)$ is the Grothendieck group of the category $\mathbf{M}(X)$ of coherent $\mathcal{O}_X$-modules. The inclusion $\mathbf{VB}(X) \subset \mathbf{M}(X)$ induces a natural map $K_0(X) \to G_0(X)$, called the *Cartan homomorphism* (see 7.1.3).

Theorem 8.2. *If X is a separated regular noetherian scheme, then every coherent $\mathcal{O}_X$-module has a finite resolution by vector bundles, and the Cartan homomorphism is an isomorphism:*

$$K_0(X) \xrightarrow{\cong} G_0(X).$$

Proof. The first assertion is [SGA6, II, 2.2.3 and 2.2.7.1]. It implies that the Resolution Theorem 7.6 applies to the inclusion $\mathbf{VB}(X) \subset \mathbf{M}(X)$. □

Proposition 8.2.1 (Nonsingular Curves). *Let X be a 1-dimensional separated regular noetherian scheme, such as a nonsingular curve. Then $SK_0(X) = 0$, and*

$$K_0(X) = H^0(X;\mathbb{Z}) \oplus \mathrm{Pic}(X).$$

Proof. Given Theorem 8.2, this does follow from Ex. 6.10 (see Example 8.2.2 below). However, we shall give a slightly different proof here.

Without loss of generality, we may assume that X is irreducible. If X is affine, this is just Corollary 2.6.3. Otherwise, choose any closed point P on X. By [Hart, Ex. IV.1.3] the complement $U = X - P$ is affine, say $U = \mathrm{Spec}(R)$. Under the isomorphism $\mathrm{Pic}(X) \cong Cl(X)$ of I.5.14, the line bundles $\mathcal{L}(P)$ correspond to the class of the Weil divisor $[P]$. Hence the right-hand square commutes in the following diagram

$$\begin{array}{cccccccc} & G_0(P) & \xrightarrow{i_*} & \widetilde{K}_0(X) & \to & \widetilde{K}_0(R) & \to 0 \\ & \vdots\downarrow & & \downarrow \det & & \cong\downarrow\det & \\ 0\to & \mathbb{Z} & \xrightarrow{\{\mathcal{L}(P)\}} & \mathrm{Pic}(X) & \to & \mathrm{Pic}(R) & \to 0. \end{array}$$

The top row is exact by 6.4.2 (and 8.2), and the bottom row is exact by I.5.14 and Ex. I.5.12. The right vertical map is an isomorphism by 2.6.2.

Now $G_0(P) \cong \mathbb{Z}$ on the class $[\mathcal{O}_P]$. From the exact sequence $0 \to \mathcal{L}(-P) \to \mathcal{O}_X \to \mathcal{O}_P \to 0$ we see that $i_*[\mathcal{O}_P] = [\mathcal{O}_X] - [\mathcal{L}(-P)]$ in $K_0(X)$, and $\det(i_*[\mathcal{O}_P]) = \det \mathcal{L}(-P)^{-1}$ in $\mathrm{Pic}(X)$. Hence the isomorphism $G_0(P) \cong \mathbb{Z}$ is compatible with the above diagram. A diagram chase yields $\widetilde{K}_0(X) \cong \mathrm{Pic}(X)$. □

EXAMPLE 8.2.2 (CLASSES OF SUBSCHEMES). Let X be a separated noetherian regular scheme. Given a subscheme Z of X, it is convenient to write $[Z]$ for the element $[\mathcal{O}_Z] \in K_0\mathbf{M}(X) = K_0(X)$. By Ex. 6.10(d) we see that $SK_0(X)$ is the subgroup of $K_0(X)$ generated by the classes $[Z]$ as Z runs through the irreducible subschemes of codimension ≥ 2. In particular, if $\dim(X) = 2$ then $SK_0(X)$ is generated by the classes $[P]$ of closed points (of codimension 2).

TRANSFER FOR FINITE AND PROPER MAPS TO REGULAR SCHEMES 8.2.3. Let $f: X \to Y$ be a finite morphism of separated noetherian schemes with Y regular. As pointed out in 6.2.5, the direct image f_* is an exact functor $\mathbf{M}(X) \to \mathbf{M}(Y)$. In this case we have a transfer map f_* on K_0 sending $[\mathcal{F}]$ to $[f_*\mathcal{F}]$: $K_0(X) \to G_0(X) \to G_0(Y) \cong K_0(Y)$.

If $f: X \to Y$ is a proper morphism of separated noetherian schemes with Y regular, we can use the transfer $G_0(X) \to G_0(Y)$ of Lemma 6.2.6 to get a functorial transfer map $f_*: K_0(X) \to K_0(Y)$, this time sending $[\mathcal{F}]$ to $\sum(-1)^i[R^i f_*\mathcal{F}]$.

A NON-SEPARATED EXAMPLE 8.2.4. Here is an example of a regular but non-separated scheme X with $K_0\mathbf{VB}(X) \neq G_0(X)$. Let X be "affine n-space with a double origin" over a field k, where $n \geq 2$. This scheme is the union of two copies of $\mathbb{A}^n = \mathrm{Spec}(k[x_1, \dots, x_n])$ along $\mathbb{A}^n - \{0\}$. Using the localization sequence for either origin and the Fundamental Theorem 6.5, one can show that $G_0(X) = \mathbb{Z} \oplus \mathbb{Z}$. However the inclusion $\mathbb{A}^n \subset X$ is known to induce an equivalence $\mathbf{VB}(X) \cong \mathbf{VB}(\mathbb{A}^n)$ (see [EGA, IV(5.9)]), so by Theorem 7.8 we have $K_0\mathbf{VB}(X) \cong K_0(k[x_1, \dots, x_n]) \cong \mathbb{Z}$.

DEFINITION 8.3. Let $\mathbf{H}(X)$ denote the category consisting of all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ such that, for each affine open subscheme $U = \mathrm{Spec}(R)$ of X, $\mathcal{F}|_U$ has a finite resolution by vector bundles. Since $\mathcal{F}|_U$ is defined by the finitely generated R-module $M = \mathcal{F}(U)$ this condition just means that M is in $\mathbf{H}(R)$.

If X is regular and separated, then we saw in Theorem 8.2 that $\mathbf{H}(X) = \mathbf{M}(X)$. If $X = \mathrm{Spec}(R)$, it is easy to see that $\mathbf{H}(X)$ is equivalent to $\mathbf{H}(R)$.

$\mathbf{H}(X)$ is an exact subcategory of $\mathcal{O}_X$-**mod**, closed under kernels of surjections, because each $\mathbf{H}(R)$ is closed under extensions and kernels of surjections in R-**mod**.

To say much more about the relation between $\mathbf{H}(X)$ and $K_0(X)$, we need to restrict our attention to quasi-compact schemes such that every $\mathcal{F}$ in $\mathbf{H}(X)$ is a quotient of a vector bundle $\mathcal{E}_0$. This implies that every module $\mathcal{F} \in \mathbf{H}(X)$ has a finite resolution $0 \to \mathcal{E}_d \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ by vector bundles. Indeed, the kernel $\mathcal{F}'$ of a quotient map $\mathcal{E}_0 \to \mathcal{F}$ is always locally of lower projective dimension than $\mathcal{F}$, and X has a finite affine cover by $U_i = \mathrm{Spec}(R_i)$, it follows that the d^{th} syzygy is a vector bundle, where $d = \max\{pd_{R_i} M_i\}$, $M_i = \mathcal{F}(U_i)$.

For this condition to hold, it is easiest to assume that X is *quasi-projective* (over a commutative ring k), *i.e.,* a locally closed subscheme of some projective space $\mathbb{P}^n_k$. By [EGA II, 4.5.5 and 4.5.10], this implies that every quasicoherent $\mathcal{O}_X$-module of finite type $\mathcal{F}$ is a quotient of some vector bundle $\mathcal{E}_0$ of the form $\mathcal{E}_0 = \bigoplus \mathcal{O}_X(n_i)$.

PROPOSITION 8.3.1. *If X is quasi-projective (over a commutative ring), then* $K_0(X) \cong K_0\mathbf{H}(X)$.

PROOF. Because $\mathbf{H}(X)$ is closed under kernels of surjections in $\mathcal{O}_X$-**mod**, and every object in $\mathbf{H}(X)$ has a finite resolution by vector bundles, the Resolution Theorem 7.6 applies to $\mathbf{VB}(X) \subset \mathbf{H}(X)$. □

TECHNICAL REMARK 8.3.2. Another assumption that guarantees that every $\mathcal{F}$ in $\mathbf{H}(X)$ is a quotient of a vector bundle is that X be quasi-separated and quasi-compact with an ample family of line bundles. Such schemes are called *divisorial* in [SGA6, II.2.2.4]. For such schemes, the proof of 8.3.1 goes through to show that we again have $K_0\mathbf{VB}(X) \cong K_0\mathbf{H}(X)$.

RESTRICTING BUNDLES 8.3.3. Given an open subscheme U of a quasi-projective scheme X, let $\mathcal{B}$ denote the full subcategory of $\mathbf{VB}(U)$ consisting of vector bundles $\mathcal{F}$ whose class in $K_0(U)$ is in the image of $j^*\colon K_0(X) \to K_0(U)$. We claim that the category $\mathcal{B}$ is cofinal in $\mathbf{VB}(U)$, so that $K_0\mathcal{B}$ is a subgroup of $K_0(U)$ by the Cofinality Lemma 7.2. To see this, note that each vector bundle $\mathcal{F}$ on U fits into an exact sequence $0 \to \mathcal{F}' \to \mathcal{E}_0 \to \mathcal{F} \to 0$, where $\mathcal{E}_0 = \bigoplus \mathcal{O}_U(n_i)$. But then $\mathcal{F} \oplus \mathcal{F}'$ is in $\mathcal{B}$, because in $K_0(U)$

$$[\mathcal{F} \oplus \mathcal{F}'] = [\mathcal{F}] + [\mathcal{F}'] = [\mathcal{E}_0] = \sum j^*[\mathcal{O}_X(n_i)].$$

Transfer Maps for Schemes

8.4 We can define a transfer map $f_*\colon K_0(X) \to K_0(Y)$ with $(gf)_* = g_*f_*$ associated to various morphisms $f\colon X \to Y$. If Y is regular, we have already done this in 8.2.3.

Suppose first that f is a finite map. In this case, the inverse image of any affine open $U = \operatorname{Spec}(R)$ of Y is an affine open $f^{-1}U = \operatorname{Spec}(S)$ of X, S is finitely generated as an R-module, and the direct image sheaf $f_*\mathcal{O}_X$ satisfies $f_*\mathcal{O}(U) = S$. Thus the direct image functor f_* is an exact functor from $\mathbf{VB}(X)$ to $\mathcal{O}_Y$-modules (as pointed out in 6.2.5).

If f is finite and $f_*\mathcal{O}_X$ is a vector bundle then f_* is an exact functor from $\mathbf{VB}(X)$ to $\mathbf{VB}(Y)$. Indeed, locally it sends each finitely generated projective S-module to a finitely generated projective R-module, as described in Example 2.8.1. Thus there is a canonical transfer map $f_*\colon K_0(X) \to K_0(Y)$ sending $[\mathcal{F}]$ to $[f_*\mathcal{F}]$.

If f is finite and $f_*\mathcal{O}_X$ is in $\mathbf{H}(X)$ then f_* sends $\mathbf{VB}(X)$ into $\mathbf{H}(X)$, because locally it is the forgetful functor $\mathbf{P}(S) \to \mathbf{H}(R)$ of (7.9.1). Therefore f_* defines a homomorphism $K_0(X) \to K_0\mathbf{H}(Y)$. If Y is quasi-projective then composition with $K_0\mathbf{H}(Y) \cong K_0(Y)$ yields a "finite" transfer map $K_0(X) \to K_0(Y)$.

Now suppose that $f\colon X \to Y$ is a proper map between quasi-projective noetherian schemes. The transfer homomorphism $f_*\colon G_0(X) \to G_0(Y)$ was constructed in Lemma 6.2.6, with $f_*[\mathcal{F}] = \sum(-1)^i[R^if_*\mathcal{F}]$.

If in addition f has finite Tor-dimension, then we can also define a transfer map $f_*\colon K_0(X) \to K_0(Y)$, following [SGA 6, IV.2.12.3]. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is called f_**-acyclic* if $R^qf_*\mathcal{F} = 0$ for all $q > 0$. Let $\mathbf{P}(f)$ denote the category of all vector bundles $\mathcal{F}$ on X such that $\mathcal{F}(n)$ is f_*-acyclic for all $n \geq 0$. By the usual yoga of homological algebra, $\mathbf{P}(f)$ is an exact category, closed under cokernels of injections, and f_* is an exact functor from $\mathbf{P}(f)$ to $\mathbf{H}(Y)$. Hence the following lemma allows us to define the transfer map as

$$K_0(X) \xleftarrow{\cong} K_0\mathbf{P}(f) \xrightarrow{f_*} K_0\mathbf{H}(Y) \xleftarrow{\cong} K_0(Y) \tag{8.4.1}$$

Lemma 8.4.2. *Every vector bundle $\mathcal{F}$ on a quasi-projective X has a finite resolution*

$$0 \to \mathcal{F} \to \mathcal{P}_0 \to \cdots \to \mathcal{P}_m \to 0$$

by vector bundles in $\mathbf{P}(f)$. *Hence by the Resolution Theorem* $K_0\mathbf{P}(f) \cong K_0(X)$.

Proof. For $n \geq 0$ the vector bundle $\mathcal{O}_X(n)$ is generated by global sections. Dualizing the resulting surjection $\mathcal{O}_X^r \to \mathcal{O}_X(n)$ and twisting n times yields a short exact sequence of vector bundles $0 \to \mathcal{O}_X \to \mathcal{O}_X(n)^r \to \mathcal{E} \to 0$. Hence for every vector bundle $\mathcal{F}$ on X we have a short exact sequence of vector bundles $0 \to \mathcal{F} \to \mathcal{F}(n)^r \to \mathcal{E} \otimes \mathcal{F} \to 0$. For all large n, the sheaf $\mathcal{F}(n)$ is f_*-acyclic (see [EGA, III.3.2.1] or [Hart, III.8.8]), and $\mathcal{F}(n)$ is in $\mathbf{P}(f)$. Repeating this process with $\mathcal{E} \otimes \mathcal{F}$ in place of $\mathcal{F}$, we obtain the desired resolution of $\mathcal{F}$. □

Like the transfer map for rings, the transfer map f_* is a $K_0(Y)$-module homomorphism. (This is the *projection formula*; see Ex. 7.10 and Ex. 8.3.)

Projective Bundles

Let $\mathcal{E}$ be a vector bundle of rank $r+1$ over a quasi-compact scheme X, and let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ denote the projective space bundle of Example I.5.8. (If $\mathcal{E}|_U$ is free over $U \subseteq X$ then $\mathbb{P}|_U$ is the usual projective space $\mathbb{P}^r_U$.) Via the structural map $\pi\colon \mathbb{P} \to X$, the base change map is a ring homomorphism $\pi^*\colon K_0(X) \to K_0(\mathbb{P})$, sending $[\mathcal{M}]$ to $[f^*\mathcal{M}]$, where $f^*\mathcal{M} = \mathcal{O}_{\mathbb{P}} \otimes_X \mathcal{M}$. In this section we give Quillen's proof [Q341, §8] of the following result, originally due to Berthelot [SGA6, VI.1.1].

Projective Bundle Theorem 8.5. *Let $\mathbb{P}$ be the projective bundle of $\mathcal{E}$ over a quasi-compact scheme X. Then $K_0(\mathbb{P})$ is a free $K_0(X)$-module with basis the twisting line bundles* $\{1 = [\mathcal{O}_{\mathbb{P}}], [\mathcal{O}_{\mathbb{P}}(-1)], \dots, [\mathcal{O}_{\mathbb{P}}(-r)]\}$.

Corollary 8.6. *As a ring,* $K_0(\mathbb{P}^r_{\mathbb{Z}}) = \mathbb{Z}[z]/(z^{r+1})$, *where* $z = [\mathbb{P}^{r-1}] = 1 - [\mathcal{O}(-1)]$. *(The relation $z^{r+1} = 0$ is Ex. 6.14(b).)*
Hence $K_0(\mathbb{P}^r_X) \cong K_0(X) \otimes K_0(\mathbb{P}^r_{\mathbb{Z}}) = K_0(X)[z]/(z^{r+1})$.

To prove Theorem 8.5, we would like to apply the direct image functor π_* to a vector bundle $\mathcal{F}$ and get a vector bundle. This requires a vanishing condition. The proof of this result rests upon the following notion, which is originally due to Castelnuovo. It is named after David Mumford, who exploited it in [Mum].

Definition 8.7.1. A quasicoherent $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F}$ is called *Mumford-regular* if for all $q > 0$ the higher derived sheaves $R^q\pi_*(\mathcal{F}(-q))$ vanish. Here $\mathcal{F}(n)$ is $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(n)$, as in Example I.5.3.1. We write **MR** for the additive category of all Mumford-regular vector bundles, and abbreviate $\otimes_X$ for $\otimes_{\mathcal{O}_X}$.

Examples 8.7.2. If $\mathcal{N}$ is a quasicoherent $\mathcal{O}_X$-module then the standard cohomology calculations on projective spaces show that $\pi^*\mathcal{N} = \mathcal{O}_{\mathbb{P}} \otimes_X \mathcal{N}$ is Mumford-regular, with $\pi_*\pi^*\mathcal{N} = \mathcal{N}$. More generally, if $n \geq 0$ then $\pi^*\mathcal{N}(n)$ is Mumford-regular, with $\pi_*\pi^*\mathcal{N}(n) = Sym_n\mathcal{E} \otimes_X \mathcal{N}$. For $n < 0$ we have $\pi_*\pi^*\mathcal{N}(n) = 0$. In particular, $\mathcal{O}_{\mathbb{P}}(n) = \pi^*\mathcal{O}_X(n)$ is Mumford-regular for all $n \geq 0$.

If X is noetherian and $\mathcal{F}$ is coherent, then for $n \gg 0$ the twists $\mathcal{F}(n)$ are Mumford-regular, because the higher derived functors $R^q\pi_*\mathcal{F}(n)$ vanish for large n and also for $q > r$ (see [Hart, III.8.8]).

The following facts were discovered by Castelnuovo when $X = \text{Spec}(\mathbb{C})$, and proven in [Mum, Lecture 14] as well as [Q341, §8]:

PROPOSITION 8.7.3. *If $\mathcal{F}$ is Mumford-regular, then*

(1) *The twists $\mathcal{F}(n)$ are Mumford-regular for all $n \geq 0$;*
(2) *Mumford-regular modules are π_*-acyclic, and in fact $R^q\pi_*\mathcal{F}(n) = 0$ for all $q > 0$ and $n \geq -q$;*
(3) *The canonical map $\varepsilon\colon \pi^*\pi_*(\mathcal{F}) \to \mathcal{F}$ is onto.*

REMARK. Suppose that X is affine. Since $\pi^*\pi_*(\mathcal{F}) = \mathcal{O}_{\mathbb{P}} \otimes_X \pi_*\mathcal{F}$, and $\pi_*\mathcal{F}$ is quasicoherent, item (3) states that Mumford-regular sheaves are generated by their global sections.

LEMMA 8.7.4. *Mumford-regular modules form an exact subcategory of $\mathcal{O}_{\mathbb{P}}$-**mod**, and π_* is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules.*

PROOF. Suppose that $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of $\mathcal{O}_{\mathbb{P}}$-modules with both $\mathcal{F}'$ and $\mathcal{F}''$ Mumford-regular. From the long exact sequence

$$R^q\pi_*\mathcal{F}'(-q) \to R^q\pi_*\mathcal{F}(-q) \to R^q\pi_*\mathcal{F}''(-q)$$

we see that $\mathcal{F}$ is also Mumford-regular. Thus Mumford-regular modules are closed under extensions, *i.e.*, they form is an exact subcategory of $\mathcal{O}_{\mathbb{P}}$-**mod**. Since $\mathcal{F}'(1)$ is Mumford-regular, $R^1\pi_*\mathcal{F}' = 0$, and so we have an exact sequence

$$0 \to \pi_*\mathcal{F}' \to \pi_*\mathcal{F} \to \pi_*\mathcal{F}'' \to 0.$$

This proves that π_* is an exact functor. □

The following results were proven by Quillen in [Q341, §8].

LEMMA 8.7.5. *Let $\mathcal{F}$ be a vector bundle on $\mathbb{P}$.*

(1) *$\mathcal{F}(n)$ is a Mumford-regular vector bundle on $\mathbb{P}$ for all large enough n;*
(2) *If $\mathcal{F}(n)$ is π_*-acyclic for all $n \geq 0$ then $\pi_*\mathcal{F}$ is a vector bundle on X.*
(3) *Hence by 8.7.3, if $\mathcal{F}$ is Mumford-regular then $\pi_*\mathcal{F}$ is a vector bundle on X.*
(4) *$\pi^*\mathcal{N} \otimes_{\mathbb{P}} \mathcal{F}(n)$ is Mumford-regular for all large enough n, and all quasicoherent $\mathcal{O}_X$-modules $\mathcal{N}$.*

DEFINITION 8.7.6 (T_n). Given a Mumford-regular $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F}$, we define a natural sequence of $\mathcal{O}_X$-modules $T_n = T_n\mathcal{F}$ and $\mathcal{O}_{\mathbb{P}}$-modules $Z_n = Z_n\mathcal{F}$, starting with $T_0\mathcal{F} = \pi_*\mathcal{F}$ and $Z_{-1} = \mathcal{F}$. Let Z_0 be the kernel of the natural map $\varepsilon\colon \pi^*\pi_*\mathcal{F} \to \mathcal{F}$ of Proposition 8.7.3. Inductively, we define $T_n\mathcal{F} = \pi_*Z_{n-1}(n)$ and define Z_n to be $\ker(\varepsilon)(-n)$, where ε is the canonical map from $\pi^*T_n = \pi^*\pi_*Z_{n-1}(n)$ to $Z_{n-1}(n)$.

Thus we have sequences (exact except possibly at $Z_{n-1}(n)$)

$$0 \to Z_n(n) \to \pi^*(T_n\mathcal{F}) \xrightarrow{\varepsilon} Z_{n-1}(n) \to 0 \tag{8.7.7}$$

whose twists fit together into the sequence of the following theorem.

Quillen Resolution Theorem 8.7.8. *Let $\mathcal{F}$ be a bundle on $\mathbb{P}(\mathcal{E})$,* $\operatorname{rank}(\mathcal{E}) = r+1$. *If $\mathcal{F}$ is Mumford-regular then $Z_r = 0$, and the sequences (8.7.7) are exact for $n \geq 0$, so there is an exact sequence*

$$0 \to (\pi^* T_r\mathcal{F})(-r) \xrightarrow{\varepsilon(-r)} \cdots \to (\pi^* T_i\mathcal{F})(-i) \xrightarrow{\varepsilon(-i)} \cdots \xrightarrow{\varepsilon(-1)} (\pi^* T_0\mathcal{F}) \xrightarrow{\varepsilon} \mathcal{F} \to 0.$$

Moreover, each $\mathcal{F} \mapsto T_i\mathcal{F}$ is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules.

Proof. We first prove by induction on $n \geq 0$ that (a) the module $Z_{n-1}(n)$ is Mumford-regular, (b) $\pi_* Z_n(n) = 0$ and (c) the canonical map $\varepsilon\colon \pi^* T_n \to Z_{n-1}(n)$ is onto, *i.e.*, that (8.7.7) is exact for n.

We are given that (a) holds for $n = 0$, so we suppose that (a) holds for n. This implies part (c) for n by Proposition 8.7.3. Inductively then, we are given that (8.7.7) is exact, so $\pi_* Z_n(n) = 0$ and the module $Z_n(n+1)$ is Mumford-regular by Ex. 8.6. That is, (b) holds for n and (a) holds for $n+1$. This finishes the first proof by induction.

Using (8.7.7), another induction on n shows that (d) each $\mathcal{F} \mapsto Z_{n-1}\mathcal{F}(n)$ is an exact functor from Mumford-regular modules to itself, and (e) each $\mathcal{F} \mapsto T_n\mathcal{F}$ is an exact functor from Mumford-regular modules to $\mathcal{O}_X$-modules. Note that (d) implies (e) by Lemma 8.7.4, since $T_n = \pi_* Z_{n-1}(n)$.

Since the canonical resolution is obtained by splicing the exact sequences (8.7.7) together for $n = 0, ..., r$, all that remains is to prove that $Z_r = 0$, or equivalently, that $Z_r(r) = 0$. From (8.7.7) we get the exact sequence

$$R^{q-1}\pi_* Z_{n+q-1}(n) \to R^q\pi_* Z_{n+q}(n) \to R^q\pi_*(\pi^* T_n(-q))$$

which allows us to conclude, starting from (b) and 8.7.2, that $R^q\pi_*(Z_{n+q}) = 0$ for all $n, q \geq 0$. Since $R^q\pi_* = 0$ for all $q > r$, this shows that $Z_r(r)$ is Mumford-regular. Since $\pi^*\pi_* Z_r(r) = 0$ by (b), we see from Proposition 8.7.3(3) that $Z_r(r) = 0$ as well. $\square$

Corollary 8.7.9. *If $\mathcal{F}$ is Mumford-regular, each $T_i\mathcal{F}$ is a vector bundle on X.*

Proof. For every $n \geq 0$, the n^{th} twist of the Quillen resolution 8.7.8 yields exact sequences of π_*-acyclic modules. Thus applying π_* yields an exact sequence of $\mathcal{O}_X$-modules, which by 8.7.2 is

$$0 \to T_n \to \mathcal{E} \otimes T_{n-1} \to \cdots \to Sym_{n-i}\mathcal{E} \otimes T_i \to \cdots \to \pi_*\mathcal{F}(n) \to 0.$$

The result follows from this sequence and induction on i, since $\pi_*\mathcal{F}(n)$ is a vector bundle by Lemma 8.7.5(3). $\square$

Let $\mathbf{MR}(n)$ denote the n^{th} twist of $\mathbf{MR}$; it is the full subcategory of $\mathbf{VB}(\mathbb{P})$ consisting of vector bundles $\mathcal{F}$ such that $\mathcal{F}(-n)$ is Mumford-regular. Since twisting is an exact functor, each $\mathbf{MR}(n)$ is an exact category. By Lemma 8.7.3 we have

$$\mathbf{MR} = \mathbf{MR}(0) \subseteq \mathbf{MR}(-1) \subseteq \cdots \subseteq \mathbf{MR}(n) \subseteq \mathbf{MR}(n-1) \subseteq \cdots .$$

Proposition 8.7.10. *The inclusions* $\mathbf{MR}(n) \subset \mathbf{VB}(\mathbb{P})$ *induce isomorphisms* $K_0\mathbf{MR} \cong K_0\mathbf{MR}(n) \cong K_0(\mathbb{P})$.

Proof. The union of the $\mathbf{MR}(n)$ is $\mathbf{VB}(\mathbb{P})$ by Lemma 8.7.5(1). By Example 7.1.7 we have $K_0\mathbf{VB}(\mathbb{P}) = \varinjlim K_0\mathbf{MR}(n)$, so it suffices to show that each inclusion $\mathbf{MR}(n) \subset \mathbf{MR}(n-1)$ induces an isomorphism on K_0. For $i > 0$, let $u_i: \mathbf{MR}(n-1) \to \mathbf{MR}(n)$ be the exact functor $\mathcal{F} \mapsto \mathcal{F}(i) \otimes_X \wedge^i\mathcal{E}$. It induces a homomorphism $u_i: K_0\mathbf{MR}(n-1) \to K_0\mathbf{MR}(n)$. By Proposition 7.5 (Additivity), applied to the Koszul resolution

$$0 \to \mathcal{F} \to \mathcal{F}(1) \otimes_X \mathcal{E} \to \cdots \to \mathcal{F}(r+1) \otimes_X \wedge^{r+1}\mathcal{E} \to 0$$

we see that the map $\sum_{i>0}(-1)^{i-1}u_i$ is an inverse to the map $\iota_n: K_0\mathbf{MR}(n) \to K_0\mathbf{MR}(n-1)$ induced by the inclusion. Hence ι_n is an isomorphism, as desired. □

Proof of Projective Bundle Theorem 8.5. Each T_n is an exact functor from $\mathbf{MR}$ to $\mathbf{VB}(X)$ by Theorem 8.7.8 and 8.7.9. Hence we have a homomorphism

$$t: K_0\mathbf{MR} \to K_0(X)^{r+1}, \qquad [\mathcal{F}] \mapsto ([T_0\mathcal{F}], -[T_1\mathcal{F}], \dots, (-1)^r[T_r\mathcal{F}]).$$

This fits into the diagram

$$K_0(\mathbb{P}) \xleftarrow{\cong} K_0\mathbf{MR} \xrightarrow{t} K_0(X)^{r+1} \xrightarrow{u} K_0(\mathbb{P}) \xleftarrow{\cong} K_0\mathbf{MR} \xrightarrow{v} K_0(X)^{r+1}$$

where $u(a_0, ..., a_r) = \pi^*a_0 + \pi^*a_1 \cdot [\mathcal{O}_{\mathbb{P}}(-1)] + \cdots + \pi^*a_r \cdot [\mathcal{O}_{\mathbb{P}}(-r)]$ and $v[\mathcal{F}] = ([\pi_*\mathcal{F}], [\pi_*\mathcal{F}(1)], \dots, [\pi_*\mathcal{F}(r)])$. The composition ut sends $[\mathcal{F}]$ to the alternating sum of the $[(\pi^*T_i\mathcal{F})(-i)]$, which equals $[\mathcal{F}]$ by Quillen's Resolution Theorem. Hence u is a surjection.

Since the (i, j) component of vu sends $\mathcal{N}_j$ to $\pi_*(\pi^*\mathcal{N}_j(i-j)) = Sym_{i-j}\mathcal{E} \otimes_X \mathcal{N}_j$ by Example 8.7.2, it follows that the composition vu is given by a lower triangular matrix with ones on the diagonal. Therefore vu is an isomorphism, so u is injective. □

λ-operations in $K_0(X)$

The following result was promised in Example 4.1.5.

Proposition 8.8. *The operations* $\lambda^k[\mathcal{F}] = [\wedge^k\mathcal{F}]$ *are well-defined on* $K_0(X)$, *and make* $K_0(X)$ *into a* λ*-ring.*

Proof. It suffices to show that the formula $\lambda_t(\mathcal{F}) = \sum[\wedge^k\mathcal{F}]t^k$ defines an additive homomorphism from $\mathbf{VB}(X)$ to the multiplicative group $1+tK_0(X)[[t]]$. Note that the constant term in $\lambda_t(\mathcal{F})$ is 1 because $\wedge^0\mathcal{F} = \mathcal{O}_X$. Suppose given an exact sequence of vector bundles $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. By Ex. I.5.4, each $\wedge^k\mathcal{F}$ has a finite filtration whose associated quotient modules are the $\wedge^i\mathcal{F}' \otimes \wedge^{k-i}\mathcal{F}''$, so in $K_0(X)$ we have

$$[\wedge^k\mathcal{F}] = \sum[\wedge^i\mathcal{F}' \otimes \wedge^{k-i}\mathcal{F}''] = \sum[\wedge^i\mathcal{F}'] \cdot [\wedge^{k-i}\mathcal{F}''].$$

Assembling these equations yields the formula $\lambda_t(\mathcal{F}) = \lambda_t(\mathcal{F}')\lambda_t(\mathcal{F}'')$ in the group $1 + tK_0(X)[[t]]$, proving that λ_t is additive. Hence λ_t (and each coefficient λ^k) is well-defined on $K_0(X)$. □

Splitting Principle 8.8.1 (see 4.2.2). *Let $f\colon \mathbb{F}(\mathcal{E}) \to X$ be the flag bundle of a vector bundle $\mathcal{E}$ over a quasi-compact scheme X. Then $K_0(\mathbb{F}(\mathcal{E}))$ is a free module over the ring $K_0(X)$, and $f^*[\mathcal{E}]$ is a sum of line bundles $\sum[\mathcal{L}_i]$.*

Proof. Let $f\colon \mathbb{F}(\mathcal{E}) \to X$ be the flag bundle of $\mathcal{E}$; by Theorem I.5.9 the bundle $f^*\mathcal{E}$ has a filtration by sub-vector bundles whose successive quotients $\mathcal{L}_i$ are line bundles. Hence $f^*[\mathcal{E}] = \sum[\mathcal{L}_i]$ in $K_0(\mathbb{F}(\mathcal{E}))$. Moreover, we saw in I.5.8 that the flag bundle is obtained from X by a sequence of projective bundle extensions, beginning with $\mathbb{P}(\mathcal{E})$. By the Projective Bundle Theorem 8.5, $K_0(\mathbb{F}(\mathcal{E}))$ is obtained from $K_0(X)$ by a sequence of finite free extensions. □

The λ-ring $K_0(X)$ has a positive structure in the sense of Definition 4.2.1. The "positive elements" are the classes $[\mathcal{F}]$ of vector bundles, and the augmentation $\varepsilon\colon K_0(X) \to H^0(X;\mathbb{Z})$ is given by Theorem 8.1. In this vocabulary, the "line elements" are the classes $[\mathcal{L}]$ of line bundles on X, and the subgroup L of units in $K_0(X)$ is just $\mathrm{Pic}(X)$. The following corollary now follows from Theorems 4.2.3 and 4.7.

Corollary 8.8.2. *$K_0(X)$ is a special λ-ring. Consequently, the first two ideals in the γ-filtration of $K_0(X)$ are $F^1_\gamma = \widetilde{K}_0(X)$ and $F^2_\gamma = SK_0(X)$. In particular,*

$$F^0_\gamma/F^1_\gamma \cong H^0(X;\mathbb{Z}) \text{ and } \quad F^1_\gamma/F^2_\gamma \cong \mathrm{Pic}(X).$$

Corollary 8.8.3. *For every commutative ring R, $K_0(R)$ is a special λ-ring.*

Proposition 8.8.4. *If X is quasi-projective, or more generally if X has an ample line bundle $\mathcal{L}$ then every element of $\widetilde{K}_0(X)$ is nilpotent. Hence $\widetilde{K}_0(X)$ is a nil ideal of $K_0(X)$.*

Proof. By Ex. 4.5, it suffices to show that $\ell = [\mathcal{L}]$ is an ample line element. Given $x = [\mathcal{E}]-[\mathcal{F}]$ in $\widetilde{K}_0(X)$, the fact that $\mathcal{L}$ is ample implies that $\mathcal{F}(n) = \mathcal{F}\otimes\mathcal{L}^{\otimes n}$ is generated by global sections for all large n. Hence there are short exact sequences

$$0 \to \mathcal{G}_n \to \mathcal{O}_X^{r_n} \to \mathcal{F}(n) \to 0$$

and therefore in $K_0(X)$ we have the required equation:

$$\ell^n x = [\mathcal{E}(n)] - [\mathcal{O}_X^{r_n}] + [\mathcal{G}_n] = [\mathcal{E}(n) \oplus \mathcal{G}_n] - r_n. \qquad \square$$

Remark 8.8.5 (Nilpotence). If X is noetherian and quasiprojective of dimension d, then $\widetilde{K}_0(X)^{d+1} = 0$, because it lies inside F^{d+1}_γ, which vanishes by [SGA6, VI.6.6] or [FL, V.3.10]. (See Example 4.8.2.)

If X is a nonsingular algebraic variety, the Chow groups $CH^i(X)$ are defined to be the quotient of $D^i(X)$, the free group on the integral codimension i subvarieties, by rational equivalence; see 6.4.3. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank n, form the projective bundle $\mathbb{P}(\mathcal{E})$ and flag bundle $\mathbb{F}(\mathcal{E})$ of $\mathcal{E}$; see I.4.10. The Projective Bundle Theorem (see [Fulton]) states that the Chow group $CH^*(\mathbb{P}(\mathcal{E}))$ is a free graded module over $CH^*(X)$ with basis $\{1, \xi, \dots, \xi^{n-1}\}$, where $\xi \in CH^1(\mathbb{P}(\mathcal{E}))$ is the class of a divisor corresponding to $\mathcal{O}(1)$. We define the Chern classes $c_i(\mathcal{E})$ in

$CH^i(X)$ to be $(-1)^i$ times the coefficients of ξ^n relative to this basis, with $c_i(\mathcal{E}) = 0$ for $i > n$, with $c_0(\mathcal{E}) = 1$. Thus we have the equation in $CH^n(\mathbb{P}(\mathcal{E}))$:

$$\xi^n - c_1\xi^{n-1} + \cdots + (-1)^i c_i \xi^{n-i} + \cdots + (-1)^n c_n = 0.$$

If $\mathcal{E}$ is trivial then $\xi^n = 0$, and all the c_i vanish except c_0; if $\mathcal{E}$ has rank 1 then there is a divisor D with $\mathcal{E} = \mathcal{L}(D)$ then $\xi = [D]$ and $c_1(\mathcal{E}) = \xi = [D]$.

PROPOSITION 8.9. *(Grothendieck, 1957) The classes $c_i(\mathcal{E})$ define Chern classes on $K_0(X)$ with values in $CH^*(X)$.*

PROOF. We have already established axioms (CC0) and (CC1); the Normalization axiom (CC3) follows from the observation that

$$c_1(\mathcal{L}(D_1) \otimes \mathcal{L}(D_2)) = [D_1] + [D_2] = c_1(\mathcal{L}(D_1)) + c_1(\mathcal{L}(D_2)).$$

We now invoke the Splitting Principle, that we may assume that $\mathcal{E}$ has a filtration with invertible sheaves $\mathcal{L}_j$ as quotients; this is because $CH^*(\mathbb{P}(\mathcal{E}))$ embeds into $CH^*(\mathbb{F}(\mathcal{E}))$, where such a filtration exists; see I.5.9. Since $\prod(\xi - c_1(\mathcal{L}_j)) = 0$ in $CH^n(\mathbb{F}(\mathcal{E}))$, expanding the product gives the coefficients $c_i(\mathcal{E})$; the coefficients of each ξ^k establish the Sum Formula (CC2). □

COROLLARY 8.9.1. *If X is nonsingular, the Chern classes induce isomorphisms $c_i : K_0^{(i)}(X) \cong CH^i(X) \otimes \mathbb{Q}$, and the Chern character induces a ring isomorphism $ch : K_0(X) \otimes \mathbb{Q} \cong CH^*(X) \otimes \mathbb{Q}$.*

PROOF. By 4.12 the Chern character $K_0(X) \to CH^*(X) \otimes \mathbb{Q}$ is a ring homomorphism. By 4.11.4, c_n vanishes on $K_0^{(i)}(X)$ for $i \neq n$, and by 4.12.1 it is a graded ring map, where $K_0(X) \otimes \mathbb{Q}$ is given the γ-filtration. Let $F^r K_0(X)$ denote the image of $K_0\mathbf{M}(X) \to K_0(X)$; it is well known (see [FL, V.3] that $F_\gamma^r K_0(X) \subseteq F^r K_0(X)$. We will prove by induction on r that $F^r K_0(X) \cong \oplus_{i \geq r} CH^i(X)$. By 6.4.3, there is a canonical surjection $CH^i(X) \to F^i K_0(X)/F^{i+1} K_0(X)$ sending $[Z]$ to the class of $[\mathcal{O}_Z]$; removing a closed subvariety of Z, we can assume that Z is a complete intersection. In that case, $c_i(\mathcal{O}_Z) \cong (-1)^i (i-1)! [Z]$ by Ex. 8.7. □

We cite the following result from [Fulton]. For any smooth X, the *Todd class* $\mathrm{td}(X)$ is defined to be the Hirzebruch character (Ex. 4.13) of the tangent bundle of X for the power series $x/(1 - e^{-x})$. If $a = a_0 + \cdots + a_d \in CH^*(X)$ with $a_i \in CH^i(X)$ and $d = \dim(X)$, we write $\deg(a)$ for the image of a_d under the degree map $CH^d(X) \to \mathbb{Z}$.

RIEMANN-ROCH THEOREM 8.10. *Let X be a nonsingular projective variety over a field k, and let $\mathcal{E}$ be a locally free sheaf of rank n on X. Then the Euler characteristic $\chi(\mathcal{E}) = \sum(-1)^i \dim H^i(X, \mathcal{E})$ equals* $\deg(ch(\mathcal{E}) \cdot td(X))$.

More generally, if $f : X \to Y$ is a smooth projective morphism, then the push-forward $f_* : K_0(X) \to K_0(Y)$ satisfies $ch(f_* x) = f_*(ch(x) \cdot td(T_f))$, where T_f is the relative tangent sheaf of f.

LIMITS OF SCHEMES 8.11. Here is the analogue for schemes of the fact that every commutative ring is the filtered union of its finitely generated (noetherian) subrings. By [EGA, IV.8.2.3], every quasi-compact separated scheme X is the inverse limit of a filtered inverse system $i \mapsto X_i$ of noetherian schemes, each finitely presented over $\mathbb{Z}$, with affine transition maps.

Let $i \mapsto X_i$ be any filtered inverse system of schemes such that the transition morphisms $X_i \to X_j$ are affine, and let X be the inverse limit scheme $\varprojlim X_i$. This scheme exists by [EGA, IV.8.2]. In fact, over an affine open subset $\mathrm{Spec}(R_j)$ of any X_j we have affine open subsets $\mathrm{Spec}(R_i)$ of each X_i, and the corresponding affine open of X is $\mathrm{Spec}(\varinjlim R_i)$. By [EGA, IV.8.5] every vector bundle on X comes from a bundle on some X_j, and two bundles on X_j are isomorphic over X just in case they are isomorphic over some X_i. Thus the filtered system of groups $K_0(X_i)$ has the property that

$$K_0(X) = \varinjlim K_0(X_i).$$

EXERCISES

8.1 Suppose that Z is a closed subscheme of a quasi-projective scheme X, with complement U. Let $\mathbf{H}_Z(X)$ denote the subcategory of $\mathbf{H}(X)$ consisting of modules supported on Z.

(a) Suppose that $U = \mathrm{Spec}(R)$ for some ring R, and that Z is locally defined by a nonzerodivisor. (The ideal $\mathcal{I}_Z$ is invertible; see I.5.12.) As in Cor. 7.7.4, show that there is an exact sequence: $K_0\mathbf{H}_Z(X) \to K_0(X) \to K_0(U)$.

(b) Suppose that Z is contained in an open subset V of X which is regular. Show that $\mathbf{H}_Z(X)$ is the abelian category $\mathbf{M}_Z(X)$ of 6.4.2, so that $K_0\mathbf{H}_Z(X) \cong G_0(Z)$. Then apply Ex. 7.12 to show that there is an exact sequence

$$G_0(Z) \to K_0(X) \to K_0(U) \to 0.$$

(c) (Deligne) Let R be a 2-dimensional noetherian domain which is not Cohen-Macauley. If M is in $\mathbf{H}(A)$ then $\mathrm{pd}(M) = 1$, by the Auslander-Buchsbaum equality [WHomo, 4.4.15]. Setting $X = \mathrm{Spec}(R)$ and $Z = \{\mathfrak{m}\}$, show that $\mathbf{H}_Z(X) = 0$. *Hint:* If M is the cokernel of $f : R^m \to R^m$, show that $\det(f)$ must lie in a height 1 prime $\mathfrak{p}$ and conclude that $M_{\mathfrak{p}} \neq 0$.

There are 2-dimensional normal domains where $K_0(R) \to G_0(R)$ is not into [We80]; for these R the sequence $K_0\mathbf{H}_Z(X) \to K_0(X) \to K_0(U)$ is not exact.

8.2 Let X be a curve over an algebraically closed field. By Ex. I.5.7, $K_0(X)$ is generated by classes of line bundles. Show that $K_0(X) = H^0(X;\mathbb{Z}) \oplus \mathrm{Pic}(X)$.

8.3 *Projection Formula for schemes.* Suppose that $f\colon X \to Y$ is a proper map between quasi-projective schemes, both of which have finite Tor-dimension.

(a) Given $\mathcal{E}$ in $\mathbf{VB}(X)$, consider the subcategory $\mathbf{L}(f)$ of $\mathbf{M}(Y)$ consisting of coherent $\mathcal{O}_Y$-modules which are Tor-independent of both $f_*\mathcal{E}$ and $f_*\mathcal{O}_X$. Show that $G_0(Y) \cong K_0\mathbf{L}(f)$.

(b) Set $x = [E] \in K_0(X)$. By (8.4.1), we can regard f_*x as an element of $K_0(Y)$. Show that $f_*(x \cdot f^*y) = f_*(x) \cdot y$ for every $y \in G_0(Y)$.

(c) Using 7.4.2 and the ring map $f^*\colon K_0(Y) \to K_0(X)$, both $K_0(X)$ and $G_0(X)$ are $K_0(Y)$-modules. Show that the transfer maps $f_*\colon G_0(X) \to G_0(Y)$ of Lemma 6.2.6 and $f_*\colon K_0(X) \to K_0(Y)$ of (8.4.1) are $K_0(Y)$-module homomorphisms, *i.e.,* that the *projection formula* holds for every $y \in K_0(Y)$:

$$f_*(x \cdot f^*y) = f_*(x) \cdot y \quad \text{for every } x \in K_0(X) \text{ or } x \in G_0(X).$$

8.4 Suppose given a commutative square of quasi-projective schemes

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with $X' = X \times_Y Y'$ and f proper. Assume that g has finite flat dimension, and that X and Y' are Tor-independent over Y, *i.e.*, for $q > 0$ and all $x \in X$, $y' \in Y'$ and $y \in Y$ with $y = f(x) = g(y')$ we have

$$\mathrm{Tor}_q^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0.$$

Show that $g^* f_* = f'_* g'^*$ as maps $G_0(X) \to G_0(Y')$.

8.5 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be vector bundles of ranks r_1 and r_2, respectively. Modify Ex. I.2.7 to show that $\det(\mathcal{F}_1 \otimes \mathcal{F}_2) \cong (\det \mathcal{F}_1)^{r_2} \otimes (\det \mathcal{F}_2)^{r_1}$. Conclude that $K_0(X) \to H^0(X; \mathbb{Z}) \oplus \mathrm{Pic}(X)$ is a ring map.

8.6 Let $\pi: \mathbb{P} \to X$ be a projective bundle as in 8.5, and let $\mathcal{F}$ be a Mumford-regular $\mathcal{O}_{\mathbb{P}}$-module. Let $\mathcal{N}$ denote the kernel of the canonical map $\varepsilon: \pi^*\pi_*\mathcal{F} \to \mathcal{F}$. Show that $\mathcal{N}(1)$ is Mumford-regular, and that $\pi_*\mathcal{N} = 0$.

8.7 Suppose that Z is a codimension i subvariety of a nonsingular X, with conormal bundle $\mathcal{E}$; $\mathcal{E} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$ is exact. Show that $c_i([\mathcal{O}_Z]) = (-1)^i(i-1)![Z]$ in $CH^i(X)$. *Hint:* Passing to a flag bundle of $\mathcal{E}$, show that $[\mathcal{O}_Z]$ is a product of divisors $[\mathcal{O}_{D_j}]$, and use the product formula.

§9. K_0 of a Waldhausen category

It is useful to be able to define the Grothendieck group $K_0(\mathcal{C})$ of a more general type of category than exact categories, by adding a notion of weak equivalence. A structure that generalizes well to higher K-theory is that of a category of cofibrations and weak equivalences, which we shall call a "Waldhausen category" for brevity. The definitions we shall use are due to Friedhelm Waldhausen, although the ideas for K_0 are due to Grothendieck and were used in [SGA6].

We need to consider two families of distinguished morphisms in a category $\mathcal{C}$, the cofibrations and the weak equivalences. For this we use the following device. Suppose that we are given a family $\mathcal{F}$ of distinguished morphisms in a category $\mathcal{C}$. We assume that these distinguished morphisms are closed under composition, and contain every identity. It is convenient to regard these distinguished morphisms as the morphisms of a subcategory of $\mathcal{C}$, which by abuse of notation we also call $\mathcal{F}$.

Definition 9.1. Let $\mathcal{C}$ be a category equipped with a subcategory $co = co(\mathcal{C})$ of morphisms in a category $\mathcal{C}$, called "cofibrations" (and indicated with feathered arrows $\rightarrowtail$). The pair $(\mathcal{C}, co)$ is called a *category with cofibrations* if the following axioms are satisfied:

(W0) Every isomorphism in $\mathcal{C}$ is a cofibration;

(W1) There is a distinguished zero object '0' in $\mathcal{C}$, and the unique map $0 \rightarrowtail A$ in $\mathcal{C}$ is a cofibration for every A in $\mathcal{C}$;

(W2) If $A \rightarrowtail B$ is a cofibration, and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \cup_A C$ of these two maps exists in $\mathcal{C}$, and moreover the map $C \rightarrowtail B \cup_A C$ is a cofibration.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \cup_A C \end{array}$$

These axioms imply that two constructions make sense in $\mathcal{C}$: (1) the coproduct $B \amalg C$ of any two objects exists in $\mathcal{C}$ (it is the pushout $B \cup_0 C$), and (2) every cofibration $i\colon A \rightarrowtail B$ in $\mathcal{C}$ has a cokernel B/A (this is the pushout $B \cup_A 0$ of i along $A \to 0$). We refer to $A \rightarrowtail B \twoheadrightarrow B/A$ as a *cofibration sequence* in $\mathcal{C}$.

For example, any abelian category is naturally a category with cofibrations: the cofibrations are the monomorphisms. More generally, we can regard any exact category as a category with cofibrations by letting the cofibrations be the admissible monics; axiom (W2) follows from Ex. 7.8(2). In an exact category, the cofibration sequences are exactly the admissible exact sequences.

Definition 9.1.1. A *Waldhausen category* $\mathcal{C}$ is a category with cofibrations, together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called "weak equivalences" (abbreviated '*w.e.*' and indicated with decorated arrows $\xrightarrow{\sim}$). Every isomorphism in $\mathcal{C}$ is to be a weak equivalence, and weak equivalences are to be closed under composition (so we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$). In addition, the following "Glueing axiom" must be satisfied:

(W3) *Glueing for weak equivalences.* For every commutative diagram of the form

$$\begin{array}{ccccc} C & \leftarrow & A & \rightarrowtail & B \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ C' & \leftarrow & A' & \rightarrowtail & B' \end{array}$$

(in which the vertical maps are weak equivalences and the two right horizontal maps are cofibrations), the induced map

$$B \cup_A C \to B' \cup_{A'} C'$$

is also a weak equivalence.

Although a Waldhausen category is really a triple $(\mathcal{C}, co, w)$, we will usually drop the (co, w) from the notation and just write $\mathcal{C}$. We say that $\mathcal{C}$ (or just $w\mathcal{C}$) is *saturated* if: whenever f, g are composable maps and fg is a weak equivalence, f is a weak equivalence if and only if g is.

DEFINITION 9.1.2 ($K_0\mathcal{C}$). Let $\mathcal{C}$ be a Waldhausen category. $K_0(\mathcal{C})$ is the abelian group presented as having one generator $[C]$ for each object C of $\mathcal{C}$, subject to the relations

(1) $[C] = [C']$ if there is a weak equivalence $C \xrightarrow{\sim} C'$
(2) $[C] = [B] + [C/B]$ for every cofibration sequence $B \rightarrowtail C \twoheadrightarrow C/B$.

Of course, in order for this to be set-theoretically meaningful, we must assume that the weak equivalence classes of objects form a set. We shall occasionally use the notation $K_0(w\mathcal{C})$ for $K_0(\mathcal{C})$ to emphasize the choice of $w\mathcal{C}$ as weak equivalences.

These relations imply that $[0] = 0$ and $[B \amalg C] = [B] + [C]$, as they did in §6 for abelian categories. Because pushouts preserve cokernels, we also have $[B \cup_A C] = [B] + [C] - [A]$. However, weak equivalences add a new feature: $[C] = 0$ whenever $0 \simeq C$.

EXAMPLE 9.1.3. Any exact category $\mathcal{A}$ becomes a Waldhausen category, with cofibrations being admissible monics and weak equivalences being isomorphisms. By construction, the Waldhausen definition of $K_0(\mathcal{A})$ agrees with the exact category definition of $K_0(\mathcal{A})$ given in §7.

More generally, any category with cofibrations $(\mathcal{C}, co)$ may be considered as a Waldhausen category in which the category of weak equivalences is the category $\text{iso}\,\mathcal{C}$ of all isomorphisms. In this case $K_0(\mathcal{C}) = K_0(\text{iso}\,\mathcal{C})$ has only the relation (2). We could of course have developed this theory in §7 as an easy generalization of the preceding paragraph.

TOPOLOGICAL EXAMPLE 9.1.4. To show that we need not have additive categories, we give a topological example due to Waldhausen. Let $\mathcal{R} = \mathcal{R}(*)$ be the category of based CW complexes with countably many cells (we need a bound on the cardinality of the cells for set-theoretic reasons). Morphisms are cellular maps, and $\mathcal{R}_f = \mathcal{R}_f(*)$ is the subcategory of finite based CW complexes. Both are Waldhausen categories: "cofibration" is a cellular inclusion, and "weak equivalence" means weak homotopy equivalence (isomorphism on homotopy groups). The coproduct $B \vee C$ is obtained from the disjoint union of B and C by identifying their basepoints.

The Eilenberg Swindle shows that $K_0\mathcal{R} = 0$. In effect, the infinite coproduct C^∞ of copies of a fixed complex C exists in $\mathcal{R}$, and equals $C \vee C^\infty$. In contrast, the finite complexes have interesting K-theory:

Proposition 9.1.5. *$K_0\mathcal{R}_f \cong \mathbb{Z}$.*

Proof. The inclusion of S^{n-1} in the n-disk D^n has $D^n/S^{n-1} \cong S^n$, so $[S^{n-1}]+[S^n] = [D^n] = 0$. Hence $[S^n] = (-1)^n[S^0]$. If C is obtained from B by attaching an n-cell, $C/B \cong S^n$ and $[C] = [B] + [S^n]$. Hence $K_0\mathcal{R}_f$ is generated by $[S^0]$. Finally, the reduced Euler characteristic $\chi(C) = \sum(-1)^i \dim \tilde{H}^i(X;\mathbb{Q})$ defines a surjection from $K_0\mathcal{R}_f$ onto $\mathbb{Z}$, which must therefore be an isomorphism. □

BiWaldhausen Categories 9.1.6. In general, the opposite $\mathcal{C}^{op}$ need not be a Waldhausen category, because the quotients $B \twoheadrightarrow B/A$ need not be closed under composition: the family quot$(\mathcal{C})$ of these quotient maps need not be a subcategory of $\mathcal{C}^{op}$. We call $\mathcal{C}$ a *category with bifibrations* if $\mathcal{C}$ is a category with cofibrations, $\mathcal{C}^{op}$ is a category with cofibrations, $co(\mathcal{C}^{op}) = \text{quot}(\mathcal{C})$, the canonical map $A \amalg B \to A \times B$ is always an isomorphism, and A is the kernel of each quotient map $B \twoheadrightarrow B/A$. We call $\mathcal{C}$ a *biWaldhausen category* if $\mathcal{C}$ is a category with bifibrations, having a subcategory $w(\mathcal{C})$ so that both $(\mathcal{C}, co, w)$ and $(\mathcal{C}^{op}, \text{quot}, w^{op})$ are Waldhausen categories. The notions of bifibrations and biWaldhausen category are self-dual, so we have:

Lemma 9.1.7. *$K_0(\mathcal{C}) \cong K_0(\mathcal{C}^{op})$ for every biWaldhausen category.*

Example 9.1.3 shows that exact categories are biWaldhausen categories. We will see in 9.2 below that chain complexes form another important family of biWaldhausen categories.

Exact Functors 9.1.8. A functor $F: \mathcal{C} \to \mathcal{D}$ between Waldhausen categories is called an *exact functor* if it preserves all the relevant structure: zero, cofibrations, weak equivalences and pushouts along a cofibration. The last condition means that the canonical map $FB \cup_{FA} FC \to F(B \cup_A C)$ is an isomorphism for every cofibration $A \rightarrowtail B$. Clearly, an exact functor induces a group homomorphism $K_0(F): K_0\mathcal{C} \to K_0\mathcal{D}$.

A *Waldhausen subcategory* $\mathcal{A}$ of a Waldhausen category $\mathcal{C}$ is a subcategory which is also a Waldhausen category in such a way that: (i) the inclusion $\mathcal{A} \subseteq \mathcal{C}$ is an exact functor, (ii) the cofibrations in $\mathcal{A}$ are the maps in $\mathcal{A}$ which are cofibrations in $\mathcal{C}$ and whose cokernel lies in $\mathcal{A}$, and (iii) the weak equivalences in $\mathcal{A}$ are the weak equivalences of $\mathcal{C}$ which lie in $\mathcal{A}$.

For example, suppose that $\mathcal{C}$ and $\mathcal{D}$ are exact categories (in the sense of §7), considered as Waldhausen categories. A functor $F: \mathcal{C} \to \mathcal{D}$ is exact in the above sense if and only if F is additive and preserves short exact sequences, *i.e.,* F is an exact functor between exact categories in the sense of §7. The routine verification of this assertion is left to the reader.

Here is an elementary consequence of the definition of exact functor. Let $\mathcal{A}$ and $\mathcal{C}$ be Waldhausen categories and F, F', F'' three exact functors from $\mathcal{A}$ to $\mathcal{C}$. Suppose moreover that there are natural transformations $F' \Rightarrow F \Rightarrow F''$ so that for all A in $\mathcal{A}$

$$F'A \rightarrowtail FA \twoheadrightarrow F''A \tag{9.1.8}$$

is a cofibration sequence in $\mathcal{C}$. Then $[FA] = [F'A] + [F''A]$ in $K_0\mathcal{C}$, so as maps from $K_0\mathcal{A}$ to $K_0\mathcal{C}$ we have $K_0(F) = K_0(F') + K_0(F'')$.

Chain complexes

9.2 Historically, one of the most important families of Waldhausen categories are those arising from chain complexes. The definition of K_0 for a category of (co)chain complexes dates to the 1960's, being used in [SGA6] to study the Riemann-Roch Theorem. We will work with chain complexes here, although by reindexing we could equally well work with cochain complexes.

Given a small abelian category $\mathcal{A}$, let $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$ denote the category of all chain complexes in $\mathcal{A}$, and let $\mathbf{Ch}^b$ denote the full subcategory of all bounded complexes. The following structure makes $\mathbf{Ch}$ into a Waldhausen category, with $\mathbf{Ch}^b(\mathcal{A})$ as a Waldhausen subcategory. We will show below that $K_0\mathbf{Ch}(\mathcal{A}) = 0$ but that $K_0\mathbf{Ch}^b(\mathcal{A}) \cong K_0(\mathcal{A})$.

A cofibration $C \to D$ is a chain map such that every map $C_n \rightarrowtail D_n$ is monic in $\mathcal{A}$. Thus a cofibration sequence is just a short exact sequence of chain complexes. A weak equivalence $C \xrightarrow{\sim} D$ is a quasi-isomorphism, *i.e.*, a chain map inducing isomorphisms on homology.

Here is a slightly more general construction, taken from [SGA6, IV(1.5.2)]. Suppose that $\mathcal{B}$ is an exact category, embedded in an abelian category $\mathcal{A}$. Let $\mathbf{Ch}(\mathcal{B})$, resp. $\mathbf{Ch}^b(\mathcal{B})$, denote the category of all (resp. all bounded) chain complexes in $\mathcal{B}$. A cofibration $A_\bullet \rightarrowtail B_\bullet$ in $\mathbf{Ch}(\mathcal{B})$ (resp. $\mathbf{Ch}^b\mathcal{B}$) is a map which is a degreewise admissible monomorphism, *i.e.*, such that each $C_n = B_n/A_n$ is in $\mathcal{B}$, yielding short exact sequences $A_n \rightarrowtail B_n \twoheadrightarrow C_n$ in $\mathcal{B}$. To define the weak equivalences, we use the notion of homology in the ambient abelian category $\mathcal{A}$: let $w\mathbf{Ch}(\mathcal{B})$ denote the family of all chain maps in $\mathbf{Ch}(\mathcal{B})$ which are quasi-isomorphisms of complexes in $\mathbf{Ch}(\mathcal{A})$. With this structure, both $\mathbf{Ch}(\mathcal{B})$ and $\mathbf{Ch}^b(\mathcal{B})$ become Waldhausen subcategories of $\mathbf{Ch}(\mathcal{A})$.

Subtraction in $K_0\mathbf{Ch}$ and $K_0\mathbf{Ch}^b$ is given by shifting indices on complexes. To see this, recall from [WHomo, 1.2.8] that the n^{th} *translate* of C is defined to be the chain complex $C[n]$ which has C_{i+n} in degree i. (If we work with cochain complexes then C^{i-n} is in degree i.) Moreover, the *mapping cone complex* $\operatorname{cone}(f)$ of a chain complex map $f\colon B \to C$ fits into a short exact sequence of complexes:

$$0 \to C \to \operatorname{cone}(f) \to B[-1] \to 0.$$

Therefore in K_0 we have $[C]+[B[-1]] = [\operatorname{cone}(f)]$. In particular, if f is the identity map on C, the cone complex is exact and hence *w.e.* to 0. Thus we have $[C] + [C[-1]] = [\operatorname{cone}(\mathrm{id})] = 0$. We record this observation as follows.

LEMMA 9.2.1. *Let $\mathcal{C}$ be any Waldhausen subcategory of $\mathbf{Ch}(\mathcal{A})$ closed under translates and the formation of mapping cones. Then $[C[n]] = (-1)^n[C]$ in $K_0(\mathcal{C})$. In particular, this is true in $K_0\mathbf{Ch}(\mathcal{B})$ and $K_0\mathbf{Ch}^b(\mathcal{B})$ for every exact subcategory $\mathcal{B}$ of $\mathcal{A}$.*

A chain complex C is called *bounded below* (resp. *bounded above*) if $C_n = 0$ for all $n \ll 0$ (resp. all $n \gg 0$). If C is bounded above, then each infinite direct sum $C_n \oplus C_{n+2} \oplus \cdots$ is finite, so the infinite direct sum of shifts

$$B = C \oplus C[2] \oplus C[4] \oplus \cdots \oplus C[2n] \oplus \cdots$$

is defined in **Ch**. From the exact sequence $0 \to B[2] \to B \to C \to 0$, we see that in $K_0\mathbf{Ch}$ we have the Eilenberg swindle: $[C] = [B] - [B[2]] = [B] - [B] = 0$. A similar argument shows that $[C] = 0$ if C is bounded below. But every chain complex C fits into a short exact sequence

$$0 \to B \to C \to D \to 0$$

in which B is bounded above and D is bounded below. (For example, take $B_n = 0$ for $n > 0$ and $B_n = C_n$ otherwise.) Hence $[C] = [B] + [D] = 0$ in $K_0\mathbf{Ch}$. This shows that $K_0\mathbf{Ch} = 0$, as asserted.

If $\mathcal{B}$ is any exact category, the natural inclusion of $\mathcal{B}$ into $\mathbf{Ch}^b(\mathcal{B})$ as the chain complexes concentrated in degree zero is an exact functor. Hence it induces a homomorphism $K_0(\mathcal{B}) \to K_0\mathbf{Ch}^b(\mathcal{B})$.

Theorem 9.2.2 ([SGA6, I.6.4]). *Let $\mathcal{A}$ be an abelian category. Then*

$$K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A}),$$

and the class $[C]$ of a chain complex C in $K_0\mathcal{A}$ is the same as its Euler characteristic, namely $\chi(C) = \sum(-1)^i[C_i]$.

Similarly, if $\mathcal{B}$ is an exact category closed under kernels of surjections in an abelian category (in the sense of 7.0.1), then $K_0(\mathcal{B}) \cong K_0\mathbf{Ch}^b(\mathcal{B})$, and again we have $\chi(C) = \sum(-1)^i[C_i]$ in $K_0(\mathcal{B})$.

Proof. We give the proof for $\mathcal{A}$; the proof for $\mathcal{B}$ is the same, except one cites 7.5 in place of 6.6. As in Proposition 6.6 (or 7.5), the Euler characteristic $\chi(C)$ of a bounded complex is the element $\sum(-1)^i[C_i]$ of $K_0(\mathcal{A})$. We saw in 6.6 (and 7.5.1) that $\chi(B) = \chi(C)$ if $B \to C$ is a weak equivalence (quasi-isomorphism). If $B \rightarrowtail C \twoheadrightarrow D$ is a cofibration sequence in $\mathbf{Ch}^b$, then from the short exact sequences $0 \to B_n \to C_n \to D_n \to 0$ in $\mathcal{A}$ we obtain $\chi(C) = \chi(B) + \chi(C/B)$ by inspection (as in 7.5.1). Hence χ satisfies the relations needed to define a homomorphism χ from $K_0(\mathbf{Ch}^b)$ to $K_0(\mathcal{A})$. If C is concentrated in degree 0 then $\chi(C) = [C_0]$, so the composite map $K_0(\mathcal{A}) \to K_0(\mathbf{Ch}^b) \to K_0(\mathcal{A})$ is the identity.

It remains to show that $[C] = \chi(C)$ in $K_0\mathbf{Ch}^b$ for every complex

$$C \colon 0 \to C_m \to \cdots \to C_n \to 0.$$

If $m = n$, then $C = C_n[-n]$ is the object C_n of $\mathcal{A}$ concentrated in degree n; we have already observed that $[C] = (-1)^n[C_n[0]] = (-1)^n[C_n]$ in this case. If $m > n$, let B denote the subcomplex consisting of C_n in degree n, and zero elsewhere. Then $B \rightarrowtail C$ is a cofibration whose cokernel C/B has shorter length than C. By induction, we have the desired relation in $K_0\mathbf{Ch}^b$, finishing the proof:

$$[C] = [B] + [C/B] = \chi(B) + \chi(C/B) = \chi(C). \qquad \square$$

Remark 9.2.3 (K_0 and derived categories). Let $\mathcal{B}$ be an exact category. Theorem 9.2.2 states that the group $K_0\mathbf{Ch}^b(\mathcal{B})$ is independent of the choice of ambient abelian category $\mathcal{A}$, as long as $\mathcal{B}$ is closed under kernels of surjections in $\mathcal{A}$. This is the group $k(\mathcal{B})$ introduced in [SGA6], Expose IV(1.5.2). (The context of [SGA6] was triangulated categories, and the main observation in *op. cit.* is that this definition only depends upon the derived category $D^b_{\mathcal{B}}(\mathcal{A})$. See Ex. 9.5 below.)

We warn the reader that if $\mathcal{B}$ is not closed under kernels of surjections in $\mathcal{A}$, then $K_0\mathbf{Ch}^b(\mathcal{B})$ can differ from $K_0(\mathcal{B})$. (See Ex. 9.11).

If $\mathcal{A}$ is an abelian category, or even an exact category, the category $\mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ has another Waldhausen structure with the same weak equivalences: we redefine cofibration so that $B \to C$ is a cofibration if and only if each $B_i \to C_i$ is a *split* injection in $\mathcal{A}$. If split $\mathbf{Ch}^b$ denotes $\mathbf{Ch}^b$ with this new Waldhausen structure, then the inclusion split $\mathbf{Ch}^b \to \mathbf{Ch}^b$ is an exact functor, so it induces a surjection $K_0(\text{split}\,\mathbf{Ch}^b) \to K_0(\mathbf{Ch}^b)$.

LEMMA 9.2.4. *If $\mathcal{A}$ is an exact category then*

$$K_0(\text{split}\,\mathbf{Ch}^b) \cong K_0(\mathbf{Ch}^b) \cong K_0(\mathcal{A}).$$

PROOF. Lemma 9.2.1 and enough of the proof of 9.2.2 go through to prove that $[C[n]] = (-1)^n[C]$ and $[C] = \sum(-1)^n[C_n]$ in $K_0(\text{split}\,\mathbf{Ch}^b)$. Hence it suffices to show that $A \mapsto [A]$ defines an additive function from $\mathcal{A}$ to $K_0(\text{split}\,\mathbf{Ch}^b)$. If A is an object of $\mathcal{A}$, let $[A]$ denote the class in $K_0(\text{split}\,\mathbf{Ch}^b)$ of the complex which is A concentrated in degree zero. Any short exact sequence $E\colon 0 \to A \to B \to C \to 0$ in $\mathcal{A}$ may be regarded as an (exact) chain complex concentrated in degrees 0, 1 and 2 so:

$$[E] = [A] - [B] + [C]$$

in $K_0(\text{split}\,\mathbf{Ch}^b)$. But E is weakly equivalent to zero, so $[E] = 0$. Hence $A \mapsto [A]$ is an additive function, defining a map $K_0(\mathcal{A}) \to K_0(\text{split}\,\mathbf{Ch}^b)$. □

EXTENSION CATEGORIES 9.3. If $\mathcal{B}$ is a category with cofibrations, the cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{B}$ form the objects of a category $\mathcal{E} = \mathcal{E}(\mathcal{B})$. A morphism $E \to E'$ in $\mathcal{E}$ is a commutative diagram:

$$\begin{array}{ccccccc} E: & & A & \rightarrowtail & B & \twoheadrightarrow & C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E': & & A' & \rightarrowtail & B' & \twoheadrightarrow & C' \end{array}$$

We make $\mathcal{E}$ into a category with cofibrations by declaring that a morphism $E \to E'$ in $\mathcal{E}$ is a cofibration if $A \to A'$, $C \to C'$ and $A' \cup_A B \to B'$ are cofibrations in $\mathcal{B}$. This is required by axiom (W2), and implies that the composite $B \rightarrowtail A' \cup_A B \rightarrowtail B'$ is a cofibration too. If $\mathcal{B}$ is a Waldhausen category then so is $\mathcal{E}(\mathcal{B})$: a weak equivalence in $\mathcal{E}$ is a morphism whose component maps $A \to A'$, $B \to B'$, $C \to C'$ are weak equivalences in $\mathcal{B}$.

Here is a useful variant. If $\mathcal{A}$ and $\mathcal{C}$ are Waldhausen subcategories of $\mathcal{B}$, the extension category $\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})$ of $\mathcal{C}$ by $\mathcal{A}$ is the Waldhausen subcategory of the extension category of $\mathcal{B}$ consisting of cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ with A in $\mathcal{A}$ and C in $\mathcal{C}$. Clearly, $\mathcal{E}(\mathcal{B}) = \mathcal{E}(\mathcal{B},\mathcal{B},\mathcal{B})$.

There is an exact functor $\amalg\colon \mathcal{A} \times \mathcal{C} \to \mathcal{B}$, sending (A, C) to $A \rightarrowtail A \amalg C \twoheadrightarrow C$. Conversely, there are three exact functors (s, t and q) from $\mathcal{E}$ to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, which send $A \rightarrowtail B \twoheadrightarrow C$ to A, B and C, respectively. By the above remarks, if $\mathcal{A} = \mathcal{B} = \mathcal{C}$ then $t_* = s_* + q_*$ as maps $K_0(\mathcal{E}) \to K_0(\mathcal{B})$.

PROPOSITION 9.3.1. $K_0(\mathcal{E}(\mathcal{A},\mathcal{B},\mathcal{C})) \cong K_0(\mathcal{A}) \times K_0(\mathcal{C})$.

PROOF. Since (s, q) is a left inverse to $\amalg$, $\amalg_*$ is a split injection from $K_0(\mathcal{A}) \times K_0(\mathcal{C})$ to $K_0(\mathcal{E})$. Thus it suffices to show that for every $E\colon A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$ we have

$[E] = [\amalg(A,0)] + [\amalg(0,C)]$ in $K_0(\mathcal{E})$. This relation follows from the fundamental relation 9.1.2(2) of K_0, given that

$$\begin{array}{ccccccc} \amalg(A,0): & & A & \overset{=}{\rightarrowtail} & A & \twoheadrightarrow & 0 \\ \downarrow & & \| & & \downarrow & & \downarrow \\ E: & & A & \rightarrowtail & B & \twoheadrightarrow & C \end{array}$$

is a cofibration in $\mathcal{E}$ with cokernel $\amalg(0,C): 0 \rightarrowtail C \twoheadrightarrow C$. □

Example 9.3.2 (Higher Extension categories). Here is a generalization of the extension category $\mathcal{E} = \mathcal{E}_2$ constructed above. Let $\mathcal{E}_n$ be the category whose objects are sequences of n cofibrations in a Waldhausen category $\mathcal{C}$:

$$A: \quad 0 = A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n.$$

A morphism $A \to B$ in $\mathcal{E}_n$ is a natural transformation of sequences, and is a weak equivalence if each component $A_i \to B_i$ is a *w.e.* in $\mathcal{C}$. It is a cofibration when for each $0 \le i < j < k \le n$ the map of cofibration sequences

$$\begin{array}{ccccc} A_j/A_i & \rightarrowtail & A_k/A_i & \twoheadrightarrow & A_k/A_j \\ \downarrow & & \downarrow & & \downarrow \\ B_j/B_i & \rightarrowtail & B_k/B_i & \twoheadrightarrow & B_k/B_j \end{array}$$

is a cofibration in $\mathcal{E}$. The reader is encouraged in Ex. 9.4 to check that $\mathcal{E}_n$ is a Waldhausen category, and to compute $K_0(\mathcal{E}_n)$.

Cofinality Theorem 9.4. *Let $\mathcal{B}$ be a Waldhausen subcategory of $\mathcal{C}$ closed under extensions. If $\mathcal{B}$ is cofinal in $\mathcal{C}$ (in the sense that for all C in $\mathcal{C}$ there is a C' in $\mathcal{C}$ so that $C \amalg C'$ is in $\mathcal{B}$), then $K_0(\mathcal{B})$ is a subgroup of $K_0(\mathcal{C})$.*

Proof. Considering $\mathcal{B}$ and $\mathcal{C}$ as symmetric monoidal categories with product $\amalg$, we have $K_0^{\amalg}(\mathcal{B}) \subset K_0^{\amalg}(\mathcal{C})$ by (1.3). The proof of cofinality for exact categories (Lemma 7.2) goes through verbatim to prove that $K_0(\mathcal{B}) \subset K_0(\mathcal{C})$. □

Remark 9.4.1. The proof shows that $K_0(\mathcal{C})/K_0(\mathcal{B}) \cong K_0^{\amalg}\mathcal{C}/K_0^{\amalg}\mathcal{B}$, and that every element of $K_0(\mathcal{C})$ has the form $[C] - [B]$ for some B in $\mathcal{B}$ and C in $\mathcal{C}$.

Products

9.5 Our discussion in 7.4 about products in exact categories carries over to the Waldhausen setting. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Waldhausen categories, and suppose given a functor $F: \mathcal{A} \times \mathcal{B} \to \mathcal{C}$. The following result is completely elementary:

Lemma 9.5.1. *If each $F(A,-): \mathcal{B} \to \mathcal{C}$ and $F(-,B): \mathcal{A} \to \mathcal{C}$ is an exact functor, then $F: \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces a bilinear map*

$$\begin{aligned} K_0\mathcal{A} \otimes K_0\mathcal{B} &\to K_0\mathcal{C} \\ [A] \otimes [B] &\mapsto [F(A,B)]. \end{aligned}$$

Note that the 3×3 diagram in $\mathcal{C}$ determined by $F(A \rightarrowtail A', B \rightarrowtail B')$ yields the following relation in $K_0(\mathcal{C})$.

$$[F(A', B')] = [F(A, B)] + [F(A'/A, B)] + [F(A, B'/B)] + [F(A'/A, B'/B)]$$

Higher K-theory will need this relation to follow from more symmetric considerations, viz. that $F(A \rightarrowtail A', B \rightarrowtail B')$ should represent a cofibration in the category $\mathcal{E}$ of all cofibration sequences in $\mathcal{C}$. With this in mind, we introduce the following definition.

Definition 9.5.2. A functor $F: \mathcal{A} \times \mathcal{B} \rightarrowtail \mathcal{C}$ between Waldhausen categories is called *biexact* if each $F(A, -)$ and $F(-, B)$ is exact, and the following condition is satisfied:

> For every pair of cofibrations ($A \rightarrowtail A'$ in $\mathcal{A}$, $B \rightarrowtail B'$ in $\mathcal{B}$) the following map must be a cofibration in $\mathcal{C}$:

$$F(A', B) \cup_{F(A,B)} F(A, B') \rightarrowtail F(A', B').$$

Our next result requires some notation. Suppose that a category with cofibrations $\mathcal{C}$ has two notions of weak equivalence, a weak one v and a stronger one w. (Every map in v belongs to w.) We write $v\mathcal{C}$ and $w\mathcal{C}$ for the two Waldhausen categories $(\mathcal{C}, co, v)$ and $(\mathcal{C}, co, w)$. The identity on $\mathcal{C}$ is an exact functor $v\mathcal{C} \to w\mathcal{C}$.

Let $\mathcal{C}^w$ denote the full subcategory of all w-acyclic objects in $\mathcal{C}$, *i.e.,* those C for which $0 \rightarrowtail C$ is in $w(\mathcal{C})$; $\mathcal{C}^w$ is a Waldhausen subcategory (9.1.8) of $v\mathcal{C}$, *i.e.,* of the category $\mathcal{C}$ with the v-notion of weak equivalence.

Recall from 9.1.1 that $w\mathcal{C}$ is called *saturated* if, whenever f, g are composable maps and fg is a weak equivalence, f is a weak equivalence if and only if g is.

Localization Theorem 9.6. *Suppose that $\mathcal{C}$ is a category with cofibrations, endowed with two notions ($v \subset w$) of weak equivalence, with $w\mathcal{C}$ saturated, and that $\mathcal{C}^w$ is defined as above.*

Assume in addition that every map $f: C_1 \to C_2$ in $\mathcal{C}$ factors as the composition of a cofibration $C_1 \rightarrowtail C$ and an equivalence $C \xrightarrow{\sim} C_2$ in $v(\mathcal{C})$.

Then the exact inclusions $\mathcal{C}^w \to v\mathcal{C} \to w\mathcal{C}$ induce an exact sequence

$$K_0(\mathcal{C}^w) \to K_0(v\mathcal{C}) \to K_0(w\mathcal{C}) \to 0.$$

Proof. Our proof of this is similar to the proof of the Localization Theorem 6.4 for abelian categories. Clearly $K_0(v\mathcal{C})$ maps onto $K_0(w\mathcal{C})$ and $K_0(\mathcal{C}^w)$ maps to zero. Let L denote the cokernel of $K_0(\mathcal{C}^w) \to K_0(v\mathcal{C})$; we will prove the theorem by showing that $\lambda(C) = [C]$ induces a map $K_0(w\mathcal{C}) \to L$ inverse to the natural surjection $L \to K_0(w\mathcal{C})$. As $v\mathcal{C}$ and $w\mathcal{C}$ have the same notion of cofibration, it suffices to show that $[C_1] = [C_2]$ in L for every equivalence $f: C_1 \to C_2$ in $w\mathcal{C}$. Our hypothesis that f factors as $C_1 \rightarrowtail C \xrightarrow{\sim} C_2$ implies that in $K_0(v\mathcal{C})$ we have $[C_2] = [C] = [C_1] + [C/C_1]$. Since $w\mathcal{C}$ is saturated, it contains $C_1 \rightarrowtail C$. The following lemma implies that C/C_1 is in $\mathcal{C}^w$, so that $[C_2] = [C_1]$ in L. This is the relation we needed to have λ define a map $K_0(w\mathcal{C}) \to L$, proving the theorem. □

LEMMA 9.6.1. *If $B \overset{\sim}{\rightarrowtail} C$ is both a cofibration and a weak equivalence in a Waldhausen category, then $0 \rightarrowtail C/B$ is also a weak equivalence.*

PROOF. Apply the Glueing Axiom (W3) to the diagram:

$$\begin{array}{ccccc} 0 & \leftarrow & B & = & B \\ \| & & \| \sim & & \downarrow\sim \\ 0 & \leftarrow & B & \rightarrowtail & C. \end{array} \quad \square$$

Here is a simple application of the Localization Theorem. Let $(\mathcal{C}, co, v)$ be a Waldhausen category, and G an abelian group. Given a surjective homomorphism $\pi\colon K_0(\mathcal{C}) \to G$, we let $\mathcal{C}^\pi$ denote the Waldhausen subcategory of $\mathcal{C}$ consisting of all objects C such that $\pi([C]) = 0$.

PROPOSITION 9.6.2. *Assume that every morphism in a Waldhausen category $\mathcal{C}$ factors as the composition of a cofibration and a weak equivalence. There is a short exact sequence*

$$0 \to K_0(\mathcal{C}^\pi) \to K_0(\mathcal{C}) \xrightarrow{\pi} G \to 0.$$

PROOF. Define $w\mathcal{C}$ to be the family of all morphisms $A \to B$ in $\mathcal{C}$ with $\pi([A]) = \pi([B])$. This satisfies axiom (W3) because $[C \cup_A B] = [B] + [C] - [A]$, and the factorization hypothesis ensures that the Localization Theorem 9.6 applies to $v \subseteq w$. Since $\mathcal{C}^\pi$ is the category $\mathcal{C}^w$ of w-acyclic objects, this yields exactness at $K_0(\mathcal{C})$. Exactness at $K_0(\mathcal{C}^\pi)$ will follow from the Cofinality Theorem 9.4, provided we show that $\mathcal{C}^\pi$ is cofinal. Given an object C, factor the map $C \to 0$ as a cofibration $C \rightarrowtail C''$ followed by a weak equivalence $C'' \xrightarrow{\sim} 0$. If C' denotes C''/C, we compute in G that

$$\pi([C \amalg C']) = \pi([C]) + \pi([C']) = \pi([C] + [C']) = \pi([C'']) = 0.$$

Hence $C \amalg C'$ is in $\mathcal{C}^\pi$, and $\mathcal{C}^\pi$ is cofinal in $\mathcal{C}$. $\square$

APPROXIMATION THEOREM 9.7. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor between two Waldhausen categories. Suppose also that F satisfies the following conditions:*

(a) A morphism f in $\mathcal{A}$ is a weak equivalence if and only if $F(f)$ is a w.e. in $\mathcal{B}$.

(b) Given any map $b\colon F(A) \to B$ in $\mathcal{B}$, there is a cofibration $a\colon A \rightarrowtail A'$ in $\mathcal{A}$ and a weak equivalence $b'\colon F(A') \xrightarrow{\sim} B$ in $\mathcal{B}$ so that $b = b' \circ F(a)$.

(c) If b is a weak equivalence, we may choose a to be a weak equivalence in $\mathcal{A}$.

Then F induces an isomorphism $K_0\mathcal{A} \cong K_0\mathcal{B}$.

PROOF. Applying (b) to $0 \rightarrowtail B$, we see that for every B in $\mathcal{B}$ there is a weak equivalence $F(A') \xrightarrow{\sim} B$. If $F(A) \xrightarrow{\sim} B$ is a weak equivalence, so is $A \xrightarrow{\sim} A'$ by (c). Therefore not only is $K_0\mathcal{A} \to K_0\mathcal{B}$ onto, but the set W of weak equivalence classes of objects of $\mathcal{A}$ is isomorphic to the set of $w.e.$ classes of objects in $\mathcal{B}$.

Now $K_0\mathcal{B}$ is obtained from the free abelian group $\mathbb{Z}[W]$ on the set W by modding out by the relations $[C] = [B] + [C/B]$ corresponding to the cofibrations $B \rightarrowtail C$ in $\mathcal{B}$. Given $F(A) \xrightarrow{\sim} B$, hypothesis (b) yields $A \rightarrowtail A'$ in $\mathcal{A}$ and a weak equivalence $F(A') \xrightarrow{\sim} C$ in $\mathcal{B}$. Finally, the Glueing Axiom (W3) applied to

$$\begin{array}{ccccc} 0 & \leftarrow & F(A) & \rightarrowtail & F(A') \\ \| & & \downarrow\sim & & \downarrow\sim \\ 0 & \leftarrow & B & \rightarrowtail & C \end{array}$$

implies that the map $F(A'/A) \to C/B$ is a weak equivalence. Therefore the relation $[C] = [B] + [C/B]$ is equivalent to the relation $[A'] = [A] + [A'/A]$ in the free abelian group $\mathbb{Z}[W]$, and already holds in $K_0\mathcal{A}$. This yields $K_0\mathcal{A} \cong K_0\mathcal{B}$, as asserted. □

Approximation for saturated categories 9.7.1. If $\mathcal{B}$ is saturated (9.1.1), then condition (c) is redundant in the Approximation Theorem, because $F(a)$ is a weak equivalence by (b) and hence by (a) the map a is a *w.e.* in $\mathcal{A}$.

Example 9.7.2. Recall from Example 9.1.4 that the category $\mathcal{R}(*)$ of based CW complexes is a Waldhausen category. Let $\mathcal{R}_{hf}(*)$ denote the Waldhausen subcategory of all based CW-complexes weakly homotopic to a finite CW complex. The Approximation Theorem applies to the inclusion of $\mathcal{R}_f(*)$ into $\mathcal{R}_{hf}(*)$; this may be seen by using the Whitehead Theorem and elementary obstruction theory. Hence

$$K_0\mathcal{R}_{hf}(*) \cong K_0\mathcal{R}_f(*) \cong \mathbb{Z}.$$

Example 9.7.3. If $\mathcal{A}$ is an exact category, the Approximation Theorem applies to the inclusion $\operatorname{split}\mathbf{Ch}^b \subset \mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ of Lemma 9.2.4, yielding a more elegant proof that $K_0(\operatorname{split}\mathbf{Ch}^b) = K_0(\mathbf{Ch}^b)$. To see this, observe that any chain complex map $f\colon A \to B$ factors through the mapping cylinder complex $\operatorname{cyl}(f)$ as the composite $A \rightarrowtail \operatorname{cyl}(f) \xrightarrow{\sim} B$, and that $\operatorname{split}\mathbf{Ch}^b$ is saturated (9.1.1).

Example 9.7.4 (Homologically bounded complexes). Fix an abelian category $\mathcal{A}$, and consider the Waldhausen category $\mathbf{Ch}(\mathcal{A})$ of all chain complexes over $\mathcal{A}$, as in (9.2). We call a complex $C_\bullet$ *homologically bounded* if it is exact almost everywhere, *i.e.,* if only finitely many of the $H_i(C)$ are nonzero. Let $\mathbf{Ch}^{hb}(\mathcal{A})$ denote the Waldhausen subcategory of $\mathbf{Ch}(\mathcal{A})$ consisting of the homologically bounded complexes, and let $\mathbf{Ch}^{hb}_-(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$ denote the Waldhausen subcategory of all bounded above, homologically bounded chain complexes $0 \to C_n \to C_{n-1} \to \cdots$. These are all saturated biWaldhausen categories (see 9.1.1 and 9.1.6). We will prove that

$$K_0\mathbf{Ch}^{hb}(\mathcal{A}) \cong K_0\mathbf{Ch}^{hb}_-(\mathcal{A}) \cong K_0\mathbf{Ch}^b(\mathcal{A}) \cong K_0(\mathcal{A}),$$

the first isomorphism being Theorem 9.2.2. From this and Proposition 6.6 it follows that if C is homologically bounded then

$$[C] = \sum (-1)^i [H_i(\mathcal{A})] \quad \text{in} \quad K_0\mathcal{A}.$$

We first claim that the Approximation Theorem 9.7 applies to $\mathbf{Ch}^b \subset \mathbf{Ch}^{hb}_-$, yielding $K_0\mathbf{Ch}^b \cong K_0\mathbf{Ch}^{hb}_-$. If $C_\bullet$ is bounded above then each good truncation $\tau_{\ge n}C = (\cdots C_{n+1} \to Z_n \to 0)$ of C is a bounded subcomplex of C such that $H_i(\tau_{\ge n}C)$ is $H_i(C)$ for $i \ge n$, and 0 for $i < n$. (See [WHomo, 1.2.7].) Therefore $\tau_{\ge n}C \xrightarrow{\sim} C$ is a quasi-isomorphism for small n ($n \ll 0$). If B is a bounded complex, any map $f\colon B \to C$ factors through $\tau_{\ge n}C$ for small n; let A denote the mapping cylinder of $B \to \tau_{\ge n}C$ (see [WHomo, 1.5.8]). Then A is bounded and f factors as the cofibration $B \rightarrowtail A$ composed with the weak equivalence $A \xrightarrow{\sim} \tau_{\ge n}C \xrightarrow{\sim} C$. Thus we may apply the Approximation Theorem, as claimed.

The Approximation Theorem does not apply to $\mathbf{Ch}_-^{hb} \subset \mathbf{Ch}^{hb}$, but rather to $\mathbf{Ch}_+^{hb} \subset \mathbf{Ch}^{hb}$, where the "+" indicates bounded below chain complexes. The argument for this is the same as for $\mathbf{Ch}^b \subset \mathbf{Ch}_-^{hb}$. Since these are biWaldhausen categories, we can apply 9.1.7 to $\mathbf{Ch}_-^{hb}(\mathcal{A})^{op} = \mathbf{Ch}_+^{hb}(\mathcal{A}^{op})$ and $\mathbf{Ch}^{hb}(\mathcal{A})^{op} = \mathbf{Ch}^{hb}(\mathcal{A}^{op})$ to get

$$K_0\mathbf{Ch}_-^{hb}(\mathcal{A}) = K_0\mathbf{Ch}_+^{hb}(\mathcal{A}^{op}) \cong K_0\mathbf{Ch}^{hb}(\mathcal{A}^{op}) = K_0\mathbf{Ch}^{hb}(\mathcal{A}).$$

This completes our calculation that $K_0(\mathcal{A}) \cong K_0\mathbf{Ch}^{hb}(\mathcal{A})$.

EXAMPLE 9.7.5 (K_0 AND PERFECT COMPLEXES). Let R be a ring. A chain complex $M_\bullet$ of R-modules is called *perfect* if there is a quasi-isomorphism $P_\bullet \xrightarrow{\sim} M_\bullet$, where $P_\bullet$ is a bounded complex of finitely generated projective R-modules, *i.e.,* $P_\bullet$ is a complex in $\mathbf{Ch}^b(\mathbf{P}(R))$. The perfect complexes form a Waldhausen subcategory $\mathbf{Ch}_{\text{perf}}(R)$ of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$. We claim that the Approximation Theorem applies to $\mathbf{Ch}^b(\mathbf{P}(R)) \subset \mathbf{Ch}_{\text{perf}}(R)$, so that

$$K_0\mathbf{Ch}_{\text{perf}}(R) \cong K_0\mathbf{Ch}^b\mathbf{P}(R) \cong K_0(R).$$

To see this, consider the intermediate Waldhausen category $\mathbf{Ch}_{\text{perf}}^b$ of bounded perfect complexes. The argument of Example 9.7.4 applies to show that $K_0\mathbf{Ch}_{\text{perf}}^b \cong K_0\mathbf{Ch}_{\text{perf}}(R)$, so it suffices to show that the Approximation Theorem applies to $\mathbf{Ch}^b\mathbf{P}(R) \subset \mathbf{Ch}_{\text{perf}}^b$. This is an elementary application of the projective lifting property, which we relegate to Exercise 9.2.

EXAMPLE 9.7.6 (G_0 AND PSEUDO-COHERENT COMPLEXES). Let R be a ring. A complex $M_\bullet$ of R-modules is called *pseudo-coherent* if there exists a quasi-isomorphism $P_\bullet \xrightarrow{\sim} M_\bullet$, where $P_\bullet$ is a bounded below complex $\cdots \to P_{n+1} \to P_n \to 0$ of finitely generated projective R-modules, *i.e.,* $P_\bullet$ is a complex in $\mathbf{Ch}_+(\mathbf{P}(R))$. For example, if R is noetherian we can consider any finitely generated module M as a pseudo-coherent complex concentrated in degree zero. Even if R is not noetherian, it follows from Example 7.1.4 that M is pseudo-coherent as an R-module if and only if it is pseudo-coherent as a chain complex. (See [SGA6, I.2.9].)

The pseudo-coherent complexes form a Waldhausen subcategory $\mathbf{Ch}_{\text{pcoh}}(R)$ of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$, and the subcategory $\mathbf{Ch}_{\text{pcoh}}^{hb}(R)$ of homologically bounded pseudo-coherent complexes is also Waldhausen. Moreover, the above remarks show that $\mathbf{M}(R)$ is a Waldhausen subcategory of both of them. We will see in Ex. 9.7 that the Approximation Theorem applies to the inclusions of $\mathbf{Ch}^b\mathbf{M}(R)$ and $\mathbf{Ch}_+^{hb}\mathbf{P}(R)$ in $\mathbf{Ch}_{\text{pcoh}}^{hb}(R)$, so that in particular we have

$$G_0(R) \cong K_0\mathbf{Ch}_+^{hb}\mathbf{P}(R) \cong K_0\mathbf{Ch}_{\text{pcoh}}^{hb}(R) \cong G_0(R).$$

Chain complexes with support

Suppose that S is a multiplicatively closed set of central elements in a ring R. Let $\mathbf{Ch}_S^b\mathbf{P}(R)$ denote the Waldhausen subcategory of $\mathcal{C} = \mathbf{Ch}^b\mathbf{P}(R)$ consisting of complexes E such that $S^{-1}E$ is exact, and write $K_0(R \text{ on } S)$ for $K_0\mathbf{Ch}_S^b\mathbf{P}(R)$.

The category $\mathbf{Ch}_S^b\mathbf{P}(R)$ is the category $\mathcal{C}^w$ of the Localization Theorem 9.6, where w is the family of all morphisms $P \to Q$ in $\mathcal{C}$ such that $S^{-1}P \to S^{-1}Q$ is a quasi-isomorphism. By Theorem 9.2.2 we have $K_0(\mathcal{C}) = K_0(R)$. Hence there is an exact sequence

$$K_0(R \text{ on } S) \to K_0(R) \to K_0(w\mathcal{C}) \to 0.$$

THEOREM 9.8. *The localization $w\mathcal{C} \to \mathbf{Ch}^b\mathbf{P}(S^{-1}R)$ induces an injection on K_0, so there is an exact sequence*

$$K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R).$$

PROOF. Let $\mathcal{B}$ denote the category of $S^{-1}R$-modules of the form $S^{-1}P$ for P in $\mathbf{P}(R)$. By Example 7.3.2 and Theorem 9.2.2, $K_0\mathbf{Ch}^b(\mathcal{B}) = K_0(\mathcal{B})$ is a subgroup of $K_0(S^{-1}R)$. Therefore the result follows from the following Proposition. □

PROPOSITION 9.8.1. *The Approximation Theorem 9.7 applies to $w\mathcal{C} \to \mathbf{Ch}^b(\mathcal{B})$.*

PROOF. Let P be a complex in $\mathbf{Ch}^b\mathbf{P}(R)$ and $b: S^{-1}P \to B$ a map in $\mathcal{B}$. Because each B_n has the form $S^{-1}Q_n$ and each $B_n \to B_{n-1}$ is $s_n^{-1}d_n$ for some $s_n \in S$ and $d_n: Q_n \to Q_{n-1}$ such that $d_n d_{n-1} = 0$, B is isomorphic to the localization $S^{-1}Q$ of a bounded complex Q in $\mathbf{P}(R)$, and some sb is the localization of a map $f: P \to Q$ in $\mathbf{Ch}^b\mathbf{P}(R)$. Hence f factors as $P \rightarrowtail \text{cyl}(f) \xrightarrow{\sim} Q$. Since b is the localization of f, followed by an isomorphism $S^{-1}Q \cong B$ in $\mathcal{B}$, it factors as desired. □

EXERCISES

9.1 *Retracts of a space.* Fix a CW complex X and let $\mathcal{R}(X)$ be the category of CW complexes Y obtained from X by attaching cells, and having a retraction $Y \to X$. Let $\mathcal{R}_f(X)$ be the subcategory of those Y obtained by attaching only finitely many cells. Let $\mathcal{R}_{fd}(X)$ be the subcategory of those Y which are finitely dominated, *i.e.*, are retracts up to homotopy of spaces in $\mathcal{R}_f(X)$. Show that $K_0\mathcal{R}_f(X) \cong \mathbb{Z}$ and $K_0\mathcal{R}_{fd}(X) \cong K_0(\mathbb{Z}[\pi_1 X])$. *Hint*: The cellular chain complex of the universal covering space $\tilde{Y}$ is a chain complex of free $\mathbb{Z}[\pi_1 X]$-modules.

9.2 Let R be a ring. Use the projective lifting property to show that the Approximation Theorem applies to the inclusion $\mathbf{Ch}^b\mathbf{P}(R) \subset \mathbf{Ch}^b_{\text{perf}}$ of Example 9.7.5. Conclude that $K_0(R) = K_0\mathbf{Ch}_{\text{perf}}(R)$.

If S is a multiplicatively closed set of central elements of R, show that the Approximation Theorem also applies to the inclusion of $\mathbf{Ch}^b_S\mathbf{P}(R)$ in $\mathbf{Ch}_{\text{perf},S}(R)$, and conclude that $K_0(R \text{ on } S) \cong K_0\mathbf{Ch}_{\text{perf},S}(R)$.

9.3 Consider the category $\mathbf{Ch}^b = \mathbf{Ch}^b(\mathcal{A})$ of Theorem 9.2.2 as a Waldhausen category in which the weak equivalences are the isomorphisms, $\text{iso}\,\mathbf{Ch}^b$, as in Example 9.1.3. Let $\mathbf{Ch}^b_{acyc}$ denote the subcategory of complexes whose differentials are all zero. Show that $\mathbf{Ch}^b_{acyc}$ is equivalent to the category $\oplus_{n\in\mathbb{Z}}\mathcal{A}$, and that the inclusion in $\mathbf{Ch}^b$ induces an isomorphism

$$K_0(\text{iso}\,\mathbf{Ch}^b) \cong \bigoplus_{n\in\mathbb{Z}} K_0(\mathcal{A}).$$

9.4 *Higher Extension categories.* Consider the category $\mathcal{E}_n$ constructed in Example 9.3.2, whose objects are sequences of n cofibrations in $\mathcal{C}$. Show that $\mathcal{E}_n$ is a category with cofibrations, that $\mathcal{E}_n$ is a Waldhausen category when $\mathcal{C}$ is, and in that case

$$K_0(\mathcal{E}_n) \cong \bigoplus_{i=1}^{n} K_0(\mathcal{C}).$$

9.5 ([SGA6, IV(1.6)]) Let $\mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{A}$, or more generally any exact subcategory of $\mathcal{A}$ closed under extensions and kernels of surjections. Let $\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A})$ denote the Waldhausen subcategory of $\mathbf{Ch}^b(\mathcal{A})$ of bounded complexes C with $H_i(C)$ in $\mathcal{B}$ for all i. Show that

$$K_0\mathcal{B} \cong K_0\mathbf{Ch}^b(\mathcal{B}) \cong K_0\mathbf{Ch}^b_{\mathcal{B}}(\mathcal{A}).$$

9.6 *Perfect injective complexes.* Let R be a ring and let $\mathbf{Ch}^+_{inj}(R)$ denote the Waldhausen subcategory of $\mathbf{Ch}(\mathbf{mod}\text{-}R)$ consisting of perfect bounded below cochain complexes of injective R-modules $0 \to I^m \to I^{m+1} \cdots$. (Recall from Example 9.7.5 that $I^\bullet$ is called *perfect* if it is quasi-isomorphic to a bounded complex $P^\bullet$ of finitely generated projective modules.) Show that

$$K_0\mathbf{Ch}^+_{inj}(R) \cong K_0(R).$$

9.7 *Pseudo-coherent complexes and $G_0(R)$.* Let R be a ring. Recall from Example 9.7.6 that $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$ denotes the Waldhausen category of all homologically bounded pseudo-coherent chain complexes of R-modules. Show that:

(a) The category $\mathbf{M}(R)$ is a Waldhausen subcategory of $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$.

(b) $K_0\mathbf{Ch}_{\mathrm{pcoh}}(R) = K_0\mathbf{Ch}_+\mathbf{P}(R) = 0$

(c) The Approximation Theorem applies to the inclusions of both $\mathbf{Ch}^{hb}_+\mathbf{M}(R)$ and $\mathbf{Ch}^{hb}_+\mathbf{P}(R)$ in $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$, and $\mathbf{Ch}^b\mathbf{M}(R) \subset \mathbf{Ch}^{hb}_-\mathbf{M}(R)$. *Hint:* See 9.7.4.

This shows that $G_0(R) \cong K_0\mathbf{Ch}^{hb}_+\mathbf{P}(R) \cong K_0\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$.

9.8 *Pseudo-coherent complexes and G_0^{der}.* Let X be a scheme. A cochain complex $E^\bullet$ of $\mathcal{O}_X$-modules is called *strictly pseudo-coherent* if it is a bounded above complex of vector bundles, and *pseudo-coherent* if it is locally quasi-isomorphic to a strictly pseudo-coherent complex, *i.e.,* if every point $x \in X$ has a neighborhood U, a strictly pseudo-coherent complex $P^\bullet$ on U and a quasi-isomorphism $P^\bullet \to E^\bullet|_U$. Let $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$ denote the Waldhausen category of all pseudo-coherent complexes $E^\bullet$ which are homologically bounded, and set $G_0^{der}(X) = K_0\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$; this is the definition used in [SGA6, Expose IV(2.2)].

(a) If X is a noetherian scheme, show that every coherent $\mathcal{O}_X$-module is a pseudo-coherent complex concentrated in degree zero, so that we may consider $\mathbf{M}(X)$ as a Waldhausen subcategory of $\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(X)$. Then show that a complex $E^\bullet$ is pseudo-coherent if and only if is it homologically bounded above and all the homology sheaves of $E^\bullet$ are coherent $\mathcal{O}_X$-modules.

(b) If X is a noetherian scheme, show that $G_0(X) \cong G_0^{der}(X)$.

(c) If $X = \mathrm{Spec}(R)$ for a ring R, show that $G_0^{der}(X)$ is isomorphic to the group $K_0\mathbf{Ch}^{hb}_{\mathrm{pcoh}}(R)$ of the previous exercise.

9.9 Let Z be a closed subscheme of X. Let $\mathbf{Ch}^{hb}_{\text{pcoh},Z}(X)$ denote the subcategory of complexes in $\mathbf{Ch}^{hb}_{\text{pcoh}}(X)$ which are acyclic on $X-Z$, and define $G_0(X \text{ on } Z)$ to be $K_0\mathbf{Ch}^{hb}_{\text{pcoh}}(X)$.

(a) If X is a noetherian scheme, show that $G_0(Z) \cong G_0(X \text{ on } Z)$.

(b) Show that there is an exact sequence $G_0(X \text{ on } Z) \to G_0^{der}(X) \to G_0^{der}(X-Z)$.

9.10 *Perfect complexes and* K_0^{der}. Let X be a scheme. A complex $E^\bullet$ of $\mathcal{O}_X$-modules is called *strictly perfect* if it is a bounded complex of vector bundles, *i.e.,* a complex in $\mathbf{Ch}^b\mathbf{VB}(X)$. A complex is called *perfect* if it is locally quasi-isomorphic to a strictly perfect complex, *i.e.,* if every point $x \in X$ has a neighborhood U, a strictly perfect complex $P^\bullet$ on U and a quasi-isomorphic $P^\bullet \to E^\bullet|_U$. Write $\mathbf{Ch}_{\text{perf}}(X)$ for the Waldhausen category of all perfect complexes, and $K_0^{der}(X)$ for $K_0\mathbf{Ch}_{\text{perf}}(X)$; this is the definition used in [SGA6], Expose IV(2.2).

(a) If $X = \text{Spec}(R)$, show that $K_0(R) \cong K_0^{der}(X)$. *Hint:* show that the Approximation Theorem 9.7 applies to $\mathbf{Ch}_{\text{perf}}(R) \subset \mathbf{Ch}_{\text{perf}}(X)$.

(b) If X is noetherian, show that the category $\mathcal{C} = \mathbf{Ch}^{qc}_{\text{perf}}$ of perfect complexes of quasi-coherent $\mathcal{O}_X$-modules also has $K_0(\mathcal{C}) = K_0^{der}(X)$.

(c) If X is a regular noetherian scheme, show that a homologically bounded complex is perfect if and only if it is pseudo-coherent, and conclude that $K_0^{der}(X) \cong G_0(X)$.

(d) Let X be the affine plane with a double origin over a field k, obtained by glueing two copies of $\mathbb{A}^2 = \text{Spec}(k[x,y])$ together; X is a regular noetherian scheme. Show that $K_0\mathbf{VB}(X) = \mathbb{Z}$ but $K_0^{der}(X) = \mathbb{Z} \oplus \mathbb{Z}$. *Hint.* Use the fact that $\mathbb{A}^2 \to X$ induces an isomorphism $\mathbf{VB}(X) \cong \mathbf{VB}(\mathbb{A}^2)$ and the identification of $K_0^{der}(X)$ with $G_0(X)$ from part (c).

9.11 Give an example of an exact subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ in which $K_0(\mathcal{B}) \neq K_0\mathbf{Ch}^b(\mathcal{B})$. Here $\mathbf{Ch}^b(\mathcal{B})$ is the Waldhausen category defined in 9.2. Note that $\mathcal{B}$ cannot be closed under kernels of surjections, by Theorem 9.2.2.

9.12 *Finitely dominated complexes.* Let $\mathcal{C}$ be a small exact category, closed under extensions and kernels of surjections in an ambient abelian category $\mathcal{A}$ (Definition 7.0.1). A bounded below complex $C_\bullet$ of objects in $\mathcal{C}$ is called *finitely dominated* if there is a bounded complex $B_\bullet$ and two maps $C_\bullet \to B_\bullet \to C_\bullet$ whose composite $C_\bullet \to C_\bullet$ is chain homotopic to the identity. Let $\mathbf{Ch}^{fd}_+(\mathcal{C})$ denote the category of finitely dominated chain complexes of objects in $\mathcal{C}$. (If $\mathcal{C}$ is abelian, this is the category $\mathbf{Ch}^{hb}_+(\mathcal{C})$ of Example 9.7.4.)

(a) Let e be an idempotent endomorphism of an object C, and let $\text{tel}(e)$ denote the nonnegative complex

$$\cdots \xrightarrow{e} C \xrightarrow{1-e} C \xrightarrow{e} C \to 0.$$

Show that $\text{tel}(e)$ is finitely dominated.

(b) Let $\hat{\mathcal{C}}$ denote the idempotent completion 7.3 of $\mathcal{C}$. Show that there is a map from $K_0(\hat{\mathcal{C}})$ to $K_0\mathbf{Ch}^{fd}_+(\mathcal{C})$ sending $[(C,e)]$ to $[\text{tel}(e)]$.

(c) Show that the map in (b) induces an isomorphism $K_0(\hat{\mathcal{C}}) \cong K_0\mathbf{Ch}^{fd}_+(\mathcal{C})$.

9.13 Let S be a multiplicatively closed set of central nonzerodivisors in a ring R. Show that $K_0\mathbf{H}_S(R) \cong K_0(R \text{ on } S)$, and compare Cor. 7.7.4 to Theorem 9.8.

9.14 (Grayson's Trick) Let $\mathcal{B}$ be a Waldhausen subcategory of $\mathcal{C}$ closed under extensions. Suppose that $\mathcal{B}$ is cofinal in $\mathcal{C}$, so that $K_0(\mathcal{B}) \subseteq K_0(\mathcal{C})$ by Cofinality 9.4. Define an equivalence relation $\sim$ on objects of $\mathcal{C}$ by $C \sim C'$ if there are B, B' in $\mathcal{B}$ with $C \amalg B \cong C' \amalg B'$.
(a) Given a cofibration sequence $C' \rightarrowtail C \twoheadrightarrow C''$ in $\mathcal{C}$, use the proof of Cofinality 7.2 to show that $C \sim C' \amalg C''$.
(b) Conclude that $C \sim C'$ if and only if $[C] - [C']$ is in $K_0(\mathcal{B}) \subseteq K_0(\mathcal{C})$. (See 9.4.1.)
(c) For each sequence $C_1, ..., C_n$ of objects in $\mathcal{C}$ such that $[C_1] = \cdots = [C_n]$ in $K_0(\mathcal{C})/K_0(\mathcal{B})$, show that there is a C' in $\mathcal{C}$ so that each $C_i \amalg C'$ is in $\mathcal{B}$.
(d) If $K_0(\mathcal{B}) = K_0(\mathcal{C})$, show that $\mathcal{B}$ is *strictly cofinal* in $\mathcal{C}$, meaning that for every C in $\mathcal{C}$ there is a B in $\mathcal{B}$ so that $C \amalg B$ is in $\mathcal{B}$.

9.15 *Triangulated Categories.* If $\mathcal{C}$ is a triangulated category, the Grothendieck group $k(\mathcal{C})$ is the free abelian group on the objects, modulo the relation that $[A] - [B] + [C] = 0$ for every triangle $A \to B \to C \to A[1]$. (a) If $\mathcal{B}$ is an additive category, regarded as a split exact category (7.1.2), show that $K_0(\mathcal{B})$ is isomorphic to $k(\mathbf{K}^b\mathcal{B})$. (b) If $\mathcal{B}$ is an exact subcategory of an abelian category, closed under kernels, show that $K_0(\mathcal{B})$ is isomorphic to $k(\mathbf{K}^b\mathcal{B})$. *Hint.* See 9.2.2. (c) If $\mathcal{C}$ has a bounded t-structure with heart $\mathcal{A}$ [BBD], show that $K_0(\mathcal{A}) \cong k(\mathcal{C})$.

Appendix. Localizing by categories of fractions

If $\mathcal{C}$ is a category and S is a collection of morphisms in $\mathcal{C}$, then the *localization of* $\mathcal{C}$ *with respect to* S is a category $\mathcal{C}_S$, together with a functor $\mathrm{loc}\colon \mathcal{C} \to \mathcal{C}_S$ such that

(1) For every $s \in S$, $\mathrm{loc}(s)$ is an isomorphism
(2) If $F\colon \mathcal{C} \to \mathcal{D}$ is any functor sending S to isomorphisms in $\mathcal{D}$, then F factors uniquely through $\mathrm{loc}\colon \mathcal{C} \to \mathcal{C}_S$.

Example. We may consider any ring R as an additive category $\mathcal{R}$ with one object. If S is a central multiplicative subset of R, there is a ring $S^{-1}R$ obtained by localizing R at S, and the corresponding category is $\mathcal{R}_S$. The useful fact that every element of the ring $S^{-1}R$ may be written in standard form $s^{-1}r = rs^{-1}$ generalizes to morphisms in a localization $\mathcal{C}_S$, provided that S is a "locally small multiplicative system" in the following sense.

Definition A.1. A collection S of morphisms in $\mathcal{C}$ is called a *multiplicative system* if it satisfies the following three self-dual axioms:

(FR1) S is closed under composition and contains the identity morphisms 1_X of all objects X of $\mathcal{C}$. That is, S forms a subcategory of $\mathcal{C}$ with the same objects.
(FR2) (Ore condition) (a) If $t : Z \to Y$ is in S, then for every $g : X \to Y$ in $\mathcal{C}$ there is a commutative diagram in $\mathcal{C}$ with $s \in S$:

$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{g} & Y. \end{array}$$

(The slogan is "$t^{-1}g = fs^{-1}$ for some f and s.") (b) The dual statement (whose slogan is "$fs^{-1} = t^{-1}g$ for some t and g") is also valid.
(FR3) (Cancellation) If $f, g\colon X \to Y$ are parallel morphisms in $\mathcal{C}$, then the following two conditions are equivalent:
(a) $sf = sg$ for some $s\colon Y \to Z$ in S
(b) $ft = gt$ for some $t\colon W \to X$ in S.

We say that S is a *right multiplicative system* if it satisfies (FR1) and (FR2a), and if (FR3a) implies (FR3b) Left multiplicative systems are defined dually.

Example A.1.1. If S is a multiplicatively closed subset of a ring R, then S forms a multiplicative system if and only if S is a "2–sided denominator set." (One-sided denominator sets (left and right) correspond to left and right multiplicative systems.) The localization of rings at denominator sets was the original application of Øystein Ore.

Example A.1.2 (Gabriel). Let $\mathcal{B}$ be a Serre subcategory (see §6) of an abelian category $\mathcal{A}$, and let S be the collection of all $\mathcal{B}$-*isos*, *i.e.,* those maps f such that $\ker(f)$ and $\mathrm{coker}(f)$ is in $\mathcal{B}$. Then S is a multiplicative system in $\mathcal{A}$; the verification of axioms (FR2), (FR3) is a pleasant exercise in diagram chasing. In this case, $\mathcal{A}_S$ is the quotient abelian category $\mathcal{A}/\mathcal{B}$ discussed in the Localization Theorem 6.4.

We would like to say that every morphism $X \to Z$ in $\mathcal{C}_S$ is of the form fs^{-1}. However, the issue of whether or this construction makes sense (in our universe) involves delicate set-theoretic questions. The following notion is designed to avoid these set-theoretic issues.

We say that S is *locally small* (on the left) if for each X in $\mathcal{C}$ there is a *set* S_X of morphisms $X' \xrightarrow{s} X$ in S such that every map $Y \to X$ in S factors as $Y \to X' \xrightarrow{s} X$ for some $s \in S_X$.

DEFINITION A.2 (FRACTIONS). A (left) *fraction* between X and Y is a chain in $\mathcal{C}$ of the form:

$$fs^{-1}: \quad X \xleftarrow{s} X_1 \xrightarrow{f} Y, \qquad s \in S.$$

Call fs^{-1} *equivalent* to $X \leftarrow X_2 \to Y$ just in case there is a chain $X \leftarrow X_3 \to Y$ fitting into a commutative diagram in $\mathcal{C}$:

$$\begin{array}{ccccc} & & X_1 & & \\ & \swarrow & \uparrow & \searrow & \\ X & \leftarrow & X_3 & \rightarrow & Y \\ & \nwarrow & \downarrow & \nearrow & \\ & & X_2 & & \end{array}$$

It is easy to see that this is an equivalence relation. Write $\mathrm{Hom}_S(X,Y)$ for the equivalence classes of such fractions between X and Y. ($\mathrm{Hom}_S(X,Y)$ is a set when S is locally small.)

We cite the following theorem without proof from [WHomo, 10.3.7], relegating its routine proof to Exercises A.1 and A.2.

GABRIEL-ZISMAN THEOREM A.3. *Let S be a locally small multiplicative system of morphisms in a category $\mathcal{C}$. Then the localization $\mathcal{C}_S$ of $\mathcal{C}$ exists, and may be constructed by the following "calculus" of left fractions.*

$\mathcal{C}_S$ has the same objects as $\mathcal{C}$, but $\mathrm{Hom}_{\mathcal{C}_S}(X,Y)$ is the set of equivalence classes of chains $X \leftarrow X' \to Y$ with $X' \to X$ in S, and composition is given by the Ore condition. The functor loc: $\mathcal{C} \to \mathcal{C}_S$ sends $X \to Y$ to the chain $X \xleftarrow{=} X \to Y$, and if $s: X \to Y$ is in S its inverse is represented by $Y \leftarrow X \xrightarrow{=} X$.

COROLLARY A.3.1. *Two parallel arrows $f, g: X \to Y$ become identified in $\mathcal{C}_S$ if and only if the conditions of (FR3) hold.*

COROLLARY A.3.2. *Suppose that $\mathcal{C}$ has a zero object, and that S is a multiplicative system in $\mathcal{C}$. Assume that S is saturated in the sense that if s and st are in S then so is t. Then for every X in $\mathcal{C}$:*

$$loc(X) \cong 0 \Leftrightarrow \textit{The zero map } \; X \xrightarrow{0} X \; \textit{is in } \; S.$$

PROOF. Since $\mathrm{loc}(0)$ is a zero object in $\mathcal{C}_S$, $\mathrm{loc}(X) \cong 0$ if and only if the parallel maps $0, 1: X \to X$ become identified in $\mathcal{C}_S$. $\square$

Now let $\mathcal{A}$ be an abelian category, and $\mathbf{C}$ a full subcategory of the category $\mathbf{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$, closed under translation and the formation of mapping cones. Let $\mathbf{K}$ be the quotient category of $\mathbf{C}$, obtained by identifying chain homotopic maps in $\mathbf{C}$. Let Q denote the family of (chain homotopy equivalence classes of) quasi-isomorphisms in $\mathbf{C}$. The following result states that Q forms a multiplicative system in $\mathbf{K}$, so that we can form the localization $\mathbf{K}_Q$ of $\mathbf{K}$ with respect to Q by the calculus of fractions.

LEMMA A.4. *The family Q of quasi-isomorphisms in the chain homotopy category* $\mathbf{K}$ *forms a multiplicative system.*

PROOF. (FR1) is trivial. To prove (FR2), consider a diagram $X \xrightarrow{u} Y \xleftarrow{s} Z$ with $s \in Q$. Set $C = \text{cone}(s)$, and observe that C is acyclic. If $f: Y \to C$ is the natural map, set $W = \text{cone}(fu)$, so that the natural map $t: W \to X[-1]$ is a quasi-isomorphism. Now the natural projections from each $W_n = Z_{n-1} \oplus Y_n \oplus X_{n-1}$ to Z_{n-1} form a morphism $v: W \to Z$ of chain complexes making the following diagram commute:

$$\begin{array}{ccccccccc} & & X & \xrightarrow{fu} & C & \longrightarrow & W & \xrightarrow{t} & X[-1] \\ & & u\downarrow & & \| & & \downarrow v & & \downarrow \\ Z & \xrightarrow{s} & Y & \xrightarrow{f} & C & \longrightarrow & Z[-1] & \xrightarrow{s[-1]} & Y[-1]. \end{array}$$

Applying $X \mapsto X[1]$ to the right square gives the first part of (FR2); the second part is dual and is proven similarly.

To prove (FR3), we suppose given a quasi-isomorphism $s: Y \to Y'$ and set $C = \text{cone}(s)$; from the long exact sequence in homology we see that C is acyclic. Moreover, if v denotes the map $C[1] \to Y$ then there is an exact sequence:

$$\text{Hom}_{\mathbf{K}}(X, C[1]) \xrightarrow{v} \text{Hom}_{\mathbf{K}}(X, Y) \xrightarrow{s} \text{Hom}_{\mathbf{K}}(X, Y')$$

(see [WHomo, 10.2.8]). Given f and g, set $h = f - g$. If $sh = 0$ in $\mathbf{K}$, there is a map $w: X \to C[1]$ such that $h = vw$. Setting $X' = \text{cone}(w)[1]$, the natural map $X' \xrightarrow{t} X$ must be a quasi-isomorphism because C is acyclic. Moreover, $wt = 0$, so we have $ht = vwt = 0$, *i.e.*, $ft = gt$. □

DEFINITION A.5. Let $\mathbf{C} \subset \mathbf{Ch}(\mathcal{A})$ be a full subcategory closed under translation and the formation of mapping cones. The *derived category* of $\mathbf{C}$, $\mathbf{D}(\mathbf{C})$, is defined to be the localization $\mathbf{K}_Q$ of the chain homotopy category $\mathbf{K}$ at the multiplicative system Q of quasi-isomorphisms. The *derived category* of $\mathcal{A}$ is $\mathbf{D}(\mathcal{A}) = \mathbf{D}(\mathbf{Ch}(\mathcal{A}))$.

Another application of calculus of fractions is Verdier's formation of quotient triangulated categories by thick subcategories. We will use Rickard's definition of thickness, which is equivalent to Verdier's.

DEFINITION A.6. Let $\mathbf{K}$ be any triangulated category (see [WHomo, 10.2.1]). A full additive subcategory $\mathcal{E}$ of $\mathbf{K}$ is called *thick* if:

(1) In any distinguished triangle $A \to B \to C \to A[1]$, if two out of A, B, C are in $\mathcal{E}$ then so is the third.
(2) if $A \oplus B$ is in $\mathcal{E}$ then both A and B are in $\mathcal{E}$.

If $\mathcal{E}$ is a thick subcategory of $\mathbf{K}$, we can form a quotient triangulated category $\mathbf{K}/\mathcal{E}$, parallel to Gabriel's construction of a quotient abelian category in A.1.2. That is, $\mathbf{K}/\mathcal{E}$ is defined to be $S^{-1}\mathbf{K}$, where S is the family of maps whose cone is in $\mathcal{E}$. By Ex. A.6, S is a saturated multiplicative system of morphisms, so $S^{-1}\mathbf{K}$ can be constructed by the calculus of fractions (theorem A.3).

To justify this definition, note that because S is saturated it follows from A.3.2 and A.6(2) that: (a) $X \cong 0$ in $\mathbf{K}/\mathcal{E}$ if and only if X is in $\mathcal{E}$, and (b) a morphism $f : X \to Y$ in $\mathbf{K}$ becomes an isomorphism in $\mathbf{K}/\mathcal{E}$ if and only if f is in S.

We conclude with a more recent application, due to M. Schlichting [Schl].

DEFINITION A.7. Let $A \subset \mathcal{B}$ be exact categories, with $\mathcal{A}$ closed under extensions, admissible subobjects and admissible quotients in $\mathcal{B}$. We say that $\mathcal{A}$ is *right filtering* in $\mathcal{B}$ if every map from an object B of $\mathcal{B}$ to an object of $\mathcal{A}$ factors through an admissible epi $B \twoheadrightarrow A$ with A in $\mathcal{A}$.

A morphism of $\mathcal{B}$ is called a *weak isomorphism* if it is a finite composition of admissible monics with cokernel in $\mathcal{A}$ and admissible epis with kernel in $\mathcal{A}$. We write $\mathcal{B}/\mathcal{A}$ for the localization of $\mathcal{B}$ with respect to the weak isomorphisms.

PROPOSITION A.7.1. *If $\mathcal{A}$ is right filtering in $\mathcal{B}$, then the class Σ of weak isomorphisms is a right multiplicative system. By the Gabriel-Zisman Theorem A.3, $\mathcal{B}/\mathcal{A}$ may be constructed using a calculus of right fractions.*

PROOF. By construction, weak isomorphisms are closed under composition, so (FR1) holds. Given an admissible $t : Z \twoheadrightarrow Y$ in $\mathcal{B}$ with kernel in $\mathcal{A}$ and $g : X \to Y$, the base change $s : Z \times_Y X \twoheadrightarrow X$ is an admissible epi in Σ and the canonical map $Z \times_Y X \to Z \to Y$ equals gs. Given an admissible monic $t : Z \rightarrowtail Y$ with kernel A' in $\mathcal{A}$, the map $X \to A'$ factors throught an admissible epi $q : X \twoheadrightarrow A$ with A in $\mathcal{A}$ because $\mathcal{A} \subset \mathcal{B}$ is right filtering; the kernel $W \rightarrowtail X$ of q is in Σ and $W \to X \to Y$ factors through a universal map $W \to Z$. An arbitrary t in Σ is a finite composition of these two types, so by induction on the length of t, we see that Σ satisfies (FR2a).

Finally, suppose that $sf = sg$ for some weak isomorphism $s : Y \to Z$ and $f, g : X \to Y$. If s is an admissible monic, then $f = g$ already. If s is an admissible epi, $f - g$ factors through the kernel $A \rightarrowtail Y$ of s. Because $\mathcal{A}$ is right filtering in $\mathcal{B}$, there is an admissible exact sequence $W \overset{t}{\rightarrowtail} X \twoheadrightarrow A$ with A in $\mathcal{A}$, such that $f - g$ factors through A. Hence t is a weak equivalence and $ft = gt$. As before, induction shows that (a) implies (b) in axiom (FR3). □

EXERCISES

A.1 Show that the construction of the Gabriel-Zisman Theorem A.3 makes $\mathcal{C}_S$ into a category by showing that composition is well-defined and associative.

A.2 If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor sending S to isomorphisms, show that F factors uniquely through the Gabriel-Zisman category $\mathcal{C}_S$ of the previous exercise as $\mathcal{C} \to \mathcal{C}_S \to \mathcal{D}$. This proves the Gabriel-Zisman Theorem A.3, that $\mathcal{C}_S$ is indeed the localization of $\mathcal{C}$ with respect to S.

A.3 Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$, and let S be a multiplicative system in $\mathcal{C}$ such that $S \cap \mathcal{B}$ is a multiplicative system in $\mathcal{B}$. Assume furthermore that one of the following two conditions holds:

(a) Whenever $s\colon C \to B$ is in S with B in $\mathcal{B}$, there is a morphism $f\colon B' \to C$ with B' in $\mathcal{B}$ such that $sf \in S$

(b) Condition (a) with the arrows reversed, for $s\colon B \to C$.

Show that the natural functor $\mathcal{B}_S \to \mathcal{C}_S$ is fully faithful, so that $\mathcal{B}_S$ can be identified with a full subcategory of $\mathcal{C}_S$.

A.4 Let $F\colon \mathcal{A} \to \mathcal{A}'$ be an exact functor between two abelian categories, and let S be the family of morphisms s in $\mathbf{Ch}(\mathcal{A})$ such that $F(s)$ is a quasi-isomorphism. Show that S is a multiplicative system in $\mathbf{Ch}\mathcal{A}$.

A.5 Suppose that $\mathbf{C}$ is a subcategory of $\mathbf{Ch}(\mathcal{A})$ closed under translation and the formation of mapping cones, and let Σ be the family of all chain homotopy equivalences in $\mathbf{C}$. Show that the localization $\mathbf{C}_\Sigma$ is the quotient category $\mathbf{K}$ of $\mathbf{C}$ described before Lemma A.4. Conclude that the derived category $\mathbf{D}(\mathbf{C})$ is the localization of $\mathbf{C}$ at the family of all quasi-isomorphisms. *Hint:* If two maps $f_1, f_2 \colon X \to Y$ are chain homotopic then they factor through a common map $f \colon \mathrm{cyl}(X) \to Y$ out of the mapping cylinder of X.

A.6 Let $\mathcal{E}$ be a thick subcategory of a triangulated category $\mathbf{K}$, and S the morphisms whose cone is in $\mathcal{E}$, as in A.6. Show that S is a multiplicative system of morphisms. Then show that S is saturated in the sense of A.3.2.