CHAPTER III

# $K_1$ AND $K_2$ OF A RING

Let $R$ be an associative ring with unit. In this chapter, we introduce the classical definitions of the groups $K_1(R)$ and $K_2(R)$. These definitions use only linear algebra and elementary group theory, as applied to the groups $GL(R)$ and $E(R)$. We also define relative groups for $K_1$ and $K_2$, as well as the negative $K$-groups $K_{-n}(R)$ and the Milnor $K$-groups $K_n^M(R)$.

In the next chapter we will give another definition: $K_n(R) = \pi_n K(R)$ for all $n \geq 0$, where $K(R)$ is a certain topological space built using the category $\mathbf{P}(R)$ of finitely generated projective $R$-modules. We will then have to prove that these topologically defined groups agree with the definition of $K_0(R)$ in chapter II, as well as with the classical constructions of $K_1(R)$ and $K_2(R)$ in this chapter.

## §1. The Whitehead Group $K_1$ of a ring

Let $R$ be an associative ring with unit. Identifying each $n \times n$ matrix $g$ with the larger matrix $\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives an embedding of $GL_n(R)$ into $GL_{n+1}(R)$. The union of the resulting sequence

$$GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_n(R) \subset GL_{n+1}(R) \subset \cdots$$

is called the *infinite general linear group* $GL(R)$.

Recall that the commutator subgroup $[G, G]$ of a group $G$ is the subgroup generated by its commutators $[g, h] = ghg^{-1}h^{-1}$. It is always a normal subgroup of $G$, and has a universal property: the quotient $G/[G, G]$ is an abelian group, and every homomorphism from $G$ to an abelian group factors through $G/[G, G]$.

DEFINITION 1.1. $K_1(R)$ is the abelian group $GL(R)/[GL(R), GL(R)]$.

The universal property of $K_1(R)$ is this: every homomorphism from $GL(R)$ to an abelian group must factor through the natural quotient $GL(R) \to K_1(R)$. Depending upon our situation, we will sometimes think of $K_1(R)$ as an additive group, and sometimes as a multiplicative group.

A ring map $R \to S$ induces a natural map from $GL(R)$ to $GL(S)$, and hence from $K_1(R)$ to $K_1(S)$. That is, $K_1$ is a functor from rings to abelian groups.

EXAMPLE 1.1.1 ($SK_1$). If $R$ happens to be commutative, the determinant of a matrix provides a group homomorphism from $GL(R)$ onto the group $R^\times$ of units of $R$. It is traditional to write $SK_1(R)$ for the kernel of the induced surjection $\det\colon K_1(R) \to R^\times$. The *special linear group* $SL_n(R)$ is the subgroup of $GL_n(R)$ consisting of matrices with determinant 1, and $SL(R)$ is their union. Since the

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natural inclusion of the units $R^\times$ in $GL(R)$ as $GL_1(R)$ is split by the homomorphism $\det\colon GL(R)\to R^\times$, we see that $GL(R)$ is the semidirect product $SL(R)\rtimes R^\times$, and there is a direct sum decomposition: $K_1(R)=R^\times\oplus SK_1(R)$.

Example 1.1.2. If $F$ is a field, then $K_1(F)=F^\times$. We will see this below (see Lemma 1.2.2 and 1.3.1 below), but it is more fun to deduce this from an 1899 theorem of L. E. J. Dickson, that $SL_n(F)$ is the commutator subgroup of both $GL_n(F)$ and $SL_n(F)$, with only two exceptions: $GL_2(\mathbb{F}_2)=SL_2(\mathbb{F}_2)\cong\Sigma_3$, which has order 6, and $GL_2(\mathbb{F}_3)$, which has center $\{\pm I\}$ and quotient $PGL_2(\mathbb{F}_3)=GL_2(\mathbb{F}_3)/\{\pm I\}$ isomorphic to $\Sigma_4$.

Example 1.1.3. If $R$ is the product $R'\times R''$ of two rings, then $K_1(R)=K_1(R')\oplus K_1(R'')$. Indeed, $GL(R)$ is the product $GL(R')\times GL(R'')$, and the commutator subgroup decomposes accordingly.

Example 1.1.4. For all $n$, the Morita equivalence between $R$ and $S=M_n(R)$ (see II.2.7.2) produces an isomorphism between $M_{mn}(R)=\mathrm{End}_R(R^m\otimes R^n)$ and $M_m(M_n(R))=\mathrm{End}_S(S^m)$. It is easy to see that the resulting isomorphism of units $GL_{mn}(R)\cong GL_m(M_n(R))$ is compatible with stablization in $m$, giving an isomorphism $GL(R)\cong GL(M_n(R))$. Hence $K_1(R)\cong K_1(M_n(R))$.

We will show that the commutator subgroup of $GL(R)$ is the subgroup $E(R)$ generated by "elementary" matrices. These are defined as follows.

Definition 1.2. If $i\neq j$ are distinct positive integers and $r\in R$ then the *elementary matrix* $e_{ij}(r)$ is the matrix in $GL(R)$ which has 1 in every diagonal spot, has $r$ in the $(i,j)$-spot, and is zero elsewhere.

$E_n(R)$ denotes the subgroup of $GL_n(R)$ generated by all elementary matrices $e_{ij}(r)$ with $1\le i,j\le n$, and the union $E(R)$ of the $E_n(R)$ is the subgroup of $GL(R)$ generated by all elementary matrices.

Example 1.2.1. A *signed permutation matrix* is one which permutes the standard basis $\{e_i\}$ up to sign, *i.e.*, it permutes the set $\{\pm e_1,\dots,\pm e_n\}$. The following signed permutation matrix belongs to $E_2(R)$:

$$\bar w_{12}=e_{12}(1)e_{21}(-1)e_{12}(1)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

By changing the subscripts, we see that the signed permutation matrices $\bar w_{ij}$ belong to $E_n(R)$ for $n\ge i,j$. Since the products $\bar w_{jk}\bar w_{ij}$ correspond to cyclic permutations of 3 basis elements, every matrix corresponding to an even permutation of basis elements belongs to $E_n(R)$. Moreover, if $g\in GL_n(R)$ then we see by Ex. I.1.11 that $E_{2n}(R)$ contains the matrix $\left(\begin{smallmatrix}g&0\\0&g^{-1}\end{smallmatrix}\right)$.

1.2.2. If we interpret matrices as linear operators on column vectors, then $e_{ij}(r)$ is the elementary row operation of adding $r$ times row $j$ to row $i$, and $E_n(R)$ is the subset of all matrices in $GL_n(R)$ which may be reduced to the identity matrix using only these row operations. The quotient set $GL_n(R)/E_n(R)$ measures the obstruction to such a reduction.

If $F$ is a field this obstruction is $F^\times$, and is measured by the determinant. That is, $E_n(F)=SL_n(F)$ for all $n\ge 1$. Indeed, standard linear algebra shows that every matrix of determinant 1 is a product of elementary matrices.

REMARK 1.2.3 (SURJECTIONS). If $I$ is an ideal of $R$, each homomorphism $E_n(R) \to E_n(R/I)$ is onto, because the generators $e_{ij}(r)$ of $E_n(R)$ map onto the generators $e_{ij}(\bar{r})$ of $E_n(R/I)$. In contrast, the maps $GL_n(R) \to GL_n(R/I)$ are usually not onto unless $I$ is a radical ideal (Ex. I.1.12(iv)). Indeed, the obstruction is measured by the group $K_0(I) = K_0(R, I)$; see Proposition 2.3 below.

DIVISION RINGS 1.2.4. The same linear algebra that we invoked for fields shows that if $D$ is a division ring (a "skew field") then every invertible matrix may be reduced to a diagonal matrix diag$(r, 1, ..., 1)$, and that $E_n(D)$ is a normal subgroup of $GL_n(D)$. Thus each $GL_n(D)/E_n(D)$ is a quotient group of the nonabelian group $D^\times$. Dieudonné proved in 1943 that in fact $GL_n(D)/E_n(D) = D^\times/[D^\times, D^\times]$ for all $n > 1$ (except for $n = 2$ when $D = \mathbb{F}_2$). In particular, $K_1(D) = GL_n(D)/E_n(D)$ for all $n \geq 3$. A proof of this result is sketched in Exercise 1.2 below.

If $D$ is a $d$-dimensional division algebra over its center $F$ (which must be a field), then $d = n^2$ for some integer $n$, and $n$ is called the *Schur index* of $D$. Indeed, there are (many) field extensions $E$ of $F$ such that $D \otimes_F E \cong M_n(E)$; such a field is called a *splitting field* for $D$. For example, any maximal subfield $E \subset D$ has $[E : F] = n$ and is a splitting field.

For any splitting field $E$, the inclusion of $D$ in $M_n(E)$, and $M_r(D)$ in $M_{nr}(E)$, induces maps $D^\times \subset GL_n(E) \xrightarrow{\det} E^\times$ and $GL_r(D) \to GL_{nr}(E) \xrightarrow{\det} E^\times$ whose image lies in the subgroup $F^\times$ of $E^\times$. (If $E/F$ is Galois, the image is fixed by the Galois group $Gal(E/F)$ and hence lies in $F^\times$.) The induced maps $D^\times \to F^\times$ and $GL_r(D) \to F^\times$ are called the *reduced norms* $N_{\text{red}}$ for $D$, and are independent of $E$. For example, if $D = \mathbb{H}$ is the quaternions then $F = \mathbb{R}$, and $N_{\text{red}}(t + ix + jy + kz) = t^2 + x^2 + y^2 + z^2$. It is easy to check here that $N_{\text{red}}$ induces $K_1(\mathbb{H}) \cong \mathbb{R}_+^\times \subset \mathbb{R}^\times$.

Now if $A$ is any central simple $F$-algebra then $A \cong M_r(D)$ for some $D$, and $M_m(A) \cong M_{mr}(D)$. The induced maps $N_{\text{red}}\colon GL_m(A) \cong GL_{mr}(D) \to F^\times$ are sometimes called the reduced norm for $A$, and the kernel of this map is written as $SL_m(A)$. We define $SK_1(A)$ to be the kernel of the induced map

$$N_{\text{red}}\colon K_1(A) \cong K_1(D) \to K_1(F) = F^\times.$$

In 1950 S. Wang showed that $SK_1(D) = 1$ if $F$ is a number field. For every real embedding $\sigma : F \hookrightarrow \mathbb{R}$, $D \otimes_F \mathbb{R}$ is a matrix algebra over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; it is called *unramified* in case $\mathbb{H}$ occurs. The Hasse-Schilling-Maass norm theorem describes the image of the reduced norm, and hence $K_1(D)$:

$$K_1(D) \xrightarrow[N_{\text{red}}]{\simeq} \{x \in F^\times : \sigma(x) > 0 \text{ in } \mathbb{R} \text{ for all ramified } \sigma\}.$$

Wang also showed that $SK_1(D) = 1$ if the Schur index of $D$ is squarefree. In 1976 V. Platanov produced the first examples of a $D$ with $SK_1(D) \neq 1$, by constructing a map from $SK_1(D)$ to a subquotient of the Brauer group $Br(F)$. We will see in 1.7.2 below that the group $SK_1(D)$ has exponent $n$.

REMARK 1.2.5. There is no a priori reason to believe that the subgroups $E_n(R)$ are normal, except in special cases. For example, we shall show in Ex. 1.3 that if $R$ has stable range $d+1$ then $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq d+2$. Vaserstein proved [V69] that $K_1(R) = GL_n(R)/E_n(R)$ for all $n \geq d + 2$.

If $R$ is commutative, we can do better: $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq 3$. This theorem was proven by A. Suslin in [Su77]; we give Suslin's proof in Ex. 1.9. Suslin also gave examples of Dedekind domains for which $E_2(R)$ is not normal in $GL_2(R)$ in [Su76]. For noncommutative rings, the $E_n(R)$ are only known to be normal for large $n$, and only then when the ring $R$ has finite stable range in the sense of Ex. I.1.5; see Ex. 1.3 below.

COMMUTATORS 1.3. Here are some easy-to-check formulas for multiplying elementary matrices. Fixing the indices, we have $e_{ij}(r)e_{ij}(s) = e_{ij}(r+s)$, and $e_{ij}(-r)$ is the inverse of $e_{ij}(r)$. The commutator of two elementary matrices is easy to compute and simple to describe (unless $j = k$ and $i = \ell$):

$$(1.3.1) \qquad [e_{ij}(r), e_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell \\ e_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell \\ e_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases}$$

Recall that a group is called *perfect* if $G = [G, G]$. If a subgroup $H$ of $G$ is perfect, then $H \subseteq [G, G]$. The group $E(R)$ is perfect, as are most of its finite versions:

LEMMA 1.3.2. *If $n \geq 3$ then $E_n(R)$ is a perfect group.*

PROOF. If $i, j, k$ are distinct then $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$.

We know from Example 1.1.2 that $E_2(R)$ is not always perfect; in fact $E_2(\mathbb{F}_2)$ and $E_2(\mathbb{F}_3)$ are solvable groups.

Rather than become enmeshed in technical issues, it is useful to "stabilize" by increasing the size of the matrices we consider. One technical benefit of stability is given in Ex. 1.4. The following stability result was proven by J.H.C. Whitehead in the 1950 paper [Wh50], and in some sense is the origin of $K$-theory.

WHITEHEAD'S LEMMA 1.3.3. *$E(R)$ is the commutator subgroup of $GL(R)$. Hence $K_1(R) = GL(R)/E(R)$.*

PROOF. The commutator subgroup contains $E(R)$ by Lemma 1.3.2. Conversely, every commutator in $GL_n(R)$ can be expressed as a product in $GL_{2n}(R)$:

$$(1.3.4) \qquad [g, h] = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix}.$$

But we saw in Example 1.2.1 that each of these terms is in $E_{2n}(R)$.

EXAMPLE 1.3.5. If $F$ is a field then $K_1(F) = F^\times$, because we have already seen that $E(R) = SL(R)$. Similarly, if $R$ is a Euclidean domain such as $\mathbb{Z}$ or $F[t]$ then it is easy to show that $SK_1(R) = 0$ and hence $K_1(R) = R^\times$; see Ex. 1.5. In particular, $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}$ and $K_1(F[t]) = F^\times$.

To get a feeling for the non-commutative situation, suppose that $D$ is a division ring. Diedonné's calculation of $GL_n(D)/E_n(D)$ (described in 1.2.4 and Ex. 1.2) gives an isomorphism $K_1(D) \cong D^\times/[D^\times, D^\times]$.

EXAMPLE 1.3.6. If $F$ is a finite field extension of $\mathbb{Q}$ (a number field) and $R$ is an integrally closed subring of $F$, then Bass, Milnor and Serre proved in [BMS, 4.3] that $SK_1(R) = 0$, so that $K_1(R) \cong R^\times$. We mention that if $R$ is finitely generated over $\mathbb{Z}$ then, by the Dirichlet Unit Theorem, $K_1(R) = R^\times$ is a finitely generated abelian group isomorphic to $\mu(F) \oplus \mathbb{Z}^{s-1}$, where $\mu(F)$ denotes the cyclic group of all roots of unity in $F$ and $s$ is the number of "places at infinity" for $R$.

EXAMPLE 1.3.7. (Vaserstein) If $r, s \in R$ are such that $1 + rs$ is a unit, then so is $1 + sr$ because $(1 + sr)(1 - s(1 + rs)^{-1}r) = 1$. The subgroup $W(R)$ of $R^\times$ generated by the $(1+rs)(1+sr)^{-1}$ belongs to $E_2(R)$ by Ex. 1.1. For $R = M_2(\mathbb{F}_2)$, $W(R) = R^\times \cong \Sigma_3$ and $K_1(R) = 0$ but $R^\times \neq [R^\times, R^\times]$. The units of the subring $T$ of upper triangular matrices in $M_2(\mathbb{F}_2)$ is even abelian ($T^\times \cong \mathbb{Z}/2$), but $K_1(T) = 0$ since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = (1 + rs)(1 + sr)^{-1}$ for $r = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $s = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Vaserstein has shown [V04] that $W(R) = [R^\times, R^\times]$ if $\Lambda = R/\text{rad}(R)$ is a product of matrix rings, none of which is $M_2(\mathbb{F}_2)$, and at most one of the factors in $\Lambda$ is $\mathbb{F}_2$. In particular, $W(R) = [R^\times, R^\times]$ for every local ring $R$. (See Ex. 1.1.)

LEMMA 1.4. *If $R$ is a semilocal ring then the natural inclusion of $R^\times = GL_1(R)$ into $GL(R)$ induces an isomorphism $K_1(R) \cong R^\times/W(R)$,*

*If $R$ is a commutative semilocal ring, then*

$$SK_1(R) = 0 \quad \textit{and} \quad K_1(R) = R^\times.$$

PROOF. By Example 1.1.1 (1.3.7 and Ex. 1.2 in the noncommutative case), it suffices to prove that $R^\times$ maps onto $K_1(R)$. This will follow by induction on $n$ once we show that $GL_n(R) = E_n(R)GL_{n-1}(R)$. Let $J$ denote the Jacobson radical of $R$, so that $R/J$ is a finite product of matrix algebras over division rings. By examples 1.1.3, 1.1.4 and 1.2.4, $(R/J)^\times$ maps onto $K_1(R/J)$; in fact by Exercise 1.3 we know that every $\bar{g} \in GL_n(R/J)$ is a product $\bar{e}\bar{g}_1$, where $\bar{e} \in E_n(R/J)$ and $\bar{g}_1 \in GL_1(R/J)$.

Given $g \in GL_n(R)$, its reduction $\bar{g}$ in $GL_n(R/J)$ may be decomposed as above: $\bar{g} = \bar{e}\bar{g}_1$. By Remark 1.2.3, we can lift $\bar{e}$ to an element $e \in E_n(R)$. The matrix $e^{-1}g$ is congruent to the diagonal matrix $\bar{g}_1$ modulo $J$, so its diagonal entries are all units and its off-diagonal entries lie in $J$. Using elementary row operations $e_{ij}(r)$ with $r \in J$, it is an easy matter to reduce $e^{-1}g$ to a diagonal matrix, say to $D = \text{diag}(r_1, ..., r_n)$. By Ex. I.1.11, the matrix $\text{diag}(1, ..., 1, r_n, r_n^{-1})$ is in $E_n(R)$. Multiplying $D$ by this matrix yields a matrix in $GL_{n-1}(R)$, finishing the induction and the proof.

*Commutative Banach Algebras*

Let $R$ be a commutative Banach algebra over the real or complex numbers. For example, $R$ could be the ring $\mathbb{R}^X$ of continuous real-valued functions of a compact space $X$. As subspaces of the metric space of $n \times n$ matrices over $R$, the groups $SL_n(R)$ and $GL_n(R)$ are topological groups.

PROPOSITION 1.5. *$E_n(R)$ is the path component of the identity matrix in the special linear group $SL_n(R)$, $n \geq 2$. Hence we may identify the group $SK_1(R)$ with the group $\pi_0 SL(R)$ of path components of the topological space $SL(R)$.*

PROOF. To see that $E_n(R)$ is path-connected, fix an element $g = \prod e_{i_\alpha j_\alpha}(r_\alpha)$. The formula $t \mapsto \prod e_{i_\alpha j_\alpha}(r_\alpha t)$, $0 \le t \le 1$ defines a path in $E_n(R)$ from the identity to $g$. To prove that $E_n(R)$ is open subset of $SL_n(R)$ (and hence a path-component), it suffices to prove that $E_n(R)$ contains $U_{n-1}$, the set of matrices $1+(r_{ij})$ in $SL_n(R)$ with $||r_{ij}|| < \frac{1}{n-1}$ for all $i,j$. We will actually show that each matrix in $U_{n-1}$ can be expressed naturally as a product of $n^2+5n-6$ elementary matrices, each of which depends continuously upon the entries $r_{ij} \in R$.

Set $u = 1 + r_{11}$. Since $\frac{n-2}{n-1} < ||u||$, $u$ has an inverse $v$ with $||v|| < \frac{n-1}{n-2}$. Subtracting $vr_{1j}$ times the first column from the $j^{th}$ we obtain a matrix $1 + r'_{ij}$ whose first row is $(u, 0, ..., 0)$ and

$$||r'_{ij}|| < \frac{1}{n-1} + \frac{n-1}{n-2}\left(\frac{1}{n-1}\right)^2 = \frac{1}{n-2}.$$

We can continue to clear out entries in this way so that after $n(n-1)$ elementary operations we have reduced the matrix to diagonal form.

By Ex. I.1.10, any diagonal matrix $\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1}\end{smallmatrix}\right)$ is the product of 6 elementary matrices. By induction, it follows that any diagonal $n \times n$ matrix of determinant 1 can be written naturally as a product of $6(n-1)$ elementary matrices.

Let $V$ denote the path component of 1 in the topological group $R^\times$, *i.e.*, the kernel of $R^\times \to \pi_0 R^\times$. By Ex. 1.12, $V$ is a quotient of the additive group $R$.

COROLLARY 1.5.1. *If $R$ is a commutative Banach algebra, there is a natural surjection from $K_1(R)$ onto $\pi_0 GL(R) = \pi_0(R^\times) \times \pi_0 SL(R)$. The kernel of this map is the divisible subgroup $V$ of $R^\times$.*

EXAMPLE 1.5.2. If $R = \mathbb{R}$ then $K_1(\mathbb{R}) = \mathbb{R}^\times$ maps onto $\pi_0 GL(\mathbb{R}) = \{\pm 1\}$, and the kernel is the uniquely divisible multiplicative group $V = (0, \infty)$. If $R = \mathbb{C}$ then $V = \mathbb{C}^\times$, because $K_1(\mathbb{C}) = \mathbb{C}^\times$ but $\pi_0 GL(\mathbb{C}) = 0$.

EXAMPLE 1.5.3. Let $X$ be a compact space with a nondegenerate basepoint. Then $SK_1(\mathbb{R}^X)$ is the group $\pi_0 SL(\mathbb{R}^X) = [X, SL(\mathbb{R})] = [X, SO]$ of homotopy classes of maps from $X$ to the infinite special orthogonal group $SO$. By Ex. II.3.11 we have $\pi_0 GL(\mathbb{R}^X) = [X, O] = KO^{-1}(X)$, and there is a short exact sequence

$$0 \to \mathbb{R}^X \xrightarrow{\exp} K_1(\mathbb{R}^X) \to KO^{-1}(X) \to 0.$$

Similarly, $SK_1(\mathbb{C}^X)$ is the group $\pi_0 SL(\mathbb{C}^X) = [X, SL(\mathbb{C})] = [X, SU]$ of homotopy classes of maps from $X$ to the infinite special unitary group $SU$. Since $\pi_0 GL(\mathbb{C}^X) = [X, U] = KU^{-1}(X)$ by II.3.5.1 and Ex. II.3.11, there is a natural surjection from $K_1(\mathbb{C}^X)$ onto $KU^{-1}(X)$, and the kernel $V$ is the divisible group of all contractible maps $X \to \mathbb{C}^\times$.

EXAMPLE 1.5.4. When $X$ is the circle $S^1$ we have $SK_1(\mathbb{R}^{S^1}) = [S^1, SO] = \pi_1 SO = \mathbb{Z}/2$. On the other hand, we have $\pi_0 SL_2(\mathbb{R}^{S^1}) = \pi_1 SL_2(\mathbb{R}) = \pi_1 SO_2 = \mathbb{Z}$, generated by the matrix $A = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$. Since $\pi_1 SO_2(\mathbb{R}) \to \pi_1 SO$ is onto, the matrix $A$ represents the nonzero element of $SK_1(\mathbb{R}^{S^1})$.

The ring $R = \mathbb{R}[x,y]/(x^2+y^2-1)$ may be embedded in the ring $\mathbb{R}^{S^1}$ by $x \mapsto \cos(\theta), y \mapsto \sin(\theta)$. Since the matrix $\left(\begin{smallmatrix} x & y \\ -y & x\end{smallmatrix}\right)$ maps to $A$, it represents a nontrivial element of $SK_1(R)$. In fact it is not difficult to show that $SK_1(R) \cong \mathbb{Z}/2$ using Mennicke symbols (Ex. 1.10).

*$K_1$ and projective modules*

Now let $P$ be a finitely generated projective $R$-module. Choosing an isomorphism $P \oplus Q \cong R^n$ gives a group homomorphism from $\mathrm{Aut}(P)$ to $GL_n(R)$. (Send $\alpha$ to $\alpha \oplus 1_Q$.)

LEMMA 1.6. *The homomorphism from* $\mathrm{Aut}(P)$ *to* $GL(R) = \bigcup GL_n(R)$ *is well-defined up to inner automorphism of* $GL(R)$. *Hence there is a well-defined homomorphism* $\mathrm{Aut}(P) \to K_1(R)$.

PROOF. First suppose that $Q$ and $n$ are fixed. Two different isomorphisms between $P \oplus Q$ and $R^n$ must differ by an automorphism of $R^n$, *i.e.*, by an element $g \in GL_n(R)$. Thus if $\alpha \in \mathrm{Aut}(P)$ maps to the matrices $A$ and $B$, respectively, we must have $A = gBg^{-1}$. Next we observe that there is no harm in stabilizing, *i.e.*, replacing $Q$ by $Q \oplus R^m$ and $P \oplus Q \cong R^n$ by $P \oplus (Q \oplus R^m) \cong R^{n+m}$. This is because $GL_n(R) \to GL(R)$ factors through $GL_{n+m}(R)$. Finally, suppose given a second isomorphism $P \oplus Q' \cong R^m$. Since $Q \oplus R^m \cong R^n \oplus Q'$, we may stabilize both $Q$ and $Q'$ to make them isomorphic, and invoke the above argument.

COROLLARY 1.6.1. *If $R$ and $S$ are rings, there is a natural external product operation* $K_0(R) \otimes K_1(S) \to K_1(R \otimes S)$.

*If $R$ is commutative and $S$ is an $R$-algebra, there is a natural product operation* $K_0(R) \otimes K_1(S) \to K_1(S)$, *making* $K_1(S)$ *into a module over the ring* $K_0(R)$.

PROOF. For each finitely generated projective $R$-module $P$ and each $m$, Lemma 1.6 provides a homomorphism $\mathrm{Aut}(P \otimes S^m) \to K_1(R \otimes S)$. For each $\beta \in GL_m(S)$, let $[P] \cdot \beta$ denote the image of the automorphism $1_P \otimes \beta$ of $P \otimes S^m$ under this map. Fixing $\beta$ and $m$, the isomorphism $(P \oplus P') \otimes S^m \cong (P \otimes S^m) \oplus (P' \otimes S^m)$ yields the identity $[P \oplus P'] \cdot \beta = [P] \cdot \beta + [P'] \cdot \beta$ in $K_1(R \otimes S)$. Hence $P \mapsto [P] \cdot \beta$ is an additive function of $P \in \mathbf{P}(R)$, so (by definition) it factors through $K_0(R)$. Now fix $P$; the map $GL_m(S) \to K_1(R \otimes S)$ given by $\beta \mapsto [P] \cdot \beta$ is compatible with stabilization in $m$. Thus it factors through a map $GL(S) \to K_1(R \otimes S)$, and through a map $K_1(S) \to K_1(R \otimes S)$. This shows that the product is well-defined and bilinear.

When $R$ is commutative, $K_0(R)$ is a ring by II, §2. If $S$ is an $R$-algebra, there is a ring map $R \otimes S \to S$. Composing the external product with $K_1(R \otimes S) \to K_1(S)$ yields a natural product operation $K_0(R) \otimes K_1(S) \to K_1(S)$. The verification that $[P \otimes_R Q] \cdot \beta = [P] \cdot ([Q] \cdot \beta)$ is routine.

Here is a homological interpretation of $K_1(R)$. Recall that the first homology $H_1(G; \mathbb{Z})$ of any group $G$ is naturally isomorphic to $G/[G, G]$. (See [WHomo, 6.1.11] for a proof.) For $G = GL(R)$ this yields

$$K_1(R) = H_1(GL(R); \mathbb{Z}) = \lim_{n \to \infty} H_1(GL_n(R); \mathbb{Z}). \tag{1.6.2}$$

By Lemma 1.6, we also have well-defined compositions

$$H_1(\mathrm{Aut}(P); \mathbb{Z}) \to H_1(GL_n(R); \mathbb{Z}) \to K_1(R),$$

which are independent of the choice of isomorphism $P \oplus Q \cong R^n$.

Here is another description of $K_1(R)$ in terms of the category $\mathbf{P}(R)$ of finitely generated projective $R$-modules. Consider the translation category $t\mathbf{P}$ of $\mathbf{P}(R)$: its objects are isomorphism classes of finitely generated projective modules, and the morphisms between $P$ and $P'$ are the isomorphism classes of $Q$ such that $P \oplus Q \cong P'$. This is a filtering category [WHomo, 2.6.13], and $P \mapsto H_1(\mathrm{Aut}(P);\mathbb{Z})$ is a well-defined functor from $t\mathbf{P}$ to abelian groups. Hence we can take the filtered direct limit of this functor. Since the free modules are cofinal in $t\mathbf{P}$, we see from (1.6.2) that we have

Corollary 1.6.3 (Bass). $K_1(R) \cong \varinjlim_{P\in t\mathbf{P}} H_1(\mathrm{Aut}(P);\mathbb{Z})$.

Recall from II.2.7 that if two rings $R$ and $S$ are *Morita equivalent* then the categories $\mathbf{P}(R)$ and $\mathbf{P}(S)$ are equivalent. By Corollary 1.6.3 we have the following:

Proposition 1.6.4 (Morita invariance of $K_1$). *The group $K_1(R)$ depends only upon the category $\mathbf{P}(R)$. That is, if $R$ and $S$ are Morita equivalent rings then $K_1(R) \cong K_1(S)$. In particular, the isomorphism of 1.1.4 arises in this way:*

$$K_1(R) \cong K_1(M_n(R)).$$

*Transfer maps*

Let $f\colon R \to S$ be a ring homomorphism. We will see later on that a *transfer homomorphism* $f_*\colon K_1(S) \to K_1(R)$ is defined whenever $S$ has a finite $R$-module resolution by finitely generated projective $R$-modules. This construction requires a definition of $K_1$ for an exact category such as $\mathbf{H}(R)$, and is analogous to the transfer map in II(7.9.1) for $K_0$. Without this machinery, we can still construct the transfer map when $S$ is finitely generated projective as an $R$-module, using the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$; this is the analogue of the method used for the $K_0$ transfer map in example II.2.8.1.

Lemma 1.7. *Any additive functor $T : \mathbf{P}(S) \to \mathbf{P}(R)$ induces a natural homomorphism $K_1(T)\colon K_1(S) \to K_1(R)$, and $T_1 \oplus T_2$ induces the sum $K_1(T_1) + K_1(T_2)$.*

Proof. The functor $T$ induces an evident functor $t\mathbf{P}(S) \to t\mathbf{P}(R)$. If $P$ is a finitely generated projective $S$-module, $T$ also induces a homomorphism $\mathrm{Aut}_S(P) \to \mathrm{Aut}_R(TP)$ and hence $H_1(\mathrm{Aut}_S(P);\mathbb{Z}) \to H_1(\mathrm{Aut}_R(TP);\mathbb{Z})$. As $P$ varies, these assemble to give a natural transformation of functors from the translation category $t\mathbf{P}(S)$ to abelian groups. Since $K_1(S) = \varinjlim_{P\in\mathbf{P}(S)} H_1(\mathrm{Aut}_S(P);\mathbb{Z})$ by Corollary 1.6.3, taking the direct limit over $t\mathbf{P}(S)$ yields the desired map

$$K_1(S) \to \varinjlim_{P\in\mathbf{P}(S)} H_1(\mathrm{Aut}_R(P);\mathbb{Z}) \to \varinjlim_{Q\in\mathbf{P}(R)} H_1(\mathrm{Aut}_R(Q);\mathbb{Z}) = K_1(R).$$

Corollary 1.7.1. *Suppose that $S$ is finitely generated projective as an $R$-module. Then the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ induces a natural transfer homomorphism $f_*\colon K_1(S) \to K_1(R)$. If $R$ is commutative, the composite*

$$K_1(R) \xrightarrow{f^*} K_1(S) \xrightarrow{f_*} K_1(R)$$

*is multiplication by $[S] \in K_0(R)$.*

Proof. When $T$ is the forgetful map, so that $K_1(S) \to K_1(R)$ is the transfer map $f_*$, we compute the composite $f_* f^*$, by computing its effect upon an element $\alpha \in GL_n(R)$. The matrix $f^*(\alpha) = 1_S \otimes_R \alpha$ lies in $GL_n(S)$. To apply $f_*$ we consider $1_S \otimes_R \alpha$ as an element of the group $\mathrm{Aut}_R(S^n) = \mathrm{Aut}_R(S \otimes_R R^n)$, which we then map into $GL(R)$. But this is just the product $[S] \cdot \alpha$ of 1.6.1.

When $j : F \to E$ is a finite field extension, it is easy to see from 1.1.2 that the transfer map $j_* : E^\times \to F^\times$ is the classical norm map. For this reason, the transfer map is sometimes called the *norm map*.

Example 1.7.2. Let $D$ be a division algebra of dimension $d = n^2$ over its center $F$, and recall from 1.2.4 that $SK_1(D)$ is the kernel of the reduced norm $N_{\mathrm{red}}$. We will show that $SK_1(D)$ has exponent $n$ by showing that $i^* N_{\mathrm{red}} : K_1(D) \to K_1(D)$ is multiplication by $n$.

To see this, choose a maximal subfield $E$ with inclusions $F \xrightarrow{j} E \xrightarrow{\sigma} D$. By the definition of $N_{\mathrm{red}}$, composing it with $j^* : F^\times \subset E^\times$ yields the transfer map $\sigma_* : K_1(D) \to K_1(E)$. Therefore, $i^* N_{\mathrm{red}} = \sigma^* j^* N_{\mathrm{red}} = \sigma^* \sigma_*$. Hence it suffices to show that $\sigma^* \sigma_* : K_1(D) \to K_1(D)$ is multiplication by $n$. By 1.7, $\sigma^* \sigma_*$ is induced by the additive self-map $T : M \mapsto M \otimes_D (D \otimes_E D)$ of $\mathbf{P}(D)$. Since $D \otimes_E D \cong D^n$ as a $D$-bimodule, $T(M) \cong M^n$ and the assertion follows from 1.7.

The transfer map $i_* : K_1(D) \to K_1(F)$ associated to $i : F \subset D$ is induced from the classical norm map $N_{D/F} : D^\times \subset GL_d(F) \to F^\times$. In fact, the norm map is $n$ times the reduced norm $N_{\mathrm{red}} : D^\times \to F^\times$ of 1.2.4; see Ex. 1.16 below. Moreover, the composition $i^* i_* : K_1(D) \to K_1(D)$ is multiplication by $d$ since it corresponds to the additive self-map $M \mapsto M \otimes_D (D \otimes_F D)$ of $\mathbf{P}(D)$, and $D \otimes_F D \cong D^d$ as a $D$-bimodule (see II.2.8.1).

Corollary 1.7.3. *$K_1(R) = 0$ for every flasque ring $R$.*

Proof. Recall from II.2.1.3 that a ring $R$ is *flasque* if there is an additive self-functor $T$ ($P \mapsto P \otimes_R M$) on $\mathbf{P}(R)$ together with a natural transformation $\theta_P : P \oplus T(P) \cong T(P)$. By 1.7, the induced self-map on $K_1(R)$ satisfies $x + T(x) = T(x)$ (and hence $x = 0$) for all $x \in K_1(R)$.

Here is an application of 1.7 that anticipates the higher $K$-theory groups with coefficients in chapter IV.

Definition 1.7.4. For each natural number $m$, we define $K_1(R; \mathbb{Z}/m)$ to be the relative group $K_0(\cdot m)$ of II.2.10, where $\cdot m$ is the endo-functor of $\mathbf{P}(R)$ sending $P$ to $P^m = P \otimes_R R^m$. Since the $P^m$ are cofinal, we see by Ex. II.2.15 and Ex. 1.14, that it fits into a universal coefficient sequence:

$$K_1(R) \xrightarrow{m} K_1(R) \to K_1(R; \mathbb{Z}/m) \to K_0(R) \xrightarrow{m} K_0(R).$$

Example 1.8 (Whitehead group $Wh_1$). If $R$ is the group ring $\mathbb{Z}[G]$ of a group $G$, the (first) Whitehead group $Wh_1(G)$ is the quotient of $K_1(\mathbb{Z}[G])$ by the subgroup generated by $\pm 1$ and the elements of $G$, considered as elements of $GL_1$. If $G$ is abelian, then $\mathbb{Z}[G]$ is a commutative ring and $\pm G$ is a subgroup of $K_1(\mathbb{Z}[G])$, so by 1.3.4 we have $Wh_1(G) = (\mathbb{Z}[G]^\times)/\pm G \oplus SK_1(\mathbb{Z}[G])$. If $G$ is finite then $Wh_1(G)$ is a finitely generated group whose rank is $r - q$, where $r$ and $q$ are the number of simple factors in $\mathbb{R}[G]$ and $\mathbb{Q}[G]$, respectively. This and other calculations related to $Wh_1(G)$ may be found in R. Oliver's excellent sourcebook [Oliver].

The group $Wh_1(G)$ arose in Whitehead's 1950 study [Wh50] of simple homotopy types. Two finite CW complexes have the same simple homotopy type if they are connected by a finite sequence of "elementary expansions and collapses." Given a homotopy equivalence $f\colon K \to L$ of complexes with fundamental group $G$, the *torsion* of $f$ is an element $\tau(f) \in Wh_1(G)$. Whitehead proved that $\tau(f) = 0$ if and only if $f$ is a simple homotopy equivalence, and that every element of $Wh_1(G)$ is the torsion of some $f$. An excellent source for the geometry behind this is [Cohen].

EXAMPLE 1.9 (THE $s$-COBORDISM THEOREM). Here is another area of geometric topology in which Whitehead torsion has played a crucial role, piecewise-linear ("PL") topology. We say that a triple $(W, M, M')$ of compact PL manifolds is an *h-cobordism* if the boundary of $W$ is the disjoint union of $M$ and $M'$, and both inclusions $M \subset W$, $M' \subset W$ are homotopy equivalences. In this case we can define the torsion $\tau$ of $M \subset W$, as an element of $Wh_1(G)$, $G = \pi_1 M$. The *s-cobordism theorem* states that if $M$ is fixed with $\dim(M) \geq 5$ then $(W, M, M') \cong (M \times [0,1], M \times 0, M \times 1)$ if and only if $\tau = 0$. Moreover, every element of $Wh_1(G)$ arises as the torsion of some $h$-cobordism $(W, M, M')$.

Here is an application. Suppose given an $h$-cobordism $(W, M, M')$, and let $N$ be the union of $W$, the cone on $M$ and the cone on $M'$. Then $N$ is PL homeomorphic to the suspension $\Sigma M$ of $M$ if and only if $(W, M, M') \cong (M \times [0,1], M \times 0, M \times 1)$ if and only if $\tau = 0$.

This gives a counterexample to the "Hauptvermutung" that two homeomorphic complexes would be PL homeomorphic. Indeed, if $(W, M, M')$ is an $h$-cobordism with nonzero torsion, then $N$ and $\Sigma M$ cannot be PL homeomorphic, yet the theory of "engulfing" implies that they must be homeomorphic manifolds.

Another application, due to Smale, is the Generalized Poincaré Conjecture. Let $N$ be an $n$-dimensional PL manifold of the homotopy type of the sphere $S^n$, $n \geq 5$. Then $N$ is PL homeomorphic to $S^n$. To see this, let $W$ be obtained by removing two small disjoint $n$-discs $D_1, D_2$ from $N$. The boundary of these discs is the boundary of $W$, and $(W, S^{n-1}, S^{n-1})$ is an $h$-cobordism. Its torsion must be zero since $\pi_1(S^{n-1}) = 0$ and $Wh_1(0) = 0$. Hence $W$ is $S^{n-1} \times [0,1]$, and this implies that $N = W \cup D_1 \cup D_2$ is $S^n$.

EXERCISES

**1.1** If $r, s, t \in R$ are such that $(1 + rs)t = 1$, show that $(1 + rs)(1 + sr)^{-1} \in E_2(R)$. *Hint:* Start by calculating $e_{12}(r + rsr)e_{21}(st + s)e_{12}(-r)e_{21}(-s)$.

If $r$ is a unit of $R$, or if $r, s \in \mathrm{rad}(R)$, show that $(1 + rs)(1 + sr)^{-1} \in [R^\times, R^\times]$. Conclude that if $R$ is a local ring then $W(R) = [R^\times, R^\times]$. *Hint:* If $r, s \in \mathrm{rad}(R)$, then $t = 1 + s - sr$ is a unit; compute $[t^{-1} + r, t]$ and $(1 + rs)(1 + r)$.

**1.2** *Semilocal rings.* Let $R$ be a noncommutative semilocal ring (Ex. II.2.6). Show that there exists a unique "determinant" map from $GL_n(R)$ onto the abelian group $R^\times/W(R)$ of Lemma 1.4 with the following properties: (i) $\det(e) = 1$ for every elementary matrix $e$, and (ii) If $\rho = \mathrm{diag}(r, 1, ..., 1)$ and $g \in GL_n(R)$ then $\det(\rho{\cdot}g) = r \cdot \det(g)$. Then show that det is a group homomorphism: $\det(gh) = \det(g)\det(h)$. Conclude that $K_1(R) \cong R^\times/W(R)$.

**1.3** Suppose that a ring $R$ has stable range $sr(R) = d + 1$ in the sense of Ex. I.1.5. (For example, $R$ could be a $d$-dimensional commutative noetherian ring.) This condition describes the action of $E_{d+2}(R)$ on unimodular rows in $R^{d+2}$.

(a) Show that $GL_n(R) = GL_{d+1}(R)E_n(R)$ for all $n > d+1$, and deduce that $GL_{d+1}(R)$ maps onto $K_1(R)$.
(b) Show that $E_n(R)$ is a normal subgroup of $GL_n(R)$ for all $n \geq d+2$. *Hint:* Conjugate $e_{nj}(r)$ by $g \in GL_{d+2}(R)$.

**1.4** Let $R$ be the polynomial ring $F[x,y]$ over a field $F$. P.M. Cohn proved that the matrix $g = \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix}$ is not in $E_2(R)$. Show that $g \in E_3(R) \cap GL_2(R)$.

**1.5** Let $R$ be a Euclidean domain, such as $\mathbb{Z}$ or the polynomial ring $F[t]$ over a field. Show that $E_n(R) = SL_n(R)$ for all $n$, and hence that $SK_1(R) = 0$.

**1.6** Here is another interpretation of the group law for $K_1$. For each $m, n$, let $\oplus_{mn}$ denote the group homomorphism $GL_m(R) \times GL_n(R) \to GL_{m+n}(R)$ sending $(\alpha, \beta)$ to the block diagonal matrix $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. Show that in $K_1(R)$ we have $[\alpha \oplus_{mn} \beta] = [\alpha][\beta]$.

**1.7** Let $E = \mathrm{End}_R(R^\infty)$ be the ring of infinite row-finite matrices over $R$ of Ex. I.1.7. Show that $K_1(E) = 0$. *Hint:* If $\alpha \in GL_n(E)$, form the block diagonal matrix $\alpha^\infty = \mathrm{diag}(\alpha, \alpha, \dots)$ in $\mathrm{Aut}(V) \cong GL(E)$, where $V$ is an infinite sum of copies of $(R^\infty)^n$, and show that $\alpha \oplus \alpha^\infty$ is conjugate to $\alpha^\infty$.

**1.8** In this exercise we show that the center of $E(R)$ is trivial. First show that any matrix in $GL_n(R)$ commuting with $E_n(R)$ must be a diagonal matrix $\mathrm{diag}(r, ..., r)$ with $r$ in the center of $R$. Conclude that no element in $E_{n-1}(R)$ is in the center of $E_n(R)$, and pass to the limit as $n \to \infty$.

**1.9** In this exercise we suppose that $R$ is a commutative ring, and give Suslin's proof that $E_n(R)$ is a normal subgroup of $GL_n(R)$ when $n \geq 3$. Let $v = \sum_{i=1}^n v_i e_i$ be a column vector, and let $u, w$ be row vectors such that $u \cdot v = 1$ and $w \cdot v = 0$.
(a) Show that $w = \sum_{i<j} r_{ij}(v_j e_i - v_i e_j)$, where $r_{ij} = w_i u_j - w_j u_i$.
(b) Conclude that the matrix $I_n + (v \cdot w)$ is in $E_n(R)$ if $n \geq 3$.
(c) If $g \in GL_n(R)$ and $i < j$, let $v$ be the $i^{th}$ column of $g$ and $w$ the $j^{th}$ row of $g^{-1}$, so that $w \cdot v = 0$. Show that $g e_{ij}(r) g^{-1} = I_n + (v \cdot rw)$ for all $r \in R$. By (b), this proves that $E_n(R)$ is normal.

**1.10** *Mennicke symbols.* Let $(r, s)$ be a unimodular row over a commutative ring $R$. We define the *Mennicke symbol* $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$ to be the class in $SK_1(R)$ of the matrix $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$, where $t, u \in R$ satisfy $ru - st = 1$. Show that this Mennicke symbol is independent of the choice of $t$ and $u$, that $\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right] = \left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} s' \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} ss' \\ r \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} s+xr \\ r \end{smallmatrix}\right]$.

If $R$ is noetherian of dimension 1, or more generally has $sr(R) \leq 2$, then we know by Ex. 1.3 that $GL_2(R)$ maps onto $K_1(R)$, and hence $SK_1(R)$ is generated by Mennicke symbols.

**1.11** *Transfer.* Suppose that $R$ is a Dedekind domain and $\mathfrak{p}$ is a prime ideal of $R$. Show that there is a map $\pi_*$ from $K_1(R/\mathfrak{p}) = (R/\mathfrak{p})^\times$ to $SK_1(R)$ sending $\bar{s} \in (R/\mathfrak{p})^\times$ to the Mennicke symbol $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$, where $s \in R$ maps to $\bar{s}$ and $r \in R$ is an element of $\mathfrak{p} - \mathfrak{p}^2$ relatively prime to $s$. Another construction of the transfer map $\pi_*$ will be given in chapter V.

**1.12** If $R$ is a commutative Banach algebra, let $\exp(R)$ denote the image of the exponential map $R \to R^\times$. Show that $\exp(R)$ is the path component of 1 in $R^\times$.

**1.13** If $H$ is a normal subgroup of a group $G$, then $G$ acts upon $H$ and hence its homology $H_*(H; \mathbb{Z})$ by conjugation. Since $H$ always acts trivially upon its homology [WHomo, 6.7.8]), the group $G/H$ acts upon $H_*(H;\mathbb{Z})$. Taking $H = E(R)$ and $G = GL(R)$, use Example 1.2.1 to show that $GL(R)$ and $K_1(R)$ act trivially upon the homology of $E(R)$.

**1.14** (Swan) Let $T : \mathbf{P}(R) \to \mathbf{P}(S)$ be an additive functor, such as the base change $f^*$ associated to a ring map $f : R \to S$. In II.2.10 we constructed a relative group $K_0(T)$. Since $K_0(T)$ is abelian, we can concatenate the $K_1$ map of lemma 1.7 to (II.2.10.2) to get a sequence which is exact at $K_0(T)$ and (if $T$ is cofinal) at $K_0(R)$:

$$K_1(R) \xrightarrow{T} K_1(S) \to K_0(T) \to K_0(R) \xrightarrow{T} K_0(S).$$

In this exercise, we show that the sequence is also exact at $K_1(S)$.

(a) We say that $(P,\alpha,Q) \sim (P',\alpha',Q')$ if there are $N, N' \in \mathbf{P}(R)$ and a commutator $\gamma$ in $\mathrm{Aut}_S\, T(Q \oplus N)$ so that $(P \oplus N, \gamma(\alpha \oplus 1), Q \oplus N)$ is isomorphic to $(P' \oplus N', \alpha' \oplus 1, Q' \oplus N')$ in $\mathbf{P}(T)$. Show that $\sim$ is an equivalence relation.

(b) Show that the equivalence classes under $\sim$ form an abelian group under $\oplus$.

(c) If $(P,\alpha,Q) \sim (P',\alpha',Q')$, show that $[(P,\alpha,Q)] \cong [(P',\alpha',Q')]$ in $K_0(T)$.

(d) If $[(P,\alpha,Q)] \cong [(P',\alpha',Q')]$ in $K_0(T)$, show that $(P,\alpha,Q) \sim (P',\alpha',Q')$. *Hint:* Show that the relations for $K_0(T)$ hold in the group of (b). To do so, write $P \cong P' \oplus P''$ and $Q \cong Q' \oplus Q''$ in the exact sequence II(2.10.1) in $\mathbf{P}(T)$.

(e) Use (d) to show that if $\alpha \in \mathrm{Aut}_S\, T(R^n)$ and $[(R^n,\alpha,R^n)] = 0$ in $K_0(T)$ then (after increasing $n$) there is an isomorphism $(p,q) : (R^n,\alpha,R^n) \cong (R^n,\gamma,R^n)$ in $\mathbf{P}(T)$. Conclude that $[\alpha]$ is the image of $[q^{-1}p] \in K_1(R)$, proving exactness of the sequence at $K_1(S)$.

**1.15** *Suspension rings.* Let $R$ be any ring. Recall from Ex. I.1.8 that the cone ring $C(R)$ is the ring of row-and-column-finite matrices over $R$. The finite matrices in $C(R)$ form a 2-sided ideal $M(R)$, and the quotient $S(R) = C(R)/M(R)$ is called the *suspension ring* of $R$. Use exercise 1.14 and 1.7.3, together with II.2.1.3 and II.2.7.2 to show that $K_1S(R) \cong K_0(R)$.

**1.16** Let $D$ be a division algebra of dimension $d = n^2$ over its center $F$. Show that the norm (or transfer) map $K_1(D) \to K_1(F)$ is $n$ times the reduced norm $N_{\mathrm{red}}$ of 1.2.4. *Hint:* Choose a maximal subfield $E$ and show that the map $K_1(D) \to K_1(E)$ induced by the norm is induced by the additive map $M \mapsto M \otimes_D (D \otimes_F E)$ from $\mathbf{P}(D)$ to $\mathbf{P}(D)$. Then show that $D \otimes_F E \cong D^n$ as a $D$-$E$ bimodule.

**1.17** Let $D$ be a division algebra, finite dimensional over its center $F$, and let $E$ be any finite extension of $F$ which is a splitting field of $D$, i.e., $E \otimes_F D \cong M_n(E)$.

(a) Show that the following three maps $\theta_E : K_1(E) \to K_1(D)$ agree.
  - (i) $K_1(E) \cong K_1(M_n(E)) = K_1(E \otimes_F D) \xrightarrow{\text{transfer}} K_1(D)$;
  - (ii) $K_1(E) \to K_1(M_r(D)) \cong K_1(D)$, where $E \subset M_r(D)$;
  - (iii) $K_1(T)$, where $T : \mathbf{P}(E) \to \mathbf{P}(D)$ is $T(M) = M \otimes_E V$ for a simple $E \otimes_F D$-module $V$.

(b) If $j : E \to L$ is a finite field map over $F$, show that $\theta_E = \theta_L j_*$.

(c) If $\sigma \in \mathrm{Aut}(E/F)$, then $\theta_E = \theta_E \sigma$.

**1.18** If $A$ is any finite-dimensional semisimple algebra over a field with center $C$, construct a reduced norm $A^\times \to C^\times$ and define $SL_n(A)$ to be the kernel of the reduced norm $GL_n(A) \to C^\times$. Show that the kernel $SK_1(A)$ of the induced map $K_1(A) \to C^\times$ is isomorphic to $SL_n(A)/E_n(A)$ for all $n \geq 3$.

## §2. Relative $K_1$

Let $I$ be an ideal in a ring $R$. We write $GL(I)$ for the kernel of the natural map $GL(R) \to GL(R/I)$; the notation reflects the fact that $GL(I)$ is independent of $R$ (see Ex. I.1.10). In addition, we define $E(R, I)$ to be the smallest normal subgroup of $E(R)$ containing the elementary matrices $e_{ij}(x)$ with $x \in I$. More generally, for each $n$ we define $E_n(R, I)$ to be the normal subgroup of $E_n(R)$ generated by the matrices $e_{ij}(x)$ with $x \in I$ and $1 \le i \ne j \le n$. Clearly $E(R, I)$ is the union of the subgroups $E_n(R, I)$.

Relative Whitehead Lemma 2.1. *$E(R, I)$ is a normal subgroup of $GL(I)$, and contains the commutator subgroup of $GL(I)$.*

Proof. For any matrix $g = 1 + \alpha \in GL_n(I)$, the identity

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g^{-1}\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g\alpha & 1 \end{pmatrix}.$$

shows that the matrix $\left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right)$ is in $E_{2n}(R, I)$. (The product of the first 3 matrices is in $E_{2n}(R, I)$.) Hence if $h \in E_n(R, I)$ then the conjugate

$$\begin{pmatrix} ghg^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}$$

is in $E(R, I)$. Finally, if $g, h \in GL_n(I)$ then $[g, h]$ is in $E_{2n}(R, I)$ by equation (1.3.4).

Definition 2.2. The relative group $K_1(R, I)$ is defined to be the quotient $GL(I)/E(R, I)$. By the Relative Whitehead Lemma, it is an abelian group.

The inclusion of $GL(I)$ in $GL(R)$ induces a map $K_1(R, I) \to K_1(R)$. More generally, if $R \to S$ is a ring map sending $I$ into an ideal $I'$ of $S$, the natural maps $GL(I) \to GL(I')$ and $E(R) \to E(S)$ induce a map $K_1(R, I) \to K_1(S, I')$.

Remark 2.2.1. Suppose that $R \to S$ is a ring map sending an ideal $I$ of $R$ isomorphically onto an ideal of $S$. The induced map $K_1(R, I) \to K_1(S, I)$ must be a surjection, as both groups are quotients of $GL(I)$. However, Swan discovered that they need not be isomorphic; a simple example is given in Ex. 2.3 below.

Vaserstein proved in [V76, 14.2] that $K_1(R, I)$ is independent of $R$ if and only if $I = I^2$. One direction is easy (Ex. 2.10): if $I = I^2$ then the commutator subgroup of $GL(I)$ is perfect, and equal to $E(R, I)$. Thus $K_1(R, I) = GL(I)/[GL(I), GL(I)]$, a group which is independent of $R$. (Cf. Ex. 2.6 when $R$ is commutative.)

Proposition 2.3. *There is an exact sequence*

$$K_1(R, I) \to K_1(R) \to K_1(R/I) \xrightarrow{\partial} K_0(I) \to K_0(R) \to K_0(R/I).$$

Proof. By Ex. II.2.3 there is an exact sequence

$$1 \to GL(I) \to GL(R) \to GL(R/I) \xrightarrow{\partial} K_0(I) \to K_0(R) \to K_0(R/I).$$

Since the $K_1$ groups are quotients of the $GL$ groups, and $E(R)$ maps onto $E(R/I)$, this gives exactness except at $K_1(R)$. Suppose $g \in GL(R)$ maps to zero under $GL(R) \to K_1(R) \to K_1(R/I)$. Then the reduction $\bar{g}$ of $g$ mod $I$ is in $E(R/I)$. Since $E(R)$ maps onto $E(R/I)$, there is a matrix $e \in E(R)$ mapping to $\bar{g}$, *i.e.*, $ge^{-1}$ is in the kernel $GL(I)$ of $GL(R) \to GL(R/I)$. Hence the class of $ge^{-1}$ in $K_1(R, I)$ is defined, and maps to the class of $g$ in $K_1(R)$. This proves exactness at the remaining spot.

*The relative group $SK_1(R, I)$*

If $R$ happens to be commutative, the determinant map $K_1(R) \to R^\times$ of Example 1.1.1 induces a relative determinant map $\det\colon K_1(R, I) \to GL_1(I)$, since the determinant of a matrix in $GL(I)$ is congruent to 1 modulo $I$. It is traditional to write $SK_1(R, I)$ for the kernel of det, so the canonical map $GL_1(I) \to K_1(R, I)$ induces a direct sum decomposition $K_1(R, I) = GL_1(I) \oplus SK_1(R, I)$ compatible with the decomposition $K_1(R) = R^\times \oplus SK_1(R)$ of Example 1.1.1. Here are two important cases in which $SK_1(R, I)$ vanishes:

Lemma 2.4. *Let $I$ be a radical ideal in $R$. Then:*

(1) *$K_1(R, I)$ is a quotient of the multiplicative group $1 + I = GL_1(I)$.*
(2) *If $R$ is a commutative ring, then $SK_1(R, I) = 0$ and $K_1(R, I) = 1 + I$.*

Proof. As in the proof of Lemma 1.4, it suffices to show that $GL_n(I) = E_n(R, I)GL_{n-1}(I)$ for $n \geq 2$. If $(x_{ij})$ is a matrix in $GL_n(I)$ then $x_{nn}$ is a unit of $R$, and for $i < n$ the entries $x_{in}, x_{ni}$ are in $I$. Multiplying by the diagonal matrix $\mathrm{diag}(1, \dots, 1, x_{nn}, x_{nn}^{-1})$, we may assume that $x_{nn} = 1$. Now multiplying on the left by the matrices $e_{in}(-x_{in})$ and on the right by $e_{ni}(-x_{ni})$ reduces the matrix to one in $GL_{n-1}(I)$.

The next theorem (and its variant) extends the calculation mentioned in Example 1.3.6 above. We cite them from [BMS, 4.3], mentioning only that their proof involves calculations with Mennicke symbols (see Ex. 1.10 and 2.5) for finitely generated $R$, *i.e.,* Dedekind rings of arithmetic type.

Bass-Milnor-Serre Theorem 2.5. *Let $R$ be an integrally closed subring of a number field $F$, and $I$ an ideal of $R$. Then*

(1) *If $F$ has any embedding into $\mathbb{R}$ then $SK_1(R, I) = 0$.*
(2) *If $F$ is "totally imaginary" (has no embedding into $\mathbb{R}$), then $SK_1(R, I) \cong C_n$ is a finite cyclic group whose order $n$ divides the order $w_1$ of the group of roots of unity in $R$. The exponent $\mathrm{ord}_p\, n$ of $p$ in the integer $n$ is the minimum over all prime ideals $\mathfrak{p}$ of $R$ containing $I$ of the integer*

$$\inf\left\{\mathrm{ord}_p\, w_1, \sup\{0, \left[\frac{\mathrm{ord}_{\mathfrak{p}}(I)}{\mathrm{ord}_{\mathfrak{p}}(p)} - \frac{1}{p-1}\right]\right\}$$

Variant 2.5.1. *Let $R$ be the coordinate ring of a smooth affine curve over a finite field. Then $SK_1(R) = 0$.*

*The Mayer-Vietoris Exact Sequence*

Suppose we are given a ring map $f\colon R \to S$ and an ideal $I$ of $R$ mapped isomorphically into an ideal of $S$. Then we have a Milnor square of rings, as in I.2:

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\bar f} & S/I \end{array}$$

THEOREM 2.6 (MAYER-VIETORIS). *Given a Milnor square as above, there is an exact sequence*

$$K_1(R) \xrightarrow{\Delta} K_1(S) \oplus K_1(R/I) \to K_1(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I).$$

PROOF. By Theorem II.2.9 we have an exact sequence

$$GL(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I).$$

Since $K_0(R)$ is abelian, we may replace $GL(S/I)$ by $K_1(S/I)$ in this sequence. This gives the sequence of the Theorem, and exactness at all the $K_0$ places. Also by II.2.9, the image of $\partial\colon K_1(S/I) \to K_0(R)$ is the double coset space

$$GL(S) \backslash GL(S/I) / GL(R/I).$$

Note that $E(S) \to E(S/I)$ is onto. Therefore the kernel of $\partial$ is the subgroup of $K_1(S/I)$ generated by the images of $GL(S)$ and $GL(R/I)$, and the sequence is exact at $K_1(S/I)$. To prove exactness at the final spot, suppose given $\bar g \in GL_n(R/I)$, $h \in GL_n(S)$ and an elementary matrix $\bar e \in E(S/I)$ such that $\bar f(\bar g)\bar e \equiv h \pmod I$. Lifting $\bar e$ to an $e \in E_n(S)$ (by Remark 1.2.3) yields $\bar f(\bar g) \equiv he^{-1} \pmod I$. Since $R$ is the pullback of $S$ and $R/I$, there is a $g \in GL_n(R)$, equivalent to $\bar g$ modulo $I$, such that $f(g) = he^{-1}$. This establishes exactness at the final spot.

EXERCISES

**2.1** Suppose we are given a Milnor square in which $R$ and $S$ are commutative rings. Using the Units-Pic sequence (I.3.10), conclude that there are exact sequences

$$SK_1(R, I) \to SK_1(R) \to SK_1(R/I) \xrightarrow{\partial} SK_0(I) \to SK_0(R) \to SK_0(R/I),$$

$$SK_1(R) \to SK_1(S) \oplus SK_1(R/I) \xrightarrow{\partial} SK_0(R) \to SK_0(S) \oplus SK_0(R/I) \to SK_0(S/I).$$

**2.2** *Rim Squares.* Let $C_p$ be a cyclic group of prime order $p$ with generator $t$, and let $\zeta = e^{2\pi i/p}$. The ring $\mathbb{Z}[\zeta]$ is the integral closure of $\mathbb{Z}$ in the number field $\mathbb{Q}(\zeta)$. Let $f\colon \mathbb{Z}C_p \to \mathbb{Z}[\zeta]$ be the ring surjection sending $t$ to $\zeta$, and let $I$ denote the kernel of the augmentation $\mathbb{Z}C_p \to \mathbb{Z}$.

(a) Show that $I$ is isomorphic to the ideal of $\mathbb{Z}[\zeta]$ generated by $\zeta - 1$, so that we have a Milnor square with the rings $\mathbb{Z}C_p$, $\mathbb{Z}[\zeta]$, $\mathbb{Z}$ and $\mathbb{F}_p$.

(b) Show that for each $k = 1, ..., p-1$ the element $(\zeta^k - 1)/(\zeta - 1) = 1 + \cdots + \zeta^{k-1}$ is a unit of $\mathbb{Z}[\zeta]$, mapping onto $k \in \mathbb{F}_p^\times$.

These elements are often called *cyclotomic units*, and generate a subgroup of finite index in $\mathbb{Z}[\zeta]^\times$. If $p \geq 3$, the Dirichlet Unit Theorem says that the units of $\mathbb{Z}[\zeta]$ split as the direct sum of the finite group $\{\pm\zeta^k\}$ of order $2p$ ($p \neq 2$) and a free abelian group of rank $(p-3)/2$.

(c) Conclude that if $p > 3$ then both $K_1(\mathbb{Z}C_p)$ and $Wh_1(C_p)$ are nonzero. In fact, $SK_1(\mathbb{Z}C_p) = 0$.

**2.3** *Failure of Excision for $K_1$.* Here is Swan's simple example to show that $K_1(R, I)$ depends upon $R$. Let $F$ be a field and let $R$ be the algebra of all upper triangular matrices $r = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ in $M_2(F)$. Let $I$ be the ideal of all such matrices with $x = z = 0$, and let $R_0$ be the commutative subalgebra $F \oplus I$. Show that $K_1(R_0, I) \cong F$ but that $K_1(R, I) = 0$. *Hint:* Calculate $e_{21}(r)e_{12}(y)e_{21}(-r)$.

**2.4** (Vaserstein) If $I$ is an ideal of $R$, and $x \in I$ and $r \in R$ are such that $(1 + rx)$ is a unit, modify Ex. 1.1 to show that $(1 + rx)(1 + xr)^{-1}$ is in $E_2(R, I)$.

If $I$ is a radical ideal and $W = W(R, I)$ denotes the subgroup of units generated by the $(1 + rx)(1 + xr)^{-1}$, use Lemma 2.4 to conclude that $(1 + I)/W$ maps onto $K_1(R, I)$. Vaserstein proved in [V69] that $K_1(R, I) \cong (1 + I)/W$ for every radical ideal.

**2.5** *Mennicke symbols.* If $I$ is an ideal of a commutative ring $R$, $r \in (1 + I)$ and $s \in I$, we define the *Mennicke symbol* $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]$ to be the class in $SK_1(R, I)$ of the matrix $\left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$, where $t \in I$ and $u \in (1 + I)$ satisfy $ru - st = 1$. Show that this Mennicke symbol is independent of the choice of $t$ and $u$, with $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} s \\ r' \end{smallmatrix}\right] = \left[\begin{smallmatrix} s \\ rr' \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} s \\ r \end{smallmatrix}\right]\left[\begin{smallmatrix} s' \\ r \end{smallmatrix}\right] = \left[\begin{smallmatrix} ss' \\ r \end{smallmatrix}\right]$. (*Hint:* Use Ex. 1.10.) Finally, show that if $t \in I$ then

$$\begin{bmatrix} s \\ r \end{bmatrix} = \begin{bmatrix} s + rt \\ r \end{bmatrix} = \begin{bmatrix} s \\ r + st \end{bmatrix}.$$

**2.6** *The obstruction to excision.* Let $R \to S$ be a map of commutative rings, sending an ideal $I$ of $R$ isomorphically onto an ideal of $S$. Given $x \in I$ and $s \in S$, let $\psi(x, s)$ denote the Mennicke symbol $\left[\begin{smallmatrix} x \\ 1 - sx \end{smallmatrix}\right]$ in $SK_1(R, I)$.

(a) Verify that $\psi(x, s)$ vanishes in $SK_1(S, I)$.

(b) Prove that $\psi$ is bilinear, and that $\psi(x, s) = 1$ if either $x \in I^2$ or $s \in R$. Thus $\psi$ induces a map from $(I/I^2) \otimes (S/R)$ to $SK_1(R, I)$.

(c) Prove that the Leibniz rule holds: $\psi(x, ss') = \psi(sx, s')\psi(s'x, s)$.

For every map $R \to S$, the $S$-module $\Omega_{S/R}$ of *relative Kähler differentials* is presented with generators $ds$, $s \in S$, subject to the following relations: $d(s + s') = ds + ds'$, $d(ss') = s\, ds' + s'\, ds$, and if $r \in R$ then $dr = 0$. (See [WHomo].)

(d) (Vorst) Show that $\Omega_{S/R} \otimes_S I/I^2$ is the quotient of $(S/R) \otimes (I/I^2)$ by the subgroup generated by the elements $s \otimes s'x + s' \otimes sx - ss' \otimes x$. Then conclude that $\psi$ induces a map $\Omega_{S/R} \otimes_S I/I^2 \to SK_1(R, I)$.

Swan proved in [Swan71] that the resulting sequence is exact:

$$\Omega_{S/R} \otimes_S I/I^2 \xrightarrow{\psi} SK_1(R, I) \to SK_1(S, I) \to 1.$$

**2.7** Suppose that the ring map $R \to R/I$ is split by a map $R/I \to R$. Show that $K_1(R) \cong K_1(R/I) \oplus K_1(R,I)$. The corresponding decomposition of $K_0(R)$ follows from the ideal sequence 2.3, or from the definition of $K_0(I)$, since $R \cong R/I \oplus I$; see Ex. II.2.4.

**2.8** Suppose that $p^r = 0$ in $R$ for some prime $p$. Show that $K_1(R, pR)$ is a $p$-group. Conclude that the kernel of the surjection $K_1(R) \to K_1(R/pR)$ is also a $p$-group.

**2.9** If $I$ is a nilpotent ideal in a $\mathbb{Q}$-algebra $R$, or even a complete radical ideal, show that $K_1(R,I) \cong I/[R,I]$, where $[R,I]$ is the subgroup spanned by all elements $[r,x] = rx - xr$, $r \in R$ and $x \in I$. In particular, this proves that $K_1(R,I)$ is uniquely divisible. *Hint:* If $[R,I] = 0$, $\ln : 1 + I \to I$ is a bijection. If not, use Ex. 2.4 and the Campbell-Hausdorff formula.

**2.10** Suppose that $I$ is an ideal satisfying $I = I^2$. Show that $[GL(I), GL(I)]$ is a perfect group. Conclude that $E(R,I) = [GL(I), GL(I)]$ and hence that $K_1(R,I)$ is independent of $R$. *Hint:* Use the commutator formulas (1.3.1).

## §3. The Fundamental Theorems for $K_1$ and $K_0$

The Fundamental Theorem for $K_1$ is a calculation of $K_1(R[t,t^{-1}])$, and describes one of the many relationships between $K_1$ and $K_0$. The core of this calculation depends upon the construction of an exact sequence (see 3.2 below and II.7.8.1):

$$K_1(R[t]) \to K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0\mathbf{H}_{\{t^n\}}(R[t]) \to 0$$

We will construct a localization sequence connecting $K_1$ and $K_0$ in somewhat greater generality first. Recall from chapter II, Theorem 9.8 that for any multiplicatively closed set $S$ of central elements in a ring $R$ there is an exact sequence $K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R)$, where $K_0(R \text{ on } S)$ denotes $K_0$ of the Waldhausen category $\mathbf{Ch}^b_S\mathbf{P}(R)$. If $S$ consists of nonzerodivisors, $K_0(R \text{ on } S)$ also equals $K_0\mathbf{H}_S(R)$ by Ex. II.9.13; see Corollary II.7.7.4.

Our first goal is to extend this sequence to the left using $K_1$, and we begin by constructing the boundary map $\partial$.

Let $\alpha$ be an endomorphism of $R^n$. We say that $\alpha$ is an *$S$-isomorphism* if $S^{-1}\ker(\alpha) = S^{-1}\text{coker}(\alpha) = 0$, or equivalently, $\alpha/1 \in GL_n(S^{-1}R)$. Write $\text{cone}(\alpha)$ for the *mapping cone* of $\alpha$, which is the chain complex $R^n \xrightarrow{-\alpha} R^n$ concentrated in degrees 0 and 1; see [WHomo, 1.5.1]. It is clear that $\alpha$ is an $S$-isomorphism if and only if $\text{cone}(\alpha) \in \mathbf{Ch}^b_S\mathbf{P}(R)$.

LEMMA 3.1. *Let $S$ be a multiplicatively closed set of central elements in a ring $R$. Then there is a group homomorphism*

$$K_1(S^{-1}R) \xrightarrow{\partial} K_0(R \text{ on } S)$$

*sending each $S$-isomorphism $\alpha$ to the class $[\text{cone}(\alpha)]$ of the mapping cone of $\alpha$. In particular, each $s \in S$ is an endomorphism of $R$ so $\partial(s)$ is the class of the chain complex $\text{cone}(s): R \xrightarrow{-s} R$.*

Before proving this lemma, we give one important special case. When $S$ consists of nonzerodivisors, every $S$-isomorphism $\alpha$ must be an injection, and $\text{coker}(\alpha)$ is a module of projective dimension one, *i.e.*, an object of $\mathbf{H}_S(R)$. Moreover, under the isomorphism $K_0\mathbf{Ch}^b_S\mathbf{P}(R) \cong K_0\mathbf{H}_S(R)$ of Ex. II.9.13, the class of $\text{cone}(\alpha)$ in $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$ corresponds to the element $[\text{coker}(\alpha)] \in K_0\mathbf{H}_S(R)$. Thus we immediately have:

COROLLARY 3.1.1. *If $S$ consists of nonzerodivisors then there is a homomorphism $K_1(S^{-1}R) \xrightarrow{\partial} K_0\mathbf{H}_S(R)$ sending each $S$-isomorphism $\alpha$ to $[\text{coker}(\alpha)]$, and sending $s \in S$ to $[R/sR]$.*

PROOF OF 3.1. If $\beta \in \text{End}(R^m)$ is also an $S$-isomorphism, then the diagram

$$\begin{array}{ccccc}
0 & \rightarrowtail & R^n & \overset{=}{\twoheadrightarrow} & R^n \\
\downarrow & & \downarrow (1,\beta) & & \downarrow \beta \\
R^n & \overset{(10)}{\rightarrowtail} & R^n \oplus R^n & \overset{(01)}{\twoheadrightarrow} & R^n \\
\alpha\beta \downarrow & & \downarrow \binom{\alpha\beta}{-\alpha} & & \downarrow \\
R^n & \overset{=}{\rightarrowtail} & R^n & \twoheadrightarrow & 0
\end{array}$$

is a short exact sequence in $\mathbf{Ch}^b_S\mathbf{P}(R)$, where we regard the columns as chain complexes. Since the middle column of the diagram is quasi-isomorphic to its subcomplex $0 \to 0 \oplus R^n \xrightarrow{-\alpha} R^n$, we get the relation

$$\big[\mathrm{cone}(\alpha)\big] - \big[\mathrm{cone}(\alpha\beta)\big] = \big[\mathrm{cone}(\beta)[-1]\big] = -\big[\mathrm{cone}(\beta)\big],$$

or

$$\big[\mathrm{cone}(\alpha)\big] + \big[\mathrm{cone}(\beta)\big] = \big[\mathrm{cone}(\alpha\beta)\big] \tag{3.1.2}$$

in $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. In particular, if $\beta$ is the diagonal matrix $\mathrm{diag}(t, ..., t)$ then $\mathrm{cone}(\beta)$ is the direct sum of $n$ copies of $\mathrm{cone}(t)$, so we have

$$\big[\mathrm{cone}(\alpha t)\big] = \big[\mathrm{cone}(\alpha)\big] + n\big[\mathrm{cone}(t)\big]. \tag{3.1.3}$$

Every $g \in GL_n(S^{-1}R)$ can be represented as $\alpha/s$ for some $S$-isomorphism $\alpha$ and some $s \in S$, and we define $\partial(g) = \partial(\alpha/s)$ to be the element $\big[\mathrm{cone}(\alpha)\big] - n\big[\mathrm{cone}(s)\big]$ of $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. By (3.1.3) we have $\partial(\alpha/s) = \partial(\alpha t/st)$, which implies that $\partial(g)$ is independent of the choice of $\alpha$ and $s$. By (3.1.2) this implies that $\partial$ is a well-defined homomorphism from each $GL_n(S^{-1}R)$ to $K_0\mathbf{Ch}^b_S\mathbf{P}(R)$. Finally, the maps $\partial$ are compatible with the inclusions $GL_n \subset GL_{n+1}$, because

$$\begin{aligned}\partial\begin{pmatrix}\alpha/s & 0\\ 0 & 1\end{pmatrix} &= \partial\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}/s\right) = \left[\mathrm{cone}\begin{pmatrix}\alpha & 0\\ 0 & s\end{pmatrix}\right] - (n+1)\big[\mathrm{cone}(s)\big]\\ &= \big[\mathrm{cone}(\alpha)\big] + \big[\mathrm{cone}(s)\big] - (n+1)\big[\mathrm{cone}(s)\big] = \partial(\alpha/s).\end{aligned}$$

Hence $\partial$ extends to $GL(S^{-1}R)$, and hence must factor through the universal map to $K_1(S^{-1}R)$.

Key Example 3.1.4. For the Fundamental Theorem, we shall need the following special case of this construction. Let $T$ be the multiplicative set $\{t^n\}$ in the polynomial ring $R[t]$. Then the map $\partial$ goes from $K_1(R[t, t^{-1}])$ to $K_0\mathbf{H}_T(R[t])$. If $\nu$ is a nilpotent endomorphism of $R^n$ then $t - \nu$ is a $T$-isomorphism, because its inverse is the polynomial $t^{-1}(1 + \nu t^{-1} + \nu^2 t^{-2} + \dots)$. If $(R^n, \nu)$ denotes the $R[t]$-module $R^n$ on which $t$ acts as $\nu$,

$$\partial(t - \nu) = \big[R[t]^n/(t - \nu)\big] = \big[(R^n, \nu)\big],$$

$$\partial(1 - \nu t^{-1}) = \partial(t - \nu) - \partial(t \cdot \mathrm{id}_n) = \big[(R^n, \nu)\big] - n\big[(R, 0)\big].$$

We can also compose $\partial$ with the product $K_0(R) \otimes K_1(\mathbb{Z}[t, t^{-1}]) \xrightarrow{\cdot} K_1(R[t, t^{-1}])$ of Corollary 1.6.1. Given a finitely generated projective $R$-module $P$, the product $[P] \cdot t$ is the image of $t \cdot \mathrm{id}_{P[t,t^{-1}]}$ under the map $\mathrm{Aut}(P[t, t^{-1}]) \to K_1(R[t, t^{-1}])$ of Lemma 1.6. To compute $\partial([P] \cdot t)$, choose $Q$ such that $P \oplus Q \cong R^n$. Since the cokernel of $t \cdot \mathrm{id}_{P[t]} \colon P[t] \to P[t]$ is the $R[t]$-module $(P, 0)$, we have an exact sequence of $R[t]$-modules:

$$0 \to R[t]^n \xrightarrow{t\cdot \mathrm{id}_{P[t]} \oplus 1\cdot \mathrm{id}_{Q[t]}} R[t]^n \to (P, 0) \to 0.$$

Therefore we have the formula $\partial([P] \cdot t) = \big[(P, 0)\big]$.

LEMMA 3.1.5. *$K_0\mathbf{Ch}^b_S\mathbf{P}(R)$ is generated by the classes $[Q.]$ of chain complexes concentrated in degrees 0 and 1, i.e., by complexes $Q.$ of the form $Q_1 \to Q_0$.*

*The kernel of $K_0\mathbf{Ch}^b_S\mathbf{P}(R) \to K_0(R)$ is generated by the complexes $R^n \xrightarrow{\alpha} R^n$ associated to $S$-isomorphisms, i.e., by the classes $\partial(\alpha) = \big[cone(\alpha)\big]$.*

PROOF. By the Shifting Lemma II.9.2.1, $K_0$ is generated by bounded complexes of the form $0 \to P_n \to \cdots \to P_0 \to 0$. If $n \geq 2$, choose a free $R$-module $F = R^N$ mapping onto $H_0(P.)$. By assumption, we have $sH_0(P.) = 0$ for some $s \in S$. By the projective lifting property, there are maps $f_0$, $f_1$ making the diagram

$$\begin{array}{ccccccc} F & \xrightarrow{s} & F & \to & F/sF & \to & 0 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & \downarrow & & \\ P_1 & \to & P_0 & \to & H_0(P) & \to & 0 \end{array}$$

commute. Thus if $Q.$ denotes the complex $F \xrightarrow{s} F$ we have a chain map $Q. \xrightarrow{f} P.$ inducing a surjection on $H_0$. The mapping cone of $f$ fits into a cofibration sequence $P. \rightarrowtail \text{cone}(f) \twoheadrightarrow Q.[-1]$ in $\mathbf{Ch}^b_S\mathbf{P}(R)$, so we have $[P.] = [Q.] + [\text{cone}(f)]$ in $K_0(R \text{ on } S)$. Moreover, $H_0(\text{cone}(f)) = 0$, so there is a decomposition $P_1 \oplus F \cong P_0 \oplus P_1'$ so that the mapping cone is the direct sum of an exact complex $P_0 \xrightarrow{\cong} P_0$ and a complex $P'.$ of the form $0 \to P_n \to \cdots \to P_3 \to P_2 \oplus F \to P_1' \to 0$. Since $P'.$ has length $n-1$, induction on $n$ implies that $[\text{cone}(f)] = [P'.]$ is a sum of terms of the form $[Q_1 \to Q_0]$.

Hence every element of $K_0$ has the form $x = [P_1 \xrightarrow{\alpha} P_0] - [Q_1 \xrightarrow{\beta} Q_0]$. Choose $s \in S$ so that $s\beta^{-1}$ is represented by an $S$-isomorphism $Q_0 \xrightarrow{\gamma} Q_1$; adding $\gamma$ to both terms of $x$, as well as the appropriate zero term $Q' \xrightarrow{=} Q'$, we may assume that $Q_1 = Q_0 = R^n$, *i.e.*, that the second term of $x$ is the mapping cone of some $S$-isomorphism $\beta \in \text{End}(R^n)$. With this reduction, the map to $K_0(R)$ sends $x$ to $[P_1] - [P_0]$. If this vanishes, then $P_1$ and $P_0$ are stably isomorphic. Adding the appropriate $P' \xrightarrow{=} P'$ makes $P_1 = P_0 = R^m$ for some $m$, and writes $x$ in the form

$$x = \text{cone}(\alpha) - \text{cone}(\beta) = \partial(\alpha) - \partial(\beta).$$

THEOREM 3.2. *Let $S$ be a multiplicatively closed set of central elements in a ring $R$. Then the map $\partial$ of Lemma 3.1 fits into an exact sequence*

$$K_1(R) \to K_1(S^{-1}R) \xrightarrow{\partial} K_0(R \text{ on } S) \to K_0(R) \to K_0(S^{-1}R).$$

PROOF. We have proven exactness at $K_0(R)$ in Theorem II.9.8, and the composition of any two consecutive maps is zero by inspection. Exactness at $K_0(R \text{ on } S)$ was proven in Lemma 3.1.5. Hence it suffices to establish exactness at $K_1(S^{-1}R)$.

For reasons of exposition, we shall give the proof when $S$ consists of nonzerodivisors, relegating the general proof (which is similar but more technical) to Exercise 3.5. The point of this simplification is that we can work with the exact category $\mathbf{H}_S(R)$. In particular, for every $S$-isomorphism $\alpha$ the class of the module $\text{coker}(\alpha)$ is simpler to manipulate than the class of the mapping cone.

Recall from the proof of Lemma 3.1 that every element of $GL_n(S^{-1}R)$ can be represented as $\alpha/s$ for some $S$-isomorphism $\alpha \in \mathrm{End}(R^n)$ and some $s \in S$, and that $\partial(\alpha/s)$ is defined to be $\big[\mathrm{coker}(\alpha)\big] - [R^n/sR^n]$. If $\partial(\alpha/s) = 0$, then from Ex. II.7.2 there are short exact sequences in $\mathbf{H}_S(R)$

$$0 \to C' \to C_1 \to C'' \to 0, \quad 0 \to C' \to C_2 \to C'' \to 0$$

such that $\mathrm{coker}(\alpha) \oplus C_1 \cong (R^n/sR^n) \oplus C_2$. By Ex. 3.4 we may add terms to $C'$, $C''$ to assume that $C' = \mathrm{coker}(\alpha')$ and $C'' = \mathrm{coker}(\alpha'')$ for appropriate $S$-isomorphisms of some $R^m$. By the Horseshoe Lemma ([WHomo, 2.2.8]) we can construct two exact sequences of projective resolutions

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & R^m & \to & R^{2m} & \to & R^m & \to & 0 \\
 & & {\scriptstyle \alpha'}\downarrow & & {\scriptstyle \alpha_i}\downarrow & & {\scriptstyle \alpha''}\downarrow & & \\
0 & \to & R^m & \to & R^{2m} & \to & R^m & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C' & \to & C_i & \to & C'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Inverting $S$ makes each $\alpha_i$ an isomorphism conjugate to $\left(\begin{smallmatrix} \alpha' & 0 \\ 0 & \alpha'' \end{smallmatrix}\right)$. Thus in $K_1(S^{-1}R)$ we have $[\alpha_1] = [\alpha'] + [\alpha''] = [\alpha_2]$. On the other hand, the two endomorphisms $\alpha \oplus \alpha_1$ and $s \cdot \mathrm{id}_n \oplus \alpha_2$ of $R^{2m+n}$ have isomorphic cokernels by construction. Lemma 3.2.1 below implies that in $K_1(S^{-1}R)$ we have

$$[\alpha/s] = [\alpha \oplus \alpha_1] - [s \cdot \mathrm{id}_n \oplus \alpha_2] = g \qquad \text{for some } g \in GL(R).$$

This completes the proof of Theorem 3.2.

LEMMA 3.2.1. *Suppose that $S$ consists of nonzerodivisors. If $\alpha, \beta \in \mathrm{End}_R(R^n)$ are $S$-isomorphisms with $R^n/\alpha R^n$ isomorphic to $R^n/\beta R^n$, then there is a $g \in GL_{4n}(R)$ such that $[\alpha] = [g][\beta]$ in $K_1(S^{-1}R)$.*

PROOF. Put $M = \mathrm{coker}(\alpha) \oplus \mathrm{coker}(\beta)$, and let $\gamma\colon R^n/\alpha R^n \cong R^n/\beta R^n$ be an automorphism. By Ex. 3.3(b) with $Q = R^{2n}$ we can lift the automorphism $\left(\begin{smallmatrix} 0 & \gamma^{-1} \\ \gamma & 0 \end{smallmatrix}\right)$ of $M$ to an automorphism $\gamma_0$ of $R^{4n}$. If $\pi_1$ and $\pi_2$ denote the projections $R^{4n} \xrightarrow{(pr,0,0,0)} \mathrm{coker}(\alpha)$, and $R^{4n} \xrightarrow{(0,pr,0,0)} \mathrm{coker}(\beta)$, respectively, then we have $\gamma\pi_1 = \pi_2\gamma_0$. This yields a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & R^{4n} & \xrightarrow{(\alpha,1,1,1)} & R^{4n} & \xrightarrow{\pi_1} & R^n/\alpha R^n & \to & 0 \\
 & & \downarrow{\scriptstyle \gamma_1} & & \cong\downarrow{\scriptstyle \gamma_0} & & \cong\downarrow{\scriptstyle \gamma} & & \\
0 & \to & R^{4n} & \xrightarrow{(1,\beta,1,1)} & R^{4n} & \xrightarrow{\pi_2} & R^n/\beta R^n & \to & 0
\end{array}$$

in which $\gamma_1$ is the induced map. Since $\gamma$ and $\gamma_0$ are isomorphisms, so is $\gamma_1$. Because $\gamma_0(\alpha,1,1,1) = (1,\beta,1,1)\gamma_1$ in $GL_{4n}(S^{-1}R)$, we have $[\gamma_0]+[\alpha] = [\beta]+[\gamma_1]$, or $[\alpha] = [\gamma_1\gamma_0^{-1}][\beta]$ in $K_1(S^{-1}R)$.

*$NK_1$ and the group* $\mathrm{Nil}_0$

Definition 3.3 ($NF$). If $F$ is any functor from rings to abelian groups, we write $NF(R)$ for the cokernel of the natural map $F(R)\to F(R[t])$; $NF$ is also a functor on rings. Moreover, the ring map $R[t]\xrightarrow{t=1} R$ provides a splitting $F(R[t])\to F(R)$ of the natural map, so we have a decomposition $F(R[t])\cong F(R)\oplus NF(R)$.

In particular, when $F$ is $K_i$ ($i=0,1$) we have functors $NK_i$ and a decomposition $K_i(R[t])\cong K_i(R)\oplus NK_i(R)$. Since the ring maps $R[t]\xrightarrow{t=r} R$ are split surjections for every $r\in R$, we see by Proposition 2.3 and Ex. 2.7 that for every $r$ we also have

$$NK_0(R)\cong K_0(R[t],(t-r)) \quad\text{and}\quad NK_1(R)\cong K_1(R[t],(t-r)).$$

We will sometimes speak about $NF$ for functors $F$ defined on any category of rings closed under polynomial extensions and containing the map "$t=1$," such as $k$-algebras or commutative rings. For example, the functors $NU$ and $N\,\mathrm{Pic}$ were discussed briefly for commutative rings in chapter I, Ex. 3.17 and 3.19.

Example 3.3.1. (Chase) Suppose that $A$ is an algebra over $\mathbb{Z}/p$. Then the group $NK_1(A)$ is a $p$-group. To see this, first observe that it is true for the algebras $A_n = \mathbb{Z}/p[x]/(x^n)$ by 2.4, since $(1+tf(x,t))^p = 1+t^pf(x^p,t^p)$. Then observe that by Higman's trick (3.5.1 below) every element of $NK_1(A)$ is the image of $1-xt\in NK_1(A_n)$ (for some $n$) under a map $A_n\to M_n(A)$, $x\mapsto\nu$.

By Ex. 2.8, $NK_1(A)$ is also a $p$-group for every $\mathbb{Z}/p^r$-algebra $A$.

Example 3.3.2. If $A$ is an algebra over a field $k$ of characeteristic zero, then $NK_1(A)$ is a uniquely divisible abelian group. In fact, $NK_1(A)$ has the structure of a $k$-vector space; see Ex. 3.7.

Definition 3.4 ($F$-regular rings). We say that a ring $R$ is *$F$-regular* if $F(R)=F(R[t_1,\dots,t_n])$ for all $n$. Since $NF(R[t]) = NF(R)\oplus N^2F(R)$, we see by induction on $p$ that $R$ is $F$-regular if and only if $N^pF(R)=0$ for all $p\ge 1$.

For example, Traverso's theorem (I.3.11) states that a commutative ring $R$ is Pic-regular if and only if $R_{\mathrm{red}}$ is seminormal. We also saw in I.3.12 that commutative rings are $U$-regular ($U$=units) if and only if $R$ is reduced.

We saw in II.6.5 that any regular ring is $K_0$-regular. We will see in theorem 3.8 below that regular rings are also $K_1$-regular, and we sill see in chapter Vthat they are $K_m$-regular for every $m$. Rosenberg has also shown that commutative $C^*$-algebras are $K_m$-regular for all $m$; see [Ro96].

Lemma 3.4.1. *Let $R = R_0\oplus R_1\oplus\cdots$ be a graded ring. Then the kernel of $F(R)\to F(R_0)$ embeds in $NF(R)$ and even in the kernel of $NF(R)\to NF(R_0)$.*

*In particular, if $NF(R)=0$ then $F(R)\cong F(R_0)$.*

Proof. Let $f$ denote the ring map $R\to R[t]$ defined by $f(r_n)=r_nt^n$ for every $r_n\in R_n$. Since the composition of $f$ and "$t=1$" is the identity on $R$, $F(f)$ is an injection. Let $Q$ denote the kernel of $F(R) \to F(R_0)$, so that $F(R) = F(R_0) \oplus Q$. Since the composition of $f$ and "$t = 0$" is the projection $R \to R_0 \to R$, $Q$ embeds into the kernel $NF(R)$ of the evaluation $F(R[t]) \to F(R)$ at $t = 0$. Similarly, since the composition of $f$ and $R[t] \to R_0[t]$ is projection $R \to R_0 \to R_0[t]$, Q embeds into the kernel of $NF(R) \to NF(R_0)$.

A typical application of this result is that if $R$ is a graded seminormal algebra with $R_0$ a field then $\mathrm{Pic}(R) = 0$.

APPLICATION 3.4.2. It follows that $NF(R)$ is a summand of $N^2F(R)$ and hence $N^pF(R)$ for all $p > 0$. Indeed, the first part is the application of 3.4.1 to $R[s]$, and the second part is obtained by induction, replacing $F$ by $N^{p-2}F$.

If $s \in S$ is central, the algebra map $S[x] \to S[x]$, $x \to sx$, induces an operation $[s] : NK_0(S) \to NK_0(S)$. (It is the multiplication by $1 - st \in W(S)$ in Ex. 3.7.) Write $S_s$ for $S[1/s]$.

THEOREM 3.4.3 (VORST). *$NK_0(S_s)$ is the "localization" of $NK_0(S)$ along $[s]$, i.e., $NK_0(S_s) \cong \varinjlim(NK_0(S) \xrightarrow{[s]} NK_0(S) \xrightarrow{[s]} \cdots)$. In particular, if $S$ is $K_0$-regular, so is $S_s$.*

PROOF. Write $I$ for the ideal $(x)$ of $S_s[x]$ and set $R = S + I$. Then $NK_0(S_s) = K_0(I)$ by Ex. II.2.3. But $R = \varinjlim(S[x] \to S[x] \to \cdots)$ and $I$ is the direct limit of $xS[x] \to xS[x] \to \cdots$, so $K_0(I) = \varinjlim(K_0(xS[x]) \to \cdots)$ as claimed.

COROLLARY 3.4.4. *If $A$ is $K_0$-regular then so is $A[s, s^{-1}]$.*

**3.5.** We are going to describe the group $NK_1(R)$ in terms of nilpotent matrices. For this, we need the following trick, which was published by Graham Higman in 1940. For clarity, if $I = fA$ is an ideal in $A$ we write $GL(A, f)$ for $GL(I)$.

HIGMAN'S TRICK 3.5.1. *For every $g \in GL(R[t], t)$ there is a nilpotent matrix $\nu$ over $R$ such that $[g] = [1 - \nu t]$ in $K_1(R[t])$.*

*Similarly, for every $g \in GL(R[t, t^{-1}], t - 1)$ there is a nilpotent matrix $\nu$ over $R$ such that $[g] = [1 - \nu(t - 1)]$ in $K_1(R[t, t^{-1}], t - 1)$.*

PROOF. Every invertible $p \times p$ matrix over $R[t]$ can be written as a polynomial $g = \gamma_0 + \gamma_1 t + \gamma_2 t^2 + \cdots + \gamma_n t^n$ with the $\gamma_i$ in $M_p(R)$. If $g$ is congruent to the identity modulo $t$, then $\gamma_0 = 1$. If $n \geq 2$ and we write $g = 1 - ht + \gamma_n t^n$, then modulo $E_{2p}(R[t], t)$ we have

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} g & \gamma_n t^{n-1} \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 - ht & \gamma_n t^{n-1} \\ -t & 1 \end{pmatrix} = 1 - \begin{pmatrix} h & -\gamma_n t^{n-2} \\ 1 & 0 \end{pmatrix} t.$$

By induction on $n$, $[g]$ is represented by a matrix of the form $1 - \nu t$. The matrix $\nu$ is nilpotent by Ex. 3.1.

Over $R[t, t^{-1}]$ we can use a similar argument. After multiplying by a power of $t$, we may write $g$ as a polynomial in $t$. Such a polynomial may be rewritten as a polynomial $\sum \gamma_i x^i$ in $x = (t - 1)$. If $g$ is congruent to the identity modulo $(t - 1)$ then again we have $\gamma_0 = 1$. By Higman's trick (applied to $x$), we may reduce $g$ to a matrix of the form $1 - \nu x$, and again $\nu$ must be nilpotent by Ex. 3.1.

We will also need the category $\mathbf{Nil}(R)$ of II.7.4.4. Recall that the objects of this category are pairs $(P, \nu)$, where $P$ is a finitely generated projective $R$-module and $\nu$ is a nilpotent endomorphism of $P$. Let $T$ denote the multiplicative set $\{t^n\}$ in $R[t]$. From II.7.8.4 we have

$$K_0(R[t] \text{ on } T) \cong K_0\mathbf{Nil}(R) \cong K_0(R) \oplus \mathrm{Nil}_0(R),$$

where $\mathrm{Nil}_0(R)$ is the subgroup generated by elements of the form $\big[(R^n, \nu)\big] - n\big[(R, 0)\big]$ for some $n$ and some nilpotent matrix $\nu$.

Lemma 3.5.2. *For every ring $R$, the product with $t \in K_1(\mathbb{Z}[t, t^{-1}])$ induces a split injection $K_0(R) \xrightarrow{\cdot t} K_1(R[t, t^{-1}])$.*

Proof. Since the forgetful map $K_0\mathbf{Nil}(R) \to K_0(R)$ sends $\big[(P, \nu)\big]$ to $[P]$, the calculation in Example 3.1.4 shows that the composition

$$K_0(R) \xrightarrow{\cdot t} K_1(R[t, t^{-1}]) \xrightarrow{\partial} K_0\mathbf{Nil}(R) \to K_0(R)$$

is the identity map. Hence the first map is a split injection.

Momentarily changing variables from $t$ to $s$, we now define an additive function $\tau$ from $\mathbf{Nil}(R)$ to $K_1(R[s])$. Given an object $(P, \nu)$, let $\tau(P, \nu)$ be the image of the automorphism $1 - \nu s$ of $P[s]$ under the natural map $\mathrm{Aut}(P[s]) \to K_1(R[s])$ of Lemma 1.6. Given a short exact sequence

$$0 \to (P', \nu') \to (P, \nu) \to (P'', \nu'') \to 0$$

in $\mathbf{Nil}(R)$, a choice of a splitting $P \cong P' \oplus P''$ allows us to write

$$(1 - \nu s) = \begin{pmatrix} 1 - \nu' s & \gamma s \\ 0 & 1 - \nu'' s \end{pmatrix} = \begin{pmatrix} 1 - \nu' s & 0 \\ 0 & 1 - \nu'' s \end{pmatrix} \begin{pmatrix} 1 & \gamma' s \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{Aut}(P[s])$. Hence in $K_1(R[s])$ we have $[1 - \nu s] = [1 - \nu' s][1 - \nu'' s]$. Therefore $\tau$ is an additive function, and induces a homomorphism $\tau\colon K_0\mathbf{Nil}(R) \to K_1(R[s])$. Since $\tau(P, 0) = 1$ for all $P$ and $1 - \nu s$ is congruent to 1 modulo $s$, we see that $\tau$ is actually a map from $\mathrm{Nil}_0(R)$ to $K_1(R[s], s)$.

Proposition 3.5.3. $\mathrm{Nil}_0(R) \cong NK_1(R)$, *and* $K_0\mathbf{Nil}(R) \cong K_0(R) \oplus NK_1(R)$.

Proof. For convenience, we identify $s$ with $t^{-1}$, so that $R[s, s^{-1}] = R[t, t^{-1}]$. Applying Lemma 3.1 to $R[t]$ and $T = \{1, t, t^2, \dots\}$, we form the composition

$$\begin{aligned} \delta : K_1(R[s], s) \to K_1(R[s]) &\to K_1(R[s, s^{-1}]) \\ &= K_1(R[t, t^{-1}]) \xrightarrow{\partial} K_0(R \text{ on } T) \to \mathrm{Nil}_0(R). \end{aligned} \tag{3.5.4}$$

Let us call this composition $\delta$. We claim that $\tau$ is the inverse of $\delta$. By Higman's Trick, every element of $K_1(R[s], s)$ is represented by a matrix $1 - \nu s$ with $\nu$ nilpotent. In Example 3.1.4 we saw that $\delta(1 - \nu s) = \big[(R^n, \nu)\big] - n\big[(R, 0)\big]$. By the construction of $\tau$ we have the desired equations: $\tau\delta(1 - \nu s) = \tau\big[(R^n, \nu)\big] = (1 - \nu s)$ and

$$\delta\tau\Big(\big[(R^n, \nu)\big] - n\big[(R, 0)\big]\Big) = \delta(1 - \nu s) = \big[(R^n, \nu)\big] - n\big[(R, 0)\big].$$

Corollary 3.5.5. *$K_1(R[s]) \to K_1(R[s,s^{-1}])$ is an injection for every ring $R$.*

Proof. By Ex. 2.7, we have $K_1(R[s]) \cong K_1(R) \oplus K_1(R[s],s)$. Since $K_1(R)$ is a summand of $K_1(R[s,s^{-1}])$, the isomorphism $\delta\colon K_1(R[s],s) \cong \mathrm{Nil}_0(R)$ of (3.5.4) factors through $K_1(R[s],s) \to K_1(R[s,s^{-1}])/K_1(R)$. This quotient map must then be an injection. The result follows.

*The Fundamental Theorems for $K_1$ and $K_0$*

Fundamental Theorem for $K_1$ 3.6. *For every ring $R$, there is a split surjection $K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(R)$, with inverse $[P] \mapsto [P]\cdot t$. This map fits into a naturally split exact sequence:*

$$0 \to K_1(R) \xrightarrow{\Delta} K_1(R[t]) \oplus K_1(R[t^{-1}]) \xrightarrow{\pm} K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(R) \to 0.$$

*Consequently, we have a natural direct sum decomposition:*

$$K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_0(R) \oplus NK_1(R) \oplus NK_1(R).$$

Proof. We merely assemble the pieces of the proof from §3.5. The first assertion is just Lemma 3.5.2. The natural maps from $K_1(R)$ into $K_1(R[t])$, $K_1(R[t^{-1}])$ and $K_1(R[t,t^{-1}])$ are injections, split by "$t=1$" (as in 3.5), so the obviously exact sequence

$$0 \to K_1(R) \xrightarrow{\Delta} K_1(R) \oplus K_1(R) \xrightarrow{\pm} K_1(R) \to 0 \tag{3.6.1}$$

is a summand of the sequence we want to prove exact. From Proposition II.7.8.1, Theorem 3.2 and Corollary 3.5.5, we have an exact sequence

$$0 \to K_1(R[t]) \to K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0\mathbf{Nil}(R) \to 0. \tag{3.6.2}$$

Since $K_0\mathbf{Nil}(R) \cong K_0(R) \oplus \mathrm{Nil}_0(R)$, the map $\partial$ in (3.6.2) is split by the maps of 3.5.2 and 3.5.3. The sequence in the Fundamental Theorem for $K_1$ is obtained by rearranging the terms in sequences (3.6.1) and (3.6.2).

In order to formulate the corresponding Fundamental Theorem for $K_0$, we define $K_{-1}(R)$ to be the cokernel of the map $K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t,t^{-1}])$. We will reprove the following result more formally in the next section.

Fundamental Theorem for $K_0$ 3.7. *For every ring $R$, there is a naturally split exact sequence:*

$$0 \to K_0(R) \xrightarrow{\Delta} K_0(R[t]) \oplus K_0(R[t^{-1}]) \xrightarrow{\pm} K_0(R[t,t^{-1}] \xrightarrow{\partial} K_{-1}(R) \to 0.$$

*Consequently, we have a natural direct sum decomposition:*

$$K_0(R[t,t^{-1}] \cong K_0(R) \oplus K_{-1}(R) \oplus NK_0(R) \oplus NK_0(R).$$

Proof. Let $s$ be a second indeterminate. The Fundamental Theorem for $K_1$, applied to the variable $t$, gives a natural decomposition

$$K_1(R[s,t,t^{-1}]) \cong K_1(R[s]) \oplus NK_1(R[s]) \oplus NK_1(R[s]) \oplus K_0(R[s]),$$

and similar decompositions for the other terms in the map

$$K_1(R[s,t,t^{-1}]) \oplus K_1(R[s^{-1},t,t^{-1}]) \to K_1(R[s,s^{-1},t,t^{-1}]).$$

Therefore the cokernel of this map also has a natural splitting. But the cokernel is $K_0(R[t,t^{-1}])$, as we see by applying the Fundamental Theorem for $K_1$ to the variable $s$.

THEOREM 3.8. *If $R$ is a regular ring, $K_1(R[t]) \cong K_1(R)$ and there is a natural isomorphism $K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_0(R)$.*

PROOF. Consider the category $\mathbf{M}_t(R[t])$ of finitely generated $t$-torsion $R[t]$-modules; by Devissage II.6.3.3, $K_0\mathbf{M}_t(R[t]) \cong K_0(R)$. Since $R$ is regular, every such module has a finite resolution by finitely generated projective $R[t]$-modules, *i.e.,* $\mathbf{M}_t(R[t])$ is the same as the category $\mathbf{H}_t(R[t])$ of II.7.7. By II.7.8.4,

$$K_0\mathbf{Nil}(R) \cong K_0\mathbf{H}_t(R[t]) \cong K_0(R).$$

Hence $\mathrm{Nil}_0(R) = 0$. By 3.5.3, $NK_1(R) = 0$ and $K_1(R[t]) \cong K_1(R)$. The description of $K_1(R[t,t^{-1}])$ now comes from the Fundamental Theorem 3.6.

EXAMPLE 3.8.1. If $R$ is a commutative regular ring, and $A = R[x]/(x^N)$, it follows from 2.4 and 3.8 that $SK_1(A[t]) = SK_1(A)$ and hence (by 3.5.3 and I.3.12) $\mathrm{Nil}_0(A) \cong NK_1(A) \cong (1+tA[t])^\times = (1+xtA[t])^\times$. This isomorphism sends $[(P,\nu)] \in \mathrm{Nil}_0(A)$ to $\det(1-\nu t) \in (1+tA[t])^\times$. By inspection, this is the restriction of the canonical $\mathrm{End}_0(A)$-module map $\mathrm{Nil}_0(A) \to \mathrm{End}_0(A)$ of II.7.4.4, followed by the inclusion $\mathrm{End}_0(A) \subset W(A)$ of II.7.4.3. It follows that $\mathrm{Nil}_0(A)$ is an ideal of the ring $\mathrm{End}_0(A)$.

EXERCISES

**3.1** Let $A$ be a ring and $a \in A$, show that the following are equivalent: (i) $a$ is nilpotent; (ii) $1-at$ is a unit of $A[t]$; (iii) $1-a(t-1)$ is a unit of $A[t,t^{-1}]$.

**3.2** Let $\alpha,\beta\colon P \to Q$ be two maps between finitely generated projective $R$-modules. If $S$ is a central multiplicatively closed set in $R$ and $S^{-1}\alpha, S^{-1}\beta$ are isomorphisms, then $g = \beta^{-1}\alpha$ is an automorphism of $S^{-1}P$. Show that $\partial(g) = [\mathrm{cone}(\alpha)] - [\mathrm{cone}(\beta)]$. In particular, if $S$ consists of nonzerodivisors then $\partial(g) = [\mathrm{coker}(\alpha)] - [\mathrm{coker}(\beta)]$.

**3.3** (Bass) Prove that every module $M$ in $\mathbf{H}(R)$ has a projective resolution $P. \to M$ such that every automorphism $\alpha$ of $M$ lifts to an *automorphism* of the chain complex $P.$. To do so, proceed as follows.

(a) Fix a surjection $\pi\colon Q \to M$, and use Ex. I.1.11 to lift the automorphism $\alpha \oplus \alpha^{-1}$ of $M \oplus M$ to an automorphism $\beta$ of $Q \oplus Q$.

(b) Defining $e\colon Q \oplus Q \to M$ to be $e(x,y) = \pi(x)$, show that every automorphism of $M$ can be lifted to an automorphism of $Q \oplus Q$.

(c) Set $P_0 = Q \oplus Q$, and repeat the construction on $Z_0 = \ker(e)$ to get a finite resolution $P.$ of $M$ with the desired property.

**3.4** Suppose that $S$ consists of nonzerodivisors, and that $M$ is a module in $\mathbf{H}_S(R)$.

(a) Prove that there is a module $M'$ and an $S$-isomorphism $\alpha \in \mathrm{End}(R^m)$ so that $\mathrm{coker}(\alpha) = M \oplus M'$. *Hint:* Modify the proof of Lemma 3.1.5, where $M$ is the cokernel of a map $P_0 \xrightarrow{\beta} P_1$.

(b) Given $S$-isomorphisms $\alpha', \alpha'' \in \mathrm{End}(R^m)$ and a short exact sequence of $S$-torsion modules $0 \to \mathrm{coker}(\alpha') \to M \to \mathrm{coker}(\alpha'') \to 0$, show that there is an $S$-isomorphism $\alpha \in R^{2m}$ with $M \cong \mathrm{coker}(\alpha)$.

**3.5** Modify the proofs of the previous two exercises to prove Theorem 3.2 when $S$ contains zerodivisors.

**3.6** *Noncommutative localization.* By definition, a multiplicatively closed set $S$ in a ring $R$ is called a *right denominator set* if it satisfies the following two conditions: (i) For any $s \in S$ and $r \in R$ there exists an $s' \in S$ and $r' \in R$ such that $sr' = rs'$; (ii) if $sr = 0$ for any $r \in R$, $s \in S$ then $rs' = 0$ for some $s' \in S$. This is the most general condition under which a (right) ring of fractions $S^{-1}R$ exists, in which every element of $S^{-1}R$ has the form $r/s = rs^{-1}$, and if $r/1 = 0$ then some $rs = 0$ in $R$.

Prove Theorem 3.2 when $S$ is a right denominator set consisting of nonzerdivisors. To do this, proceed as follows.

(a) Show that for any finite set of elements $x_i$ in $S^{-1}R$ there is an $s \in S$ and $r_i \in R$ so that $x_i = r_i/s$ for all $i$.
(b) Reprove II.7.7.3 and II.9.8 for denominator sets, using (a); this yields exactness at $K_0(R)$.
(c) Modify the proof of Lemma 3.1 and 3.1.5 to construct the map $\partial$ and prove exactness at $K_0\mathbf{H}_S(R)$.
(d) Modify the proof of Theorem 3.2 to prove exactness at $K_1(S^{-1}R)$.

**3.7** Let $A$ be an algebra over a commutative ring $R$. Recall from Ex. II.7.18 that $NK_1(A) = \mathrm{Nil}_0(A)$ is a module over the ring $W(R) = 1 + tR[[t]]$ of Witt vectors II.4.3. In this exercise we develop a little of the structure of $W(R)$, which yields information about the structure of $NK_1(A)$ and hence (by theorem 3.7) the structure of $NK_0(R)$.

(a) If $1/p \in R$ for some prime integer $p$, show that $W(R)$ is an algebra over $\mathbb{Z}[1/p]$. Conclude that $NK_1(A)$ and $NK_0(A)$ are uniquely $p$-divisible abelian groups. *Hint:* use the fact that the coefficients in the power series expansion for $r(t) = (1+t)^{1/p}$ only involve powers of $p$.
(b) If $\mathbb{Q} \subseteq R$, consider the exponential map $\prod_{i=1}^{\infty} R \to W(R)$, sending $(r_1, ...)$ to $\prod_{i=1}^{\infty} \exp(-r_i t^i/i)$. This is an isomorphism of abelian groups, whose inverse (the "ghost map") is given by the coefficients of $f \mapsto -t\, d/dt(\ln f)$. Show that this is a ring isomorphism. Conclude that $NK_1(A)$ and $NK_0(A)$ have the structure of $R$-modules.
(c) If $n \in \mathbb{Z}$ is nonzero, Stienstra showed that $NK_1(A)[1/n] \cong NK_1(A[1/n])$. Use this to show that if $G$ is a finite group of order $n$ then $NK_1(\mathbb{Z}[G])$ is annihilated by some power of $n$.

**3.8** If $I$ is a nilpotent ideal in a $\mathbb{Q}$-algebra $A$, show that $NK_1(A, I) \to K_1(A, I)$ is onto. Thus Ex. 3.7 gives another proof that $K_1(A, I)$ is divisible (Ex. 2.9).

**3.9** If $s \in S$ is central, show that $NK_1(S_s)$ is a localization of $NK_1(S)$. Conclude that if $S$ is $K_1$-regular then $S$ is $K_0$-regular. *Hint:* Use the sequence of Ex. 2.6

**3.10** (Karoubi) Let $S$ be a multiplicatively closed set of central nonzerodivisors in a ring $A$. We say that a ring homomorphism $f : A \to B$ is an *analytic isomorphism* along $S$ if $f(S)$ consists of central nonzerodivisors in $B$, and if $A/sA \cong B/sB$ for every $s \in S$. (This implies that the $s$-adic completions of $A$ and $B$ are isomorphism, whence the name.)

If $f$ is an analytic isomorphism along $S$, show that $M \mapsto M \otimes_A B$ defines an equivalence of categories $\mathbf{H}_S(A) \cong \mathbf{H}_S(B)$. (One proof is given in V.7.5 below.) Using Theorem 3.2 and Ex. II.9.13, this shows that we have an exact sequence

$$K_1(S^{-1}A) \oplus K_1(B) \to K_1(S^{-1}B) \to K_0(A) \to K_0(S^{-1}A) \oplus K_0(B) \to K_0(S^{-1}B).$$

*Hint:* For $M$ in $\mathbf{H}_S^1(A)$, show that $\mathrm{Tor}_1^A(M,B)=0$, so that $\mathbf{H}_S^1(A)\to\mathbf{H}_S^1(B)$ is exact. Then use Lemma II.7.7.1 to show that $\mathbf{H}_S(A)$ is the category of modules having a finite resolution by modules in $\mathbf{H}_S^1(A)$, and similarly for $\mathbf{H}_S(B)$.

## §4. Negative $K$-theory

In the last section, we defined $K_{-1}(R)$ to be the cokernel of the map $K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t, t^{-1}])$. Of course we can keep going, and define all the negative $K$-groups by induction on $n$:

DEFINITION 4.1. For $n > 0$, we inductively define $K_{-n}(R)$ to be the cokernel of the map

$$K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \to K_{-n+1}(R[t, t^{-1}]).$$

Clearly, each $K_{-n}$ is a functor from rings to abelian groups. It follows from Theorem II.7.8 that if $R$ is regular noetherian then $K_n(R) = 0$ for all $n < 0$.

To describe the properties of these negative $K$-groups, it is convenient to cast the Fundamental Theorems above in terms of Bass' notion of *contracted functors*. With this in mind, we make the following definitions.

DEFINITION 4.1.1 (CONTRACTED FUNCTORS). Let $F$ be a functor from rings to abelian groups. For each $R$, we define $LF(R)$ to be the cokernel of the map $F(R[t]) \oplus F(R[t^{-1}]) \to F(R[t, t^{-1}])$. We write $Seq(F, R)$ for the following sequence, where $\Delta(a) = (a, a)$ and $\pm(b, c) = b - c$:

$$0 \to F(R) \xrightarrow{\Delta} F(R[t]) \oplus F(R[t^{-1}]) \xrightarrow{\pm} F(R[t, t^{-1}]) \to LF(R) \to 0.$$

We say that $F$ is *acyclic* if $Seq(F, R)$ is exact for all $R$. We say that $F$ is a *contracted functor* if $F$ is acyclic and in addition there is a splitting $h = h_{t,R}$ of the defining map $F(R[t, t^{-1}]) \to LF(R)$, a splitting which is natural in both $t$ and $R$.

By iterating this definition, we can speak about the functors $NLF$, $L^2F$, etc. For example, Definition 4.1 states that $K_{-n} = L^n(K_0)$.

As with the definition of $NF$ (3.3), it will occasionally be useful to define $LF$ etc. on a more restricted class of rings, such as commutative algebras. Suppose that $\mathcal{R}$ is a category of rings such that if $R$ is in $\mathcal{R}$ then so are $R[t]$, $R[t, t^{-1}]$ and the maps $R \to R[t] \rightrightarrows R[t, t^{-1}]$. Then the definitions of $LF$, $L^nF$ and $Seq(F, R)$ in 4.1.1 make sense for any functor $F$ from $\mathcal{R}$ to any abelian category.

EXAMPLE 4.1.2 (FUNDAMENTAL THEOREM FOR $K_{-n}$). The Fundamental Theorems for $K_1$ and $K_0$ may be restated as the assertions that these are contracted functors. It follows from Proposition 4.2 below that each $K_{-n}$ is a contracted functor; by Definition 4.1, this means that there is a naturally split exact sequence:

$$0 \to K_{-n}(R) \xrightarrow{\Delta} K_{-n}(R[t]) \oplus K_{-n}(R[t^{-1}]) \xrightarrow{\pm} K_{-n}(R[t, t^{-1}]) \xrightarrow{\partial} K_{-n-1}(R) \to 0.$$

EXAMPLE 4.1.3 (UNITS). Let $U(R) = R^\times$ denote the group of units in a commutative ring $R$. By Ex. I.3.17, $U$ is a contracted functor with contraction $LU(R) = [\mathrm{Spec}(R), \mathbb{Z}]$; the splitting map $LU(R) \to U(R[t, t^{-1}])$ sends a function $f\colon \mathrm{Spec}(R) \to \mathbb{Z}$ to the unit $t^f$ of $R[t, t^{-1}]$. From Ex. 4.2 below we see that the functors $L^2U$ and $NLU$ are zero. Thus we can write a simple formula for the units of any extension $R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. If $R$ is reduced, so that $NU(R)$ vanishes (Ex. I.3.17), then we just have

$$U(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]) = U(R) \times \coprod_{i=1}^{n} [\mathrm{Spec}(R), \mathbb{Z}] \cdot t_i.$$

EXAMPLE 4.1.4 (Pic). Recall from chapter I, §3 that the Picard group $\mathrm{Pic}(R)$ of a commutative ring is a functor, and that $N\,\mathrm{Pic}(R) = 0$ exactly when $R_{\mathrm{red}}$ is seminormal. By Ex. I.3.18 the sequence $Seq(\mathrm{Pic}, R)$ is exact. In fact Pic is a contracted functor with $NL\,\mathrm{Pic} = L^2\,\mathrm{Pic} = 0$; see [We91]. The group $L\,\mathrm{Pic}(R)$ is the étale cohomology group $H^1_{et}(\mathrm{Spec}(R), \mathbb{Z})$.

A *morphism of contracted functors* is a natural transformation $\eta: F \Rightarrow F'$ between two contracted functors such that the following square commutes for all $R$.

$$\begin{array}{ccc} LF(R) & \xrightarrow{h} & F(R[t,t^{-1}]) \\ {\scriptstyle (L\eta)_R}\downarrow & & \downarrow{\scriptstyle \eta_{R[t,t^{-1}]}} \\ LF'(R) & \xrightarrow{h'} & F'(R[t,t^{-1}]) \end{array}$$

PROPOSITION 4.2. *Let $\eta: F \Rightarrow F'$ be a morphism of contracted functors. Then both* $\ker(\eta)$ *and coker($\eta$) are also contracted functors.*

*In particular, if $F$ is contracted, then $NF$ and $LF$ are also contracted functors. Moreover, there is a natural isomorphism of contracted functors $NLF \cong LNF$.*

PROOF. If $C \xrightarrow{\phi} D$ is a morphism between split exact sequences, which have compatible spittings, then the sequences $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are always split exact, with splittings induced from the splittings of $C$ and $D$. Applying this remark to $Seq(F, R) \to Seq(F', R)$ shows that $\ker(\eta)$ and $\mathrm{coker}(\eta)$ are contracted functors: both $Seq(\ker(\eta), R)$ and $Seq(\mathrm{coker}(\eta), R)$ are split exact. It also shows that

$$0 \to \ker(\eta)(R) \to F(R) \xrightarrow{\eta_R} F'(R) \to \mathrm{coker}(\eta)(R) \to 0$$

is an exact sequence of contracted functors.

Since $NF(R)$ is the cokernel of the morphism $F(R) \to F'(R) = F(R[t])$ and $LF(R)$ is the cokernel of the morphism $\pm$ in $Seq(F, R)$, both $NF$ and $LF$ are contracted functors. Finally, the natural isomorphism $NLF(R) \cong LNF(R)$ arises from inspecting one corner of the large commutative diagram represented by

$$0 \to Seq(F, (R[s], s)) \to Seq(F, R[s]) \to Seq(F, R) \to 0.$$

EXAMPLE 4.2.1 ($SK_1$). If $R$ is a commutative ring, it follows from Examples 1.1.1 and 4.1.3 that $\det: K_1(R) \to U(R)$ is a morphism of contracted functors. Hence $SK_1$ is a contracted functor. The contracted map $L\det$ is the map $\mathrm{rank}: K_0(R) \to H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}]$ of II.2.3; it follows that $L(SK_1)(R) = \widetilde{K}_0(R)$. From Ex. 4.2 we also have $L^2(SK_1)(R) = L\widetilde{K}_0(R) = K_{-1}(R)$.

We can give an elegant formula for $F(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$, using the following notation. If $p(N, L) = \sum m_{ij} N^i L^j$ is any formal polynomial in $N$ and $L$ with integer coefficients $m_{ij} > 0$, and $F$ is a functor from rings to abelian groups, we set $p(N, L)F$ equal to the direct sum of $m_{ij}$ copies of each group $N^i L^j F(R)$.

COROLLARY 4.2.2. *$F(R[t_1, \dots, t_n]) \cong (1 + N)^n F(R)$ for every $F$. If $F$ is a contracted functor, then $F(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]) \cong (1 + 2N + L)^n F(R)$.*

PROOF. The case $n = 1$ follows from the definitions; the general case follows by induction. □

For example, if $L^2F = 0$ and $R$ is $F$-regular, then $(1+2N+L)^nF(R)$ stands for $F(R) \oplus n\,LF(R)$. In particular, the formula for units in Example 4.1.3 is just the case $F = U$ of 4.2.2.

EXAMPLE 4.2.3. Since $L^jK_0 = K_{-j}$, $K_0(R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}])$ is the direct sum of many pieces $N^iK_{-j}(R)$, including $K_{-n}(R)$ and $\binom{n}{j}$ copies of $K_{-j}(R)$.

From 3.4.4 we see that if $R$ is $K_n$-regular for some $n \le 0$ then $R$ is also $K_{n-1}$-regular. In particular, if $R$ is $K_0$-regular then $R$ is also $K_n$-regular for all $n < 0$.

CONJECTURE 4.2.4. Let $R$ be a commutative noetherian ring of Krull dimension $d$. It is conjectured that $K_{-j}(R)$ vanishes for all $j > d$, and that $R$ is $K_{-d}$-regular; see [We80]. This is so for $d = 0, 1$ by exercises 4.3 and 4.4, and Example 4.3.1 below shows that the bound is best possible. It was recently shown to be true for $\mathbb{Q}$-algebras in [CHSW].

*The Mayer-Vietoris sequence*

Suppose that $f : R \to S$ is a ring map, and $I$ is an ideal of $R$ mapped isomorphically into an ideal of $S$. By Theorem 2.6 there is an exact "Mayer-Vietoris" sequence:

$$K_1(R) \xrightarrow{\Delta} K_1(S) \oplus K_1(R/I) \to K_1(S/I) \xrightarrow{\partial} K_0(R) \xrightarrow{\Delta} K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I).$$

Applying the contraction operation $L$ to this sequence gives a sequence relating $K_0$ to $K_{-1}$, whose first three terms are identical to the last three terms of the displayed sequence. Splicing these together yields a longer sequence. Repeatedly applying $L$ and splicing sequences leads to the following result.

THEOREM 4.3 (MAYER-VIETORIS). *Suppose we are given a ring map $f : R \to S$ and an ideal $I$ of $R$ mapped isomorphically into an ideal of $S$. Then the Mayer-Vietoris sequence of Theorem 2.6 continues as a long exact Mayer-Vietoris sequence of negative $K$-groups.*

$$\xrightarrow{\Delta} \begin{bmatrix} K_0(S) \oplus \\ K_0(R/I) \end{bmatrix} \xrightarrow{\pm} K_0(S/I) \xrightarrow{\partial} K_{-1}(R) \xrightarrow{\Delta} \begin{bmatrix} K_{-1}(S) \oplus \\ K_{-1}(R/I) \end{bmatrix} \xrightarrow{\pm} K_{-1}(S/I) \xrightarrow{\partial} K_{-2}(R) \to$$

$$\cdots \to K_{-n+1}(S/I) \xrightarrow{\partial} K_{-n}(R) \xrightarrow{\Delta} \begin{bmatrix} K_{-n}(S) \oplus \\ K_{-n}(R/I) \end{bmatrix} \xrightarrow{\pm} K_{-n}(S/I) \xrightarrow{\partial} K_{-n-1}(R) \to \cdots$$

EXAMPLE 4.3.1 (B. DAYTON). Fix a regular ring $R$, and let $\Delta^n(R)$ denote the coordinate ring $R[t_0, \dots, t_n]/(f)$, $f = t_0 \cdots t_n(1 - \sum t_i)$ of the $n$-dimensional tetrahehron over $R$. Using $I = (1 - \sum t_i)\Delta^n(R)$ and $\Delta^n(R)/I \cong R[t_1, \dots, t_n]$ via $t_0 \mapsto 1 - (t_1 + \cdots + t_n)$, we have a Milnor square

$$\begin{array}{ccc} \Delta^n(R) & \to & A_n \\ \downarrow & & \downarrow \\ R[t_1, \dots, t_n] & \to & \Delta^{n-1}(R) \end{array}$$

where $A_n = R[t_0, \dots, t_n]/(t_0 \cdots t_n)$. By Ex. 4.8, the negative $K$-groups of $A_n$ vanish and $K_i(A_n) = K_i(R)$ for $i = 0, 1$. Thus $K_0(\Delta^n(R)) \cong K_0(R) \oplus K_1(\Delta^{n-1}(R))/K_1(R)$ for $n > 0$, and $K_{-j}(\Delta^n(R)) \cong K_{1-j}(\Delta^{n-1}(R))$ for $j > 0$. These groups vanish for $j > n$, with $K_{-n}(\Delta^n(R)) \cong K_0(R)$. In particular, if $F$ is a field then $\Delta^n(F)$ is an $n$-dimensional noetherian ring with $K_{-n}(\Delta^n(F)) \cong \mathbb{Z}$; see Conjecture 4.2.4.

When we have introduced higher $K$-theory, we will see that in fact $K_0(\Delta^n(R)) \cong K_n(R)$ and $K_1(\Delta^n(R)) \cong K_{n+1}(R)$. (See IV, Ex. 12.1.) This is just one way in which higher $K$-theory appears in classical $K$-theory.

*Theories of Negative K-theory*

Here is an alternative approach to defining negative $K$-theory, due to Karoubi and Villamayor [KV71].

DEFINITION 4.4. A *theory of negative K-theory* for (nonunital) rings consists of a sequence of functions $K_n$ ($n \le 0$) from nonunital rings to abelian groups, together with natural boundary maps $\partial : K_n(R/I) \to K_{n-1}(I)$ for every 2-sided ideal $I \subset R$, satisfying the following axioms.

(1) $K_0(R)$ is the Grothendieck group of chapter II;
(2) $K_n(I) \to K_n(R) \to K_n(R/I) \xrightarrow{\partial} K_{n-1}(I) \to K_{n-1}(R)$ is exact for every $I \subset R$;
(3) If $\Lambda$ is a flasque ring (II.2.1.3), then $K_n(\Lambda) = 0$ for all $n \le 0$;
(4) The inclusion $R \subset M(R) = \cup M_m(R)$ induces an isomorphism $K_n(R) \cong K_n M(R)$ for each $n \le 0$.

EXAMPLE 4.4.1. Bass' negative $K$-groups (4.1) form a theory of negative $K$-theory for rings. This follows from the contraction of 2.3 (see Ex. 4.5), Ex. 4.9 and the contraction of Morita Invariance 1.6.4.

EXAMPLE 4.4.2. Embedding $M(R)$ as an ideal in a flasque ring $\Lambda$, axiom (2) shows that $K_{-1}R \cong K_0(\Lambda/M(R))$. This was the approach used by Karoubi and Villamayor in [KV71] to inductively define a theory of negative $K$-theory; see Ex. 4.10.

EXAMPLE 4.4.3. If $A$ is a hensel local ring then $K_{-1}(A) = 0$. This was proven by Drinfeld in [Drin], using a *Calkin category* model for negative $K$-theory.

THEOREM 4.5. *Every theory of negative K-theory for rings is canonically isomorphic to the negative K-theory.*

PROOF. Suppose that $\{K'_n\}$ is another theory of negative $K$-theory for rings. We will show that there are natural isomorphisms $h_n(A) : K_n(A) \to K'_n(A)$ commuting with the boundary operators. By induction, we may assume that $h_n$ is given. Since $C(R)$ is flasque (II.2.1.3), and $S(R) = C(R)/M(R)$, the axioms yield isomorphisms $\partial : K_n S(R) \cong K_{n-1}(R)$ and $\partial' : K'_n S(R) \cong K'_{n-1}(R)$. We define $h_{n-1}(R) : K_{n-1}(R) \to K'_{n-1}(R)$ to be $\partial' \circ h_n(SR) \circ \partial^{-1}$.

It remains to check that the $h_n$ commute with the boundary maps associated to an ideal $I \subset R$. Since $M(I)$ is an ideal in $C(I)$, $C(R)$ and $M(R)$, the axioms yield $K_n S(I) \cong K_n C(R)/M(I)$ and similarly for $K'_n$. The naturality of $\partial$ and $\partial'$ relative to $M(R/I) = M(R)/M(I) \to C(R)/M(I)$ yield the diagram

$$\begin{array}{ccccccccc}
K_n(R/I) & \xrightarrow{\cong} & K_n M(R/I) & \to & K_n C(R)/M(I) & \xleftarrow{\cong} & K_n S(I) & \xrightarrow{\partial} & K_{n-1}(I) \\
{\scriptstyle h_n(R/I)}\downarrow\cong & & \downarrow\cong & & \cong\downarrow & & \cong\downarrow & & \downarrow{\scriptstyle h_{n-1}(I)} \\
K'_n(R/I) & \xrightarrow{\cong} & K'_n M(R/I) & \to & K'_n C(R)/M(I) & \xleftarrow{\cong} & K'_n S(I) & \xrightarrow{\partial'} & K'_{n-1}(I).
\end{array}$$

Since the horizontal composites are the given maps $\partial : K_n(R/I) \to K_{n-1}(I)$ and $\partial' : K'_n(R/I) \to K'_{n-1}(I)$, we have the desired relation: $\partial' h_n(R/I) \cong h_{n-1}(I)\,\partial$.

EXERCISES

**4.1** Suppose that $0 \to F' \to F \to F'' \to 0$ is an exact sequence of functors, with $F'$ and $F''$ contracted. Show that $F$ is acyclic, but need not be contracted.

**4.2** For a commutative ring $R$, let $H_0(R)$ denote the group $[\mathrm{Spec}(R), \mathbb{Z}]$ of all continuous functions from $\mathrm{Spec}(R)$ to $\mathbb{Z}$. Show that $NH_0 = LH_0 = 0$, *i.e.*, that $H_0(R) = H_0(R[t]) = H_0(R[t, t^{-1}])$.

**4.3** Let $R$ be an Artinian ring. Show that $R$ is $K_0$-regular, and that $K_{-n}(R) = 0$ for all $n > 0$.

**4.4** (Bass-Murthy) Let $R$ be a 1-dimensional commutative noetherian ring with finite normalization $\widetilde{R}$ and conductor ideal $I$. Show that $R$ is $K_{-1}$-regular, and that $K_{-n}(R) = 0$ for all $n \geq 2$. If $h_0(R)$ denotes the rank of the free abelian group $H_0(R) = [\mathrm{Spec}(R), \mathbb{Z}]$, show that $K_{-1}(R) \cong L\,\mathrm{Pic}(R) \cong \mathbb{Z}^r$, where $r = h_0(R) - h_0(\widetilde{R}) + h_0(\widetilde{R}/I) - h_0(R/I)$.

Now suppose that $R$ is any 1-dimensional commutative noetherian ring. Even if its normalization is not finitely generated over $R$, show that $R$ is $K_{-1}$-regular, and that $K_{-n}(R) = 0$ for all $n \geq 2$.

**4.5** (Carter) Let $f : R \to R'$ be a ring homomorphism. In II.2.10 we defined a group $K_0(f)$ and showed in Ex. 1.14 that it fits into an exact sequence

$$K_1(R) \xrightarrow{f^*} K_1(R') \to K_0(f) \to K_0(R) \xrightarrow{f^*} K_0(R').$$

Show that $A \mapsto K_0(f \otimes A)$ defines a functor on commutative rings $A$, and define $K_{-n}(f)$ to be $L^n K_0(f \otimes -)$. Show that each $K_{-n}(f)$ is an acyclic functor, and that the above sequence continues into negative $K$-theory as:

$$\cdots \to K_0(R) \to K_0(R') \xrightarrow{\partial} K_{-1}(f) \to K_{-1}(R) \to$$
$$K_{-1}(R') \xrightarrow{\partial} K_{-2}(f) \to K_{-2}(R) \to \cdots$$

With the help of higher $K$-theory to define $K_1(f)$ and to construct the product "$\cdot t$", it will follow that $K_0(f)$ and hence every $K_{-n}(f)$ is a contracted functor.

**4.6** Let $T : \mathbf{P}(R) \to \mathbf{P}(R')$ be any cofinal additive functor. Show that the functor $K_0(T)$ of II.2.10 and its contractions $K_{-n}(T)$ are acyclic, and that they extend the sequence of Ex. 1.14 into a long exact sequence, as in the previous exercise.

When $T$ is the endofunctor $\cdot m$ of 1.7.4, we write $K_{-n}(R; \mathbb{Z}/m)$ for $L^{n+1}K_0(\cdot m)$. Show that the sequence of 1.7.4 extends to a long exact sequence

$$K_0(R) \xrightarrow{m} K_0(R) \to K_0(R; \mathbb{Z}/m) \to K_{-1}(R) \xrightarrow{m} K_{-1}(R) \to K_{-1}(R; \mathbb{Z}/m) \to \cdots .$$

**4.7** Let $G$ be a finite group of order $n$, and let $\widetilde{R}$ be a "maximal order" in $\mathbb{Q}[G]$. It is well known that $\widetilde{R}$ is a regular ring containing $\mathbb{Z}[G]$, and that $I = n\widetilde{R}$ is an ideal of $\mathbb{Z}[G]$; see [Bass, p. 560]. Show that $K_{-n}\mathbb{Z}[G] = 0$ for $n \geq 2$, and that $K_{-1}$ has the following resolution by free abelian groups:

$$0 \to \mathbb{Z} \to H_0(\widetilde{R}) \oplus H_0(\mathbb{Z}/n[G]) \to H_0(\widetilde{R}/n\widetilde{R}) \to K_{-1}(\mathbb{Z}[G]) \to 0.$$

D. Carter has shown in [Carter] that $K_{-1}\mathbb{Z}[G] \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})^s$, where $s$ equals the number of simple components $M_{n_i}(D_i)$ of the semisimple ring $\mathbb{Q}[G]$ such that the Schur index of $D$ is even (see 1.2.4), but the Schur index of $D_p$ is odd at each prime $p$ dividing $n$. In particular, if $G$ is abelian then $K_{-1}\mathbb{Z}[G]$ is torsionfree (see [Bass, p. 695]).

**4.8** *Coordinate hyperplanes.* Let $R$ be a regular ring. By induction on $n$, show that the graded rings $A_n = R[t_0, \dots, t_n]/(t_0 \cdots t_n)$ are $K_i$-regular for all $i \le 1$. Conclude that $K_1(A_n) = K_1(R)$, $K_0(A_n) = K_0(R)$ and $K_i(A_n) = 0$ for all $i < 0$.

Show that the rings $\Delta^n(R)$ of 4.4.3 are also $K_1$-regular.

**4.9** Let $\Lambda$ be a flasque ring. Show that $\Lambda[t, t^{-1}]$ is also flasque, and conclude that $K_n(\Lambda) = 0$ for all $n \le 0$.

**4.10** (Karoubi) Recall from Ex. 1.15 that the suspension ring $S(R)$ satisfies $\partial : K_1(S(R)) \cong K_0(R)$. For each $n \ge 0$, set $K_0S^n(R) = K_0(S^n(R))$. Show that the functors $\{K'_n = K_0S^{-n}\}$ form a theory of negative $K$-theory for rings, and conclude that $K_n(R) \cong K_0(S^n(R))$.

**4.11** (Karoubi) Let $f : A \to B$ be an analytic isomorphism along $S$ in the sense of Ex. 3.10. Using Ex. 4.5, show that there is an exact sequence for all $n \le 0$, continuing the sequence of Ex. 3.10:

$$\begin{aligned} \cdots \to K_{n+1}(S^{-1}A) \oplus K_{n+1}(B) \to K_{n+1}(S^{-1}B) \to \\ K_n(A) \to K_n(S^{-1}A) \oplus K_n(B) \to K_n(S^{-1}B) \to \cdots . \end{aligned}$$

**4.12** (Reid) Let $f = y^2 - x^3 + x^2$ in $k[x, y]$ and set $B = k[x, y]/(f)$. Using Theorem 4.3, show that $K_{-1}(B) \cong K_{-1}(B_{(x,y)}) \cong \mathbb{Z}$. Let $A$ be the subring $k + \mathfrak{m}$ of $k[x, y]$, where $\mathfrak{m} = fk[X, y]$; show that $K_{-2}(A) \cong K_{-2}(A_{\mathfrak{m}}) \cong \mathbb{Z}$. Writing the integrally closed ring $A$ as the union of finitely generated normal subrings $k[f, xf, yf, ...]$, conclude that there is a 2-dimensional normal ring $A_0$, finitely generated over $k$, with $K_{-2}(A_0) \ne 0$.

**4.13** (Reid) We saw in Example 4.4.3 that $K_{-1}(A) = 0$ for every hensel local ring. In this exercise we construct a complete local 2-dimensional ring with $K_{-2}(\hat{A}) \ne 0$. Let $A$ be the ring of Exercise 4.12, and $\hat{A}$ its completion at the maximal ideal $\mathfrak{m}$. Let $\hat{A_f}$ denote the completion of $A$ at the ideal $Af$. Using Ex. 4.10, show that $K_{-2}(A) \cong K_{-2}(\hat{A_f}) \cong K_{-2}(\hat{A})$, and hence that $K_{-2}(\hat{A}) \ne 0$.

## §5. $K_2$ of a ring

The group $K_2$ of a ring was defined by J. Milnor in 1967, following a 1962 paper by R. Steinberg on Universal Central Extensions of Chevalley groups. Milnor's 1971 book [Milnor] is still the best source for the fundamental theorems about it. In this section we will give an introduction to the subject, but we will not prove the harder theorems.

Following Steinberg, we define a group in terms of generators and relations designed to imitate the behavior of the elementary matrices, as described in (1.2.1). To avoid technical complications, we shall avoid any definition of $St_2(R)$.

DEFINITION 5.1. For $n \geq 3$ the *Steinberg group* $St_n(R)$ of a ring $R$ is the group defined by generators $x_{ij}(r)$, with $i, j$ a pair of distinct integers between 1 and $n$ and $r \in R$, subject to the following "Steinberg relations"

(5.1.1) $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$

(5.1.2) $[x_{ij}(r), x_{k\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq \ell \\ x_{i\ell}(rs) & \text{if } j = k \text{ and } i \neq \ell \\ x_{kj}(-sr) & \text{if } j \neq k \text{ and } i = \ell. \end{cases}$

As observed in (1.3.1), the Steinberg relations are also satisfied by the elementary matrices $e_{ij}(r)$ which generate the subgroup $E_n(R)$ of $GL_n(R)$. Hence there is a canonical group surjection $\phi_n\colon St_n(R) \to E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$.

The Steinberg relations for $n+1$ include the Steinberg relations for $n$, so there is an obvious map $St_n(R) \to St_{n+1}(R)$. We write $St(R)$ for $\varinjlim St_n(R)$, and observe that by stabilizing the $\phi_n$ induce a surjection $\phi\colon St(R) \to E(R)$.

DEFINITION 5.2. The group $K_2(R)$ is the kernel of $\phi\colon St(R) \to E(R)$. Thus there is an exact sequence of groups

$$1 \to K_2(R) \to St(R) \xrightarrow{\phi} GL(R) \to K_1(R) \to 1.$$

It will follow from Theorem 5.3 below that $K_2(R)$ is an abelian group. Moreover, it is clear that $St$ and $K_2$ are both covariant functors from rings to groups, just as $GL$ and $K_1$ are.

THEOREM 5.2.1 (STEINBERG). *$K_2(R)$ is an abelian group. In fact it is precisely the center of $St(R)$.*

PROOF. If $x \in St(R)$ commutes with every element of $St(R)$, then $\phi(x)$ must commute with all of $E(R)$. But the center of $E(R)$ is trivial (by Ex. 1.8) so $\phi(x) = 1$, *i.e.*, $x \in K_2(R)$. Thus the center of $St(R)$ is contained in $K_2(R)$.

Conversely, suppose that $y \in St(R)$ satisfies $\phi(y) = 1$. Then in $E(R)$ we have

$$\phi([y,p]) = \phi(y)\phi(p)\phi(y)^{-1}\phi(p)^{-1} = \phi(p)\phi(p)^{-1} = 1$$

for every $p \in St(R)$. Choose an integer $n$ large enough that $y$ can be expressed as a word in the symbols $x_{ij}(r)$ with $i, j < n$. For each element $p = x_{kn}(s)$ with $k < n$ and $s \in R$, the Steinberg relations imply that the commutator $[y,p]$ is an element of the subgroup $P_n$ of $St(R)$ generated by the symbols $x_{in}(r)$ with $i < n$. On the other hand, we know by Ex. 5.2 that $\phi$ maps $P_n$ injectively into $E(R)$. Since

$\phi([y,p]) = 1$ this implies that $[y,p] = 1$. Hence $y$ commutes with every generator $x_{kn}(s)$ with $k < n$.

By symmetry, this proves that $y$ also commutes with every generator $x_{nk}(s)$ with $k < n$. Hence $y$ commutes with all of $St_n(R)$, since it commutes with every $x_{kl}(s) = [x_{kn}(s), x_{nl}(1)]$ with $k, l < n$. Since $n$ can be arbitrarily large, this proves that $y$ is in the center of $St(R)$.

EXAMPLE 5.2.2. The group $K_2(\mathbb{Z})$ is cyclic of order 2. This calculation uses the Euclidean Algorithm to rewrite elements of $St(\mathbb{Z})$, and is given in §10 of [Milnor]. In fact, Milnor proves that the symbol $\{-1,-1\} = \{x_{12}(1)x_{21}(-1)x_{12}(1)\}^4$ is the only nontrivial element of $\ker(\phi_n)$ for all $n \geq 3$. It is easy to see that $\{-1,-1\}$ is in the kernel of each $\phi_n$, because the $2 \times 2$ matrix $e_{12}(1)e_{21}(-1)e_{12}(1) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ has order 4 in $GL_n(\mathbb{Z})$. We will see in Example 6.2.1 below that $\{-1,-1\}$ is still nonzero in $K_2(\mathbb{R})$.

Tate has used the same Euclidean Algorithm type techniques to show that $K_2(\mathbb{Z}[\sqrt{-7}])$ and $K_2(\mathbb{Z}[\sqrt{-15}])$ are also cyclic of order 2, generated by the symbol $\{-1,-1\}$, while $K_2(R) = 1$ for the imaginary quadratic rings $R = \mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-3}]$, $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-11}]$. See the appendix to [BT] for details.

EXAMPLE 5.2.3. For every field $F$ we have $K_2(F[t]) = K_2(F)$. This was originally proven by K. Dennis and J. Sylvester using the same Euclidean Algorithm type techniques as in the provious example. We shall not describe the details, because we shall see in chapter V that $K_2(R[t]) = K_2(R)$ for every regular ring.

*Universal Central Extensions*

The Steinberg group $St(R)$ can be described in terms of universal central extensions, and the best exposition of this is [Milnor, §5]. Properly speaking, this is a subject in pure group theory; see [Suz, 2.9]. However, since extensions of a group $G$ are classified by the cohomology group $H^2(G)$, the theory of universal central extensions is also a part of homological algebra; see [WHomo, §6.9]. Here are the relevant definitions.

Let $G$ be a group and $A$ an abelian group. A *central extension* of $G$ by $A$ is a short exact sequence of groups $1 \to A \to X \xrightarrow{\pi} G \to 1$ such that $A$ is in the center of $X$. We say that a central extension is *split* if it is isomorphic to an extension of the form $1 \to A \to A \times G \xrightarrow{pr} G \to 1$, where $pr(a,g) = g$.

If $A$ or $\pi$ is clear from the context, we may omit it from the notation. For example, $1 \to K_2(R) \to St(R) \to E(R) \to 1$ is a central extension by Steinberg's Theorem 5.2.1, but we usually just say that $St(R)$ is a central extension of $E(R)$.

Two extensions $X$ and $Y$ of $G$ by $A$ are said to be *equivalent* if there is an isomorphism $f\colon X \to Y$ which is the identity on $A$ and which induces the identity map on $G$. It is well-known that the equivalence classes of central extensions of $G$ by a fixed group $A$ are in 1–1 correspondence with the elements of the cohomology group $H^2(G;A)$; see [WHomo, §6.6].

More generally, by a *homomorphism over* $G$ from $X \xrightarrow{\pi} G$ to another central extension $1 \to B \to Y \xrightarrow{\tau} G \to 1$ we mean a group map $f\colon X \to Y$ such that $\pi = \tau f$.

DEFINITION 5.3.1. A *universal central extension* of $G$ is a central extension $X \xrightarrow{\pi} G$ such that for every other central extension $Y \xrightarrow{\tau} G$ there is a unique homomorphism $f$ over $G$ from $X$ to $Y$. Clearly a universal central extension is unique up to isomorphism over $G$, provided it exists.

LEMMA 5.3.2. *If $G$ has a universal central extension $X$, then both $G$ and $X$ must be perfect groups.*

PROOF. Otherwise $B = X/[X,X]$ is nontrivial, and there would be two homomorphisms over $G$ from $X$ to the central extension $1 \to B \to B \times G \to G \to 1$, namely the maps $(0, \pi)$ and $(pr, \pi)$, where $pr$ is the natural projection $X \to B$.

LEMMA 5.3.3. *If $X$ and $Y$ are central extensions of $G$, and $X$ is a perfect group, there is at most one homomorphism over $G$ from $X$ to $Y$.*

PROOF. If $f$ and $f_1$ are two such homomorphisms, then for any $x$ and $x'$ in $X$ we can write $f_1(x) = f(x)c$, $f_1(x') = f(x')c'$ for elements $c$ and $c'$ in the center of $Y$. Therefore $f_1(xx'x^{-1}(x')^{-1}) = f(xx'x^{-1}(x')^{-1})$. Since the commutators $[x,x'] = xx'x^{-1}(x')^{-1}$ generate $X$ we must have $f_1 = f$.

EXAMPLE 5.3.4. Every presentation of $G$ gives rise to two natural central extensions as follows. A presentation corresponds to the choice of a free group $F$ mapping onto $G$, and a description of the kernel $R \subset F$. Since $[R,F]$ is a normal subgroup of $F$, we may form the following central extensions:

$$1 \to R/[R,F] \to F/[R,F] \to G \to 1,$$

$$1 \to (R \cap [F,F])/[R,F] \to [F,F]/[R,F] \to [G,G] \to 1. \tag{5.3.5}$$

The group $(R \cap [F,F])/[R,F]$ in (5.3.5) is the homology group $H_2(G;\mathbb{Z})$; this identity was discovered in 1941 by Hopf [WHomo, 6.8.8]. If $G = [G,G]$, both are extensions of $G$, and (5.3.5) is the universal central extension by the following theorem.

RECOGNITION THEOREM 5.4. *Every perfect group $G$ has a universal central extension, namely the extension (5.3.5):*

$$1 \to H_2(G;\mathbb{Z}) \to [F,F]/[R,F] \to G \to 1.$$

*Let $X$ be any central extension of $G$, the following are equivalent: (1) $X$ is a universal central extension; (2) $X$ is perfect, and every central extension of $X$ splits; (3) $H_1(X;\mathbb{Z}) = H_2(X;\mathbb{Z}) = 0$.*

PROOF. Given any central extension $X$ of $G$, the map $F \to G$ lifts to a map $h: F \to X$ because $F$ is free. Since $h(R)$ is in the center of $X$, $h([R,F]) = 1$. Thus $h$ induces a map from $[F,F]/[R,F]$ to $X$ over $G$. This map is unique by Lemma 5.3.3. This proves that (5.3.5) is a universal central extension, and proves the equivalence of (1) and (3). The implication (1) $\Rightarrow$ (2) is Lemma 5.3.2 and Ex. 5.7, and (2) $\Rightarrow$ (1) is immediate.

THEOREM 5.5 (KERVAIRE, STEINBERG). *The Steinberg group $St(R)$ is the universal central extension of $E(R)$. Hence*

$$K_2(R) \cong H_2(E(R); \mathbb{Z}).$$

This theorem follows immediately from the Recognition Theorem 5.4, and the following splitting result:

PROPOSITION 5.5.1. *If $n \geq 5$, every central extension $Y \xrightarrow{\pi} St_n(R)$ is split. Hence $St_n(R)$ is the universal central extension of $E_n(R)$.*

PROOF. We first show that if $j \neq k$ and $l \neq i$ then every two elements $y, z \in Y$ with $\pi(y) = x_{ij}(r)$ and $\pi(z) = x_{kl}(s)$ must commute in $Y$. Pick $t$ distinct from $i, j, k, l$ and choose $y', y'' \in Y$ with $\pi(y') = x_{it}(1)$ and $\pi(y'') = x_{tj}(r)$. The Steinberg relations imply that both $[y', z]$ and $[y'', z]$ are in the center of $Y$, and since $\pi(y) = \pi[y', y'']$ this implies that $z$ commutes with $[y', y'']$ and $y$.

We now choose distinct indices $i, j, k, l$ and elements $u, v, w \in Y$ with

$$\pi(u) = x_{ij}(1), \quad \pi(v) = x_{jk}(s) \quad \text{and } \pi(w) = x_{kl}(r).$$

If $G$ denotes the subgroup of $Y$ generated by $u, v, w$ then its commutator subgroup $[G, G]$ is generated by elements mapping under $\pi$ to $x_{ik}(s)$, $x_{jl}(sr)$ or $x_{il}(sr)$. From the first paragraph of this proof it follows that $[u, w] = 1$ and that $[G, G]$ is abelian. By Ex. 5.3 we have $[[u, v], w] = [u, [v, w]]$. Therefore if $\pi(y) = x_{ik}(s)$ and $\pi(z) = x_{jl}(sr)$ we have $[y, w] = [u, z]$. Taking $s = 1$, this identity proves that the element

$$y_{il}(r) = [u, z], \qquad \text{where } \pi(u) = x_{ij}(1),\ \pi(z) = x_{jl}(r)$$

doesn't depend upon the choice of $j$, nor upon the lifts $u$ and $z$ of $x_{ij}(1)$ and $x_{jl}(r)$.

We claim that the elements $y_{ij}(r)$ satisfy the Steinberg relations, so that there is a group homomorphism $St_n(R) \to Y$ sending $x_{ij}(r)$ to $y_{ij}(r)$. Such a homomorphism will provide the desired splitting of the extension $\pi$. The first paragraph of this proof implies that if $j \neq k$ and $l \neq i$ then $y_{ij}(r)$ and $y_{kl}(s)$ commute. The identity $[y, w] = [u, z]$ above may be rewritten as

$$[y_{ik}(r), y_{kl}(s)] = y_{il}(rs) \quad \text{for } i, k, l \text{ distinct.}$$

The final relation $y_{ij}(r)y_{ij}(s) = y_{ij}(r+s)$ is a routine calculation with commutators left to the reader.

REMARK 5.5.2 (STABILITY FOR $K_2$). The kernel of $St_n(R) \to E_n(R)$ is written as $K_2(n, R)$, and there are natural maps $K_2(n, R) \to K_2(R)$. If $R$ is noetherian of dimension $d$, or more generally has $sr(R) = d+1$, then the following stability result holds: $K_2(n, R) \cong K_2(R)$ for all $n \geq d+3$. This result evolved in the mid-1970's as a sequence of results by Dennis, Vaserstein, van der Kallen and Suslin-Tulenbaev. We refer the reader to section 19C25 of Math Reviews for more details.

*Transfer maps on $K_2$*

Here is a description of $K_2(R)$ in terms of the translation category $t\mathbf{P}(R)$ of finitely generated projective $R$-modules, analogous to the description given for $K_1$ in Corollary 1.6.3.

PROPOSITION 5.6 (BASS). $K_2(R) \cong \varinjlim_{P\in t\mathbf{P}} H_2([\mathrm{Aut}(P), \mathrm{Aut}(P)]; \mathbb{Z})$.

PROOF. If $G$ is a group, then $G$ acts by conjugation upon $[G, G]$ and hence upon the homology $H_2([G, G]; \mathbb{Z})$. Taking coinvariants, we obtain the functor $H_2'$ from groups to abelian groups defined by $H_2'(G) = H_0(G; H_2([G, G]; \mathbb{Z}))$. By construction, $G$ acts trivially upon $H_2'(G)$ and commutes with direct limits of groups.

Note that if $G$ acts trivially upon $H_2([G, G]; \mathbb{Z})$ then $H_2'(G) = H_2([G, G]; \mathbb{Z})$. For example, $GL(R)$ acts trivially upon the homology of $E(R) = [GL(R), GL(R)]$ by Ex. 1.13. By Theorem 5.5 this implies that $H_2'(GL(R)) = H_2(E(R); \mathbb{Z}) = K_2(R)$.

Since morphisms in the translation category $t\mathbf{P}(R)$ are well-defined up to isomorphism, it follows that $P \mapsto H_2'(\mathrm{Aut}(P))$ is a well-defined functor from $t\mathbf{P}(R)$ to abelian groups. Hence we can take the filtered colimit of this functor, as we did in 1.6.3. Since the free modules are cofinal in $t\mathbf{P}(R)$, the result follows from the identification of the colimit as

$$\lim_{n\to\infty} H_2'(GL_n(R)) \cong H_2'(GL(R)) = K_2(R).$$

COROLLARY 5.6.1 (MORITA INVARIANCE OF $K_2$). *The group $K_2(R)$ depends only upon the category $\mathbf{P}(R)$. That is, if $R$ and $S$ are Morita equivalent rings (see II.2.7) then $K_2(R) \cong K_2(S)$. In particular, the maps $R \to M_n(R)$ induce isomorphisms on $K_2$:*

$$K_2(R) \cong K_2(M_n(R)).$$

COROLLARY 5.6.2. *Any additive functor $T : \mathbf{P}(S) \to \mathbf{P}(R)$ induces a homomorphism $K_2(T)\colon K_2(S) \to K_2(R)$, and $T_1 \oplus T_2$ induces the sum $K_2(T_1) + K_2(T_2)$.*

PROOF. The proof of 1.7 goes through, replacing $H_1(\mathrm{Aut}\, P)$ by $H_2'(\mathrm{Aut}\, P)$.

COROLLARY 5.6.3 (FINITE TRANSFER). *Let $f\colon R \to S$ be a ring homomorphism such that $S$ is finitely generated projective as an $R$-module. Then the forgetful functor $\mathbf{P}(S) \to \mathbf{P}(R)$ induces a natural transfer homomorphism $f_*\colon K_2(S) \to K_2(R)$.*

*If $R$ is commutative, so that $K_2(R)$ is a $K_0(R)$-module by Ex. 5.4, the composition $f_*f^*\colon K_2(R) \to K_2(S) \to K_2(R)$ is multiplication by $[S] \in K_0(R)$. In particular, if $S$ is free of rank $n$, then $f_*f^*$ is multiplication by $n$.*

PROOF. The composite $f_*f^*$ is obtained from the self-map $T(P) = P \otimes_R S$ of $\mathbf{P}(S)$. It induces the self-map $\otimes_R S$ on $t\mathbf{P}(R)$ giving rise to multiplication by $[S]$ on $K_2(R)$ in Ex. 5.5.

We will see in chapter Vthat we can also define a transfer map $K_2(S) \to K_2(R)$ when $S$ is a finite $R$-algebra of finite projective dimension over $R$.

EXAMPLE 5.6.4. Let $D$ be a division algebra of dimension $d = n^2$ over its center $F$. As in Example 1.7.2, the transfer $i_* : K_2(D) \to K_2(F)$ has a kernel of exponent $n^2$, since $i^* i_*$ is induced by the functor $T(M) = M \otimes_D (D \otimes_F D) \cong M^d$ and hence is multiplication by $n^2$.

If $E$ is a splitting field for $D$, the construction of Ex. 1.17 yields a natural map $\theta_E : K_2(E) \to K_2(D)$. If $n$ is squarefree, Merkurjev and Suslin construct a reduced norm $N_{\text{red}} : K_2(D) \to K_2(F)$ such that $N_{\text{red}}\theta_E = N_{E/F}$; see [MS]. If $K_2(F) \to K_2(F)$ is injective, it is induced by the norm map $K_2(D) \to K_2(E)$, as in 1.2.4.

*Relative $K_2$ and relative Steinberg groups*

Given an ideal $I$ in a ring $R$, we may construct the augmented ring $R \oplus I$, with multiplication $(r, x)(s, y) = (rs, ry + xs + xy)$. This ring is equipped with two natural maps $pr, add \colon R \oplus I \to R$, defined by $pr(r, x) = r$ and $add(r, x) = r + x$. This "double" ring was used to define the relative group $K_0(I)$ in Ex. II.2.3.

Let $St'(R, I)$ denote the normal subgroup of $St(R \oplus I)$ generated by all $x_{ij}(0, v)$ with $v \in I$. Clearly there is a map from $St'(R, I)$ to the subgroup $E(R \oplus I, 0 \oplus I)$ of $GL(R \oplus I)$ (see Lemma 2.1), and an exact sequence

$$1 \to St'(R, I) \to St(R \oplus I) \xrightarrow{pr} St(R) \to 1.$$

The following definition is taken from [Keu78] and [Lo78], and modifies [Milnor].

DEFINITION 5.7. The *relative Steinberg group* $St(R, I)$ is defined to be the quotient of $St'(R, I)$ by the normal subgroup generated by all "cross-commutators" $[x_{ij}(0, u), x_{kl}(v, -v)]$ with $u, v \in I$.

Clearly the homomorphism $St(R \oplus I) \xrightarrow{add} St(R)$ sends these cross-commutators to 1, so it induces a homomorphism $St(R, I) \xrightarrow{add} St(R)$ whose image is the normal subgroup generated by the $x_{ij}(v)$, $v \in I$. By the definition of $E(R, I)$, the surjection $St(R) \to E(R)$ maps $St(R, I)$ onto $E(R, I)$. We define $K_2(R, I)$ to be the kernel of the map $St(R, I) \to E(R, I)$.

THEOREM 5.7.1. *If $I$ is an ideal of a ring $R$, then the exact sequence of Proposition 2.3 extends to an exact sequence*

$$K_2(R, I) \to K_2(R) \to K_2(R/I) \to K_1(R, I) \to K_1(R) \to K_1(R/I) \to K_0(I) \cdots$$

PROOF. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
K_2(R, I) & \longrightarrow & St(R, I) & \longrightarrow & GL(I) & \longrightarrow & K_1(R, I) \\
\downarrow & & {\scriptstyle add}\downarrow & & {\scriptstyle into}\downarrow & & \downarrow \\
K_2(R) & \longrightarrow & St(R) & \longrightarrow & GL(R) & \longrightarrow & K_1(R) \\
\downarrow & & {\scriptstyle onto}\downarrow & & \downarrow & & \downarrow \\
K_2(R/I) & \longrightarrow & St(R/I) & \longrightarrow & GL(R/I) & \longrightarrow & K_1(R/I)
\end{array}$$

The exact sequence now follows from the Snake Lemma and Ex. 5.1.

If $I$ and $J$ are ideals in a ring $R$ with $I \cap J = 0$, we may also consider $I$ as an ideal of $R/J$. As in §1, these rings form a Milnor square:

$$\begin{array}{ccc} R & \longrightarrow & R/J \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & R/(I+J). \end{array}$$

Theorem 5.8 (Mayer-Vietoris). *If $I$ and $J$ are ideals of $R$ with $I \cap J = 0$, then the Mayer-Vietoris sequence of Theorem 2.6 can be extended to $K_2$:*

$$\begin{array}{l} K_2(R) \xrightarrow{\Delta} K_2(R/I) \oplus K_2(R/J) \xrightarrow{\pm} K_2(R/I+J) \xrightarrow{\partial} \\ K_1(R) \xrightarrow{\Delta} K_1(R/I) \oplus K_1(R/J) \xrightarrow{\pm} K_1(R/I+J) \xrightarrow{\partial} K_0(R) \to \cdots \end{array}$$

Proof. By Ex. 5.10, we have the following commutative diagram:

$$\begin{array}{ccccccccccc} K_2(R,I) & \to & K_2(R) & \to & K_2(R/I) & \to & K_1(R,I) & \to & K_1(R) & \to & K_1(R/I) \\ {\scriptstyle onto}\downarrow & & \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\ K_2(R/J,I) & \to & K_2(R/J) & \to & K_2(\frac{R}{I+J}) & \to & K_1(R/J,I) & \to & K_1(R/J) & \to & K_1(\frac{R}{I+J}) \end{array}$$

By chasing this diagram, we obtain the exact Mayer-Vietoris sequence.

*Commutative Banach Algebras*

Let $R$ be a commutative Banach algebra over the real or complex numbers. Just as $SK_1(R) = \pi_0 SL(R)$ and $K_1(R)$ surjects onto $\pi_0 GL(R)$ (by 1.5 and 1.5.1), there is a relation between $K_2(R)$ and $\pi_1 GL(R)$.

Proposition 5.9. *Let $R$ be a commutative Banach algebra. Then there is a surjection from $K_2(R)$ onto $\pi_1 SL(R) = \pi_1 E(R)$.*

Proof. ([Milnor, p.59]) By Proposition 1.5, we know that $E_n(R)$ is the path component of the identity in the topological group $SL_n(R)$, so $\pi_1 SL(R) = \pi_1 E(R)$. Using the exponential map $M_n(R) \to GL_n(R)$, we see that $E_n(R)$ is locally contractible, so it has a universal covering space $\widetilde{E}_n$. The group map $\widetilde{E}_n \to E_n(R)$ is a central extension with kernel $\pi_1 E_n(R)$. Taking the direct limit as $n \to \infty$, we get a central extension $1 \to \pi_1 E(R) \to \widetilde{E} \to E(R) \to 1$. By universality, there is a unique homomorphism $\tilde{\phi}\colon St(R) \to \widetilde{E}$ over $E(R)$, and hence a unique map $K_2(R) \to \pi_1 E(R)$. Thus it suffices to show that $\tilde{\phi}$ is onto.

The map $\tilde{\phi}$ may be constructed explicitly as follows. Let $\tilde{e}_{ij}(r) \in \widetilde{E}$ be the endpoint of the path which starts at 1 and lifts the path $t \mapsto e_{ij}(tr)$ in $E(R)$. We claim that the map $\tilde{\phi}$ sends $x_{ij}(r)$ to $\tilde{e}_{ij}(r)$. To see this, it suffices to show that the Steinberg relations (5.1) are satisfied. But the paths $\tilde{e}_{ij}(tr)\tilde{e}_{ij}(ts)$ and $[\tilde{e}_{ij}(tr), \tilde{e}_{kl}(s)]$ cover the two paths $e_{ij}(tr)e_{ij}(s)$ and $[e_{ij}(tr), e_{kl}(s)]$ in $E(R)$. Evaluating at $t = 1$ yields the Steinberg relations.

By Proposition 1.5 there is a neighborhood $U_n$ of 1 in $SL_n(R)$ in which we may express every matrix $g$ as a product of elementary matrices $e_{ij}(r)$, where $r$ depends continuously upon $g$. Replacing each $e_{ij}(r)$ with $\tilde{e}_{ij}(r)$ defines a continuous lifting of $U_n$ to $\widetilde{E}_n$. Therefore the image of each map $\tilde{\phi}\colon St_n(R) \to \widetilde{E}_n$ contains a neighborhood $\widetilde{U}_n$ of 1. Since any open subset of a connected group (such as $\widetilde{E}_n$) generates the entire group, this proves that each $\tilde{\phi}_n$ is surjective. Passing to the limit as $n \to \infty$, we see that $\tilde{\phi}\colon St(R) \to \widetilde{E}$ is also surjective.

Example 5.9.1. If $R = \mathbb{R}$ then $\pi_1 SL(\mathbb{R}) \cong \pi_1 SO$ is cyclic of order 2. It follows that $K_2(\mathbb{R})$ has at least one nontrivial element. In fact, the symbol $\{-1,-1\}$ of Example 5.1.1 maps to the nonzero element of $\pi_1 SO$. We will see in 6.8.3 below that the kernel of $K_2(\mathbb{R}) \to \pi_1 SO$ is a uniquely divisible abelian group with uncountably many elements.

Example 5.9.2. Let $X$ be a compact space with a nondegenerate basepoint. By Ex. II.3.11, we have $KO^{-2}(X) \cong [X, \Omega SO] = \pi_1 SL(\mathbb{R}^X)$, so $K_2(\mathbb{R}^X)$ maps onto the group $KO^{-2}(X)$.

Similarly, since $\Omega U \simeq \mathbb{Z} \times \Omega SU$, we see by Ex. II.3.11 that $KU^{-2}(X) \cong [X, \Omega U] = [X, \mathbb{Z}] \times [X, \Omega SU]$. Since $\pi_1 SL(\mathbb{C}^X) = \pi_1(SU^X) = [X, \Omega SU]$ and $[X, \mathbb{Z}]$ is a subgroup of $\mathbb{C}^X$, we can combine Proposition 5.9 with Example 1.5.3 to obtain the exact sequence

$$K_2(\mathbb{C}^X) \to KU^{-2}(X) \to \mathbb{C}^X \xrightarrow{\exp} K_1(\mathbb{C}^X) \to KU^{-1}(X) \to 0.$$

*Steinberg symbols*

If two matrices $A, B \in E(R)$ commute, we can construct an element in $K_2(R)$ by lifting their commutator to $St(R)$. To do this, choose $a, b \in St(R)$ with $\phi(a) = A$, $\phi(b) = B$ and define $A \bigstar B = [a, b] \in K_2(R)$. This definition is independent of the choice of $a$ and $b$ because any other lift will equal $ac$, $bc'$ for central elements $c, c'$, and $[ac, bc'] = [a, b]$.

If $P \in GL(R)$ then $(PAP^{-1}) \bigstar (PBP^{-1}) = A \bigstar B$. To see this, suppose that $A, B, P \in GL_n(R)$ and let $g \in St_{2n}(R)$ be a lift of the block diagonal matrix $D = \text{diag}(P, P^{-1})$. Since $gag^{-1}$ and $gbg^{-1}$ lift $PAP^{-1}$ and $PBP^{-1}$ and $[a, b]$ is central we have the desired relation: $[gag^{-1}, gbg^{-1}] = g[a, b]g^{-1} = [a, b]$.

The $\bigstar$ symbol is also skew-symmetric and bilinear: $(A \bigstar B)(B \bigstar A) = 1$ and $(A_1 A_2) \bigstar B = (A_1 \bigstar B)(A_2 \bigstar B)$. These relations are immediate from the commutator identies $[a, b][b, a] = 1$ and $[a_1 a_2, b] = [a_1, [a_2, b]][a_2, b][a_1, b]$.

Definition 5.10. If $r, s$ are commuting units in a ring $R$, we define the Steinberg symbol $\{r, s\} \in K_2(R)$ to be

$$\{r, s\} = \begin{pmatrix} r & & \\ & r^{-1} & \\ & & 1 \end{pmatrix} \bigstar \begin{pmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{pmatrix} = \begin{pmatrix} r & & \\ & 1 & \\ & & r^{-1} \end{pmatrix} \bigstar \begin{pmatrix} s & & \\ & s^{-1} & \\ & & 1 \end{pmatrix}.$$

Because the $\bigstar$ symbols are skew-symmetric and bilinear, so are the Steinberg symbols: $\{r, s\}\{s, r\} = 1$ and $\{r_1 r_2, s\} = \{r_1, s\}\{r_2, s\}$.

EXAMPLE 5.10.1. For any unit $r$ of $R$ we set $w_{ij}(r) = x_{ij}(r)x_{ji}(-r^{-1})x_{ij}(r)$ and $h_{ij}(r) = w_{ij}(r)w_{ij}(-1)$. In $GL(R)$, $\phi w_{ij}(r)$ is the monomial matrix with $r$ and $-r^{-1}$ in the $(i,j)$ and $(j,i)$ places, while $\phi h_{ij}(r)$ is the diagonal matrix with $r$ and $r^{-1}$ in the $i^{th}$ and $j^{th}$ diagonal spots. By definition we then have:

$$\{r,s\} = [h_{12}(r), h_{13}(s)] = [h_{ij}(r), h_{ik}(s)].$$

LEMMA 5.10.2. *If both $r$ and $1-r$ are units of $R$, then in $K_2(R)$ we have:*

$$\{r, 1-r\} = 1 \qquad \textit{and} \qquad \{r, -r\} = 1.$$

PROOF. By Ex. 5.8, $w_{12}(-1) = w_{21}(1) = x_{21}(1)x_{12}(-1)x_{21}(1)$, $w_{12}(r)x_{21}(1) = x_{12}(-r^2)w_{12}(r)$ and $x_{21}(1)w_{12}(s) = w_{12}(s)x_{12}(-s^2)$. If $s = 1-r$ we can successively use the identities $r - r^2 = rs$, $r + s = 1$, $s - s^2 = rs$ and $\frac{1}{r} + \frac{1}{s} = \frac{1}{rs}$ to obtain:

$$\begin{aligned} w_{12}(r)w_{12}(-1)w_{12}(s) &= x_{12}(-r^2)w_{12}(r)x_{12}(-1)\ w_{12}(s)x_{12}(-s^2) \\ &= x_{12}(rs)x_{21}(-r^{-1})x_{12}(0)x_{21}(-s^{-1})x_{12}(rs) \\ &= x_{12}(rs)x_{21}(\frac{-1}{rs})x_{12}(rs) \\ &= w_{12}(rs). \end{aligned}$$

Multiplying by $w_{12}(-1)$ yields $h_{12}(r)h_{12}(s) = h_{12}(rs)$ when $r + s = 1$. By Ex. 5.9, this yields the first equation $\{r,s\} = 1$. Since $-r = (1-r)/(1-r^{-1})$, the first equation implies the second equation:

$$\{r, -r\} = \{r, 1-r\}\{r, 1-r^{-1}\}^{-1} = \{r^{-1}, 1-r^{-1}\} = 1. \tag{5.10.3}$$

REMARK 5.10.4. The equation $\{r, -r\} = 1$ holds more generally for every unit $r$, even if $1-r$ is not a unit. This follows from the fact that $K_2(\mathbb{Z}[r, \frac{1}{r}])$ injects into $K_2(\mathbb{Z}[r, \frac{1}{r}, \frac{1}{1-r}])$, a fact we shall establish in chapter V, 6.1.3. For a direct proof, see [Milnor, 9.8].

The following useful result was proven for fields and division rings in §9 of [Milnor]. It was extended to commutative semilocal rings by Dennis and Stein [DS], and we cite it here for completeness.

THEOREM 5.10.5. *If $R$ is a field, division ring, local ring, or even a semilocal ring, then $K_2(R)$ is generated by the Steinberg symbols $\{r,s\}$.*

DEFINITION 5.11 (DENNIS-STEIN SYMBOLS). If $r, s \in R$ commute and $1 - rs$ is a unit then the element

$$\langle r,s\rangle = x_{ji}\big(-s(1-rs)^{-1}\big)x_{ij}(-r)x_{ji}(s)x_{ij}\big((1-rs)^{-1}r\big)h_{ij}(1-rs)^{-1}$$

of $St(R)$ belongs to $K_2(R)$, because $\phi\langle r,s\rangle = 1$. By Ex. 5.11, it is independent of the choice of $i \neq j$, and if $r$ is a unit of $R$ then $\langle r,s\rangle = \{r, 1-rs\}$. If $I$ is an ideal of $R$ and $s \in I$ then we can even consider $\langle r,s\rangle$ as an element of $K_2(R,I)$; see 5.7.

These elements are called *Dennis-Stein symbols* because they were first studied in [DS], where the following identities were established.

(D1) $\langle r,s\rangle\langle s,r\rangle = 1$
(D2) $\langle r,s\rangle\langle r,t\rangle = \langle r, s+t-rst\rangle$
(D3) $\langle r,st\rangle = \langle rs,t\rangle\langle tr,s\rangle$ (this holds in $K_2(R,I)$ if any of $r$, $s$, or $t$ are in $I$.)

We warn the reader that the meaning of the symbol $\langle r,s\rangle$ changed circa 1980. We use the modern definition of this symbol, which equals $\langle -r,s\rangle^{-1}$ in the old literature, including that of *loc. cit.* By (D3) of our definition, $\langle r,1\rangle = 0$ for all $r$.

The following result is essentially due to Maazen, Stienstra and van der Kallen. However, their work preceded the correct definition of $K_2(R,I)$ so the correct historical reference is [Keune].

THEOREM 5.11.1. *(a) Let $R$ be a commutative local ring, or a field. Then $K_2(R)$ may be presented as the abelian group generated by the symbols $\langle r,s\rangle$ with $r,s\in R$ such that $1-rs$ is a unit, subject only to the relations (D1), (D2) and (D3).*

*(b) Let $I$ be a radical ideal, contained in a commutative ring $R$. Then $K_2(R,I)$ may be presented as the abelian group generated by the symbols $\langle r,s\rangle$ with either $r\in R$ and $s\in I$, or else $r\in I$ and $s\in R$. These generators are subject only to the relations (D1), (D2), and the relation (D3) whenever $r$, $s$, or $t$ is in $I$.*

*The product $K_1(R)\otimes K_1(R)\to K_2(R)$*

Let $R$ be a commutative ring, and suppose given two invertible matrices $g\in GL_m(R)$, $h\in GL_n(R)$. Identifying the tensor product $R^m\otimes R^n$ with $R^{m+n}$, then $g\otimes 1_n$ and $1_m\otimes h$ are commuting automorphisms of $R^m\otimes R^n$. Hence there is a ring homomorphism from $A=\mathbb{Z}[x,x^{-1},y,y^{-1}]$ to $E=\mathrm{End}_R(R^m\otimes R^n)\cong M_{m+n}(R)$ sending $x$ and $y$ to $g\otimes 1_n$ and $1_m\otimes h$. Recall that by Morita Invariance 5.6.1 the natural map $K_2(R)\to K_2(E)$ is an isomorphism.

DEFINITION 5.12. The element $\{g,h\}$ of $K_2(R)$ is defined to be the image of the Steinberg symbol $\{x,y\}$ under the homomorphism $K_2(A)\to K_2(E)\cong K_2(R)$.

Note that if $m=n=1$ this agrees with the definition of the usual Steinberg symbol in 5.10, because $R=E$.

LEMMA 5.12.1. *The symbol $\{g,h\}$ is independent of the choice of $m$ and $n$, and is skew-symmetric. Moreover, for each $\alpha\in GL_m(R)$ we have $\{g,h\}=\{\alpha g\alpha^{-1},h\}$.*

PROOF. If we embed $GL_m(R)$ and $GL_n(R)$ in $GL_{m'}(R)$ and $GL_{n'}(R)$, respectively, then we embed $E$ into the larger ring $E'=\mathrm{End}_R(R^{m'}\otimes R^{n'})$, which is also Morita equivalent to $R$. Since the natural maps $K_2(R)\to K_2(E)\to K_2(E')$ are isomorphisms, and $K_2(A)\to K_2(E)\to K_2(E')\cong K_2(R)$ defines the symbol with respect to the larger embedding, the symbol is independent of $m$ and $n$.

Any linear automorphism of $R^{m+n}$ induces an inner automorphism of $E$. Since the composition of $R\to E$ with such an automorphism is still $R\to E$, the symbol $\{g,h\}$ is unchanged by such an operation. Applying this to $\alpha\otimes 1_n$, the map $A\to E\to E$ sends $x$ and $y$ to $\alpha g\alpha^{-1}\otimes 1_n$ and $1_m\otimes h$, so $\{g,h\}$ must equal $\{\alpha g\alpha^{-1},h\}$.

As another application, note that if $m=n$ the inner automorphism of $E$ induced by $R^m\otimes R^n\cong R^n\otimes R^m$ sends $\{h,g\}$ to the image of $\{y,x\}$ under $K_2(A)\to K_2(E)$. This proves skew-symmetry, since $\{y,x\}=\{x,y\}^{-1}$.

THEOREM 5.12.2. *For every commutative ring $R$, there is a skew-symmetric bilinear pairing $K_1(R) \otimes K_1(R) \to K_2(R)$ induced by the symbol $\{g, h\}$.*

PROOF. We first show that the symbol is bimultiplicative when $g$ and $g'$ commute in $GL_m(R)$. Mapping $A[z, z^{-1}]$ into $E$ by $z \mapsto g' \otimes 1_n$ allows us to deduce $\{gg', h\} = \{g, h\}\{g', h\}$ from the corresponding property of Steinberg symbols. If $g$ and $g'$ do not commute, the following trick establishes bimultiplicativity:

$$\{gg', h\} = \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g' \end{pmatrix}, h \right\} = \left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, h \right\} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & g' \end{pmatrix}, h \right\} = \{g, h\}\{g', h\}.$$

If either $g$ or $h$ is a commutator, this implies that the symbol $\{g, h\}$ vanishes in the abelian group $K_2(R)$. Since the symbol $\{g, h\}$ is compatible with stabilization, it describes a function $K_1(R) \times K_1(R) \to K_2(R)$ which is multiplicative in each entry: $\{gg', h\} = \{g, h\}\{g', h\}$. If we write $K_1$ and $K_2$ additively the function is additive in each entry, i.e., bilinear.

EXERCISES

**5.1** *Relative Steinberg groups.* Let $I$ be an ideal in a ring $R$. Show that there is an exact sequence $St(R, I) \xrightarrow{add} St(R) \to St(R/I) \to 1$.

**5.2** Consider the function $\rho_n \colon R^{n-1} \to St_n(R)$ sending $(r_1, ..., r_{n-1})$ to the product $x_{1n}(r_1)x_{2n}(r_2) \cdots x_{n-1,n}(r_{n-1})$. The Steinberg relations show that this is a group homomorphism.

Show that $\rho$ is an injection by showing that the composite $\phi\rho \colon R^{n-1} \to St_n(R) \to GL_n(R)$ is an injection. Then show that the elements $x_{ij}(r)$ with $i, j < n$ normalize the subgroup $P_n = \rho(R^n)$ of $St_n(R)$, *i.e.*, that $x_{ij}(r)P_n x_{ij}(-r) = P_n$.

Use this and induction to show that the subgroup $T_n$ of $St_n(R)$ generated by the $x_{ij}(r)$ with $i < j$ maps isomorphically onto the subgroup of lower triangular matrices in $GL_n(R)$.

**5.3** Let $G$ be a group whose commutator group $[G, G]$ is abelian. Prove that the Jacobi identity holds for every $u, v, w \in G$:

$$[u, [v, w]][v, [w, u]][w, [u, v]] = 1.$$

If in addition $[u, w] = 1$ this implies that $[[u, v], w] = [u, [v, w]]$.

**5.4** *Product with $K_0$.* Construct a product operation $K_0(R) \otimes K_2(A) \to K_2(A)$, assuming that $R$ is commutative and $A$ is an associative $R$-algebra. To do this, fix a finitely generated projective $R$-module $P$. Each isomorphism $P \oplus Q = R^n$ gives rise to a homomorphism $h^P \colon GL_m(A) \to GL_{mn}(A) \subset GL(A)$ sending $\alpha$ to $\alpha \otimes 1$ and $E_m(A)$ to $E(A)$. Show that $h^P$ is well-defined up to conjugation by an element of $E(A)$. Since conjugation acts trivially on homology, this implies that the induced map $h^P{}_* \colon H_2(E_m(A); \mathbb{Z}) \to H_2(E(A); \mathbb{Z}) = K_2(A)$ is well-defined. Then show that $h^{P \oplus Q}{}_* = h^P{}_* \oplus h^Q{}_*$ and pass to the limit as $m \to \infty$ to obtain the required endomorphism $[P]\cdot$ of $K_2(A)$.

**5.5** If $R$ is commutative and $P \in \mathbf{P}(R)$, show that $Q \mapsto Q \otimes_R P$ defines a functor from the translation category $t\mathbf{P}(A)$ to itself for every $R$-algebra $A$, and that the resulting endomorphism of $K_2(A) = \varinjlim H_2([\mathrm{Aut}(Q), \mathrm{Aut}(Q)])$ is the map $h_*^P$ of the previous exercise. Use this description to show that the product makes $K_2(A)$ into a module over the ring $K_0(R)$.

**5.6** *Projection Formula.* Suppose that $f: R \to S$ is a finite map of commutative rings, with $S \in \mathbf{P}(R)$. Show that for all $r \in K_i(R)$ and $s \in K_j(S)$ with $i + j = 2$ we have $f_*(f^*(r) \cdot s) = r \cdot f_*(s)$ in $K_2(R)$. The case $i = 0$ states that the transfer $f_*: K_2(S) \to K_2(R)$ is $K_0(R)$-linear, while the case $i = 1$ yields the useful formula $f_*\{r, s\} = \{r, Ns\}$ for Steinberg symbols in $K_2(R)$, where $r \in R^\times$, $s \in S^\times$ and $Ns = f_*(s) \in R^\times$ is the norm of $s$.

**5.7** If $Y \xrightarrow{\rho} X$ and $X \xrightarrow{\pi} G$ are central extensions, show that the "composition" $Y \xrightarrow{\pi\rho} G$ is also a central extension. If $X$ is a universal central extension of $G$, conclude that every central extension $Y \xrightarrow{\rho} X$ splits.

**5.8** Show that the following identies hold in $St(R)$ (for $i$, $j$ and $k$ distinct).

(a) $w_{ij}(r)w_{ij}(-r) = 1$;
(b) $w_{ik}(r)x_{ij}(s)w_{ik}(-r) = x_{kj}(-r^{-1}s)$;
(c) $w_{ij}(r)x_{ij}(s)w_{ij}(-r) = x_{ji}(-r^{-1}sr^{-1})$;
(d) $w_{ij}(r)x_{ji}(s)w_{ij}(-r) = x_{ij}(-rsr)$;
(e) $w_{ij}(r)w_{ji}(r^{-1}) = 1$;

**5.9** Use the previous exercise to show that $\{r, s\} = h_{ij}(rs)h_{ij}(s)^{-1}h_{ij}(r)^{-1}$. *Hint:* Conjugate $h_{ij}(s)$ by $w_{ik}(r)w_{ik}(-1)$.

**5.10** *Excision.* If $I$ and $J$ are ideals in a ring $R$ with $I \cap J = 0$, we may also consider $I$ as an ideal of $R/J$. Show that $St(R, I)$ surjects onto $St(R/J, I)$, while the subgroups $E(R, I)$ and $E(R/J, I)$ of $GL(I)$ are equal. Use the 5-lemma to conclude that $K_1(R, I) \cong K_1(R/J, I)$ and that $K_2(R, I) \to K_2(R/J, I)$ is onto.

In fact, the sequence $I/I^2 \otimes_{R\otimes R} J/J^2 \to K_2(R, I) \to K_2(R/J, I) \to 0$ is exact, where the first map sends $x \otimes y$ to $\langle x, y\rangle$; see [Swan71].

**5.11** *Dennis-Stein symbols.* Let $\langle r, s\rangle_{ij}$ denote the element of $St(R)$ given in Definition 5.11. Show that this element is in $K_2(R)$. Then use Ex. 5.8 to show that if $w = w_{ik}(1)w_{j\ell}(1)w_{k\ell}^2(1)$ (so that $\phi(w)$ is the permutation matrix sending $i, j$ to $k, \ell$) the permutation matrix send $i, j$ to $k, \ell$ then $w\,\langle r, s\rangle_{ij}\, w^{-1} = \langle r, s\rangle_{k\ell}$. This shows that the Dennis-Stein symbol is independent of the choice of indices $i, j$.

**5.12** Let $A$ be an abelian group and $F$ a field. Show that, for all $n \geq 5$, homomorphisms $K_2(F) \xrightarrow{c} A$ are in 1–1 correspondence with central extensions of $SL_n(F)$ having kernel $A$.

**5.13** If $p$ is an odd prime, use Theorem 5.11.1 to show that $K_2(\mathbb{Z}/p^n) = 1$. If $n \geq 2$, show that $K_2(\mathbb{Z}/2^n) \cong K_2(\mathbb{Z}/4) \cong \{\pm 1\}$ on $\{-1, -1\} = \langle -1, -2\rangle = \langle 2, 2\rangle$.

Using the Mayer-Vietoris sequence 5.8, conclude that $K_1(R) = R^\times = \{\pm 1\}$ for the ring $R = \mathbb{Z}[x]/(x^2 - p^{2n})$. Note that $R/(x \pm p^n)R = \mathbb{Z}$.

**5.14** Let $R$ be a commutative ring, and let $\Omega_R$ denote the module of Kähler differentials of $R$ over $\mathbb{Z}$, as in Ex. 2.6.

(a) If $I$ is a radical ideal of $R$, show that there is a surjection from $K_2(R, I)$ onto $I \otimes_R \Omega_{R/I}$, sending $\langle x, r\rangle$ to $x \otimes dr$ ($r \in R$, $x \in I$).
(b) If $I^2 = 0$, show that the kernel of the map in (a) is generated by the Dennis-Stein symbols $\langle x, y\rangle$ with $x, y \in I$.

(c) (Van der Kallen) The *dual numbers* over $R$ is the ring $R[\varepsilon]$ with $\varepsilon^2 = 0$. If $\frac{1}{2} \in R$, show that the map $K_2(R[\varepsilon], \varepsilon) \to \Omega_R$ of part (a) is an isomorphism.

(d) Let $k$ be a field. Show that the group $K_2(k[[t]], t)$ is $\ell$-divisible for every $\ell$ invertible in $k$. If $\text{char}(K) = p > 0$, show that this group is *not* $p$-divisible.

**5.15** Assume the fact that $\mathbb{Z}/2$ is $K_2$-regular (see 3.4 and chapter V). Show that:

(a) $K_2(\mathbb{Z}/4[x])$ is an elementary abelian 2-group with basis $\langle 2, 2\rangle$, $\langle 2x^n, x\rangle$, and $\langle 2x^{2n+1}, 2\rangle$, $n \geq 0$. *Hint:* Split the map $K_2(A, 2) \to \Omega_{A/2} \cong \mathbb{Z}/2[x]$ of Ex. 5.14 and use $0 = \langle 2f, 1\rangle = \langle 2(f + f^2), 2\rangle$.

(b) The group $K_2(\mathbb{Z}/4[x, y])$ is an elementary abelian 2-group with basis $\langle 2, 2\rangle$, $\langle 2x^m y^n, x\rangle$, $\langle 2x^m y^n, y\rangle$ $(m, n \geq 0)$ and $\langle 2x^m y^n, x\rangle$ (one of $m, n$ odd).

(c) Consider the maps $\partial_1 : NK_2(\mathbb{Z}/4) \to K_2(\mathbb{Z}/4)$ and $\partial_2 : N^2K_2(\mathbb{Z}/4) \to NK_2(\mathbb{Z}/4)$ induced by the maps $\mathbb{Z}/4[x] \to \mathbb{Z}/4$ sending $x$ to 1, and $\mathbb{Z}/4[x, y] \to \mathbb{Z}/4[x]$ sending $y$ to $1 - x$, respectively. Show that the following sequence is exact: $N^2K_2(\mathbb{Z}/4) \to NK_2(\mathbb{Z}/4) \to K_2(\mathbb{Z}/4) \to 0$.

## §6. $K_2$ of fields

The following theorem was proven by Hideya Matsumoto in 1969. We refer the reader to [Milnor, §12] for a self-contained proof.

MATSUMOTO'S THEOREM 6.1. *If $F$ is a field then $K_2(F)$ is the abelian group generated by the set of Steinberg symbols $\{x, y\}$ with $x, y \in F^\times$, subject only to the relations:*

(1) *(Bilinearity)* $\{xx', y\} = \{x, y\}\{x', y\}$ *and* $\{x, yy'\} = \{x, y\}\{x, y'\}$*;*
(2) *(Steinberg Relation)* $\{x, 1 - x\} = 1$ *for all* $x \neq 0, 1$*.*

In other words, $K_2(F)$ is the quotient of $F^\times \otimes F^\times$ by the subgroup generated by the elements $x \otimes (1 - x)$. Note that the calculation (5.10.3) implies that $\{x, -x\} = 1$ for all $x$, and this implies that the Steinberg symbols are skew-symmetric: $\{x, y\}\{y, x\} = \{x, -xy\}\{y, -xy\} = \{xy, -xy\} = 1$.

COROLLARY 6.1.1. $K_2(\mathbb{F}_q) = 1$ *for every finite field* $\mathbb{F}_q$*.*

PROOF. If $x$ generates the cyclic group $\mathbb{F}_q^\times$, we must show that the generator $x \otimes x$ of the cyclic group $\mathbb{F}_q^\times \otimes \mathbb{F}_q^\times$ vanishes in $K_2$. If $q$ is even, then $\{x, x\} = \{x, -x\} = 1$, so we may suppose that $q$ is odd. Since $\{x, x\}^2 = 1$ by skew-symmetry, we have $\{x, x\} = \{x, x\}^{mn} = \{x^m, x^n\}$ for every odd $m$ and $n$. Since odd powers of $x$ are the same as non-squares, it suffices to find a non-square $u$ such that $1 - u$ is also a non-square. But such a $u$ exists because $u \mapsto (1 - u)$ is an involution on the set $\mathbb{F}_q - \{0, 1\}$, and this set consists of $(q-1)/2$ non-squares but only $(q-3)/2$ squares.

EXAMPLE 6.1.2. Let $F(t)$ be a rational function field in one variable $t$ over $F$. Then $K_2(F)$ is a direct summand of $K_2F(t)$.

To see this, we construct a map $\lambda\colon K_2F(t) \to K_2(F)$ inverse to the natural map $K_2(F) \to K_2F(t)$. To this end, we define the *leading coefficient* of the rational function $f(t) = (a_0t^n + \cdots + a_n)/(b_0t^m + \cdots + b_m)$ to be $\text{lead}(f) = a_0/b_0$ and set $\lambda(\{f, g\}) = \{\text{lead}(f), \text{lead}(g)\}$. To see that this defines a homomorphism $K_2F(t) \to K_2(F)$, we check the presentation in Matsumoto's Theorem. Bilinearity is clear from $\text{lead}(f_1f_2) = \text{lead}(f_1)\text{lead}(f_2)$, and $\{\text{lead}(f), \text{lead}(1 - f)\} = 1$ holds in $K_2(F)$ because $\text{lead}(1-f)$ is either $1$, $1-\text{lead}(f)$ or $-\text{lead}(f)$, according to whether $m > n$, $m = n$ or $m < n$.

Because $K_2$ commutes with filtered colimits, it follows that $K_2(F)$ injects into $K_2F(T)$ for *every* purely transcendental extension $F(T)$ of $F$.

LEMMA 6.1.3. *For every field extension $F \subset E$, the kernel of $K_2(F) \to K_2(E)$ is a torsion subgroup.*

PROOF. $E$ is an algebraic extension of some purely transcendental extension $F(X)$ of $F$, and $K_2(F)$ injects into $K_2F(X)$ by Example 6.1.2. Thus we may assume that $E$ is algebraic over $F$. Since $E$ is the filtered union of finite extensions, we may even assume that $E/F$ is a finite field extension. But in this case the result holds because (by 5.6.3) the composite $K_2(F) \to K_2(E) \to K_2(F)$ is multiplication by the integer $[E : F]$.

The next result is useful for manipulations with symbols.

LEMMA 6.1.4 (BASS-TATE). *If $E = F(u)$ is a field extension of $F$, then every symbol of the form $\{b_1u - a_1, b_2u - a_2\}$ $(a_i, b_i \in F)$ is a product of symbols $\{c_i, d_i\}$ and $\{c_i, u - d_i\}$ with $c_i, d_i \in F$.*

PROOF. Bilinearity allows us to assume that $b_1 = b_2 = 1$. Set $x = u - a_1$, $y = u - a_2$ and $a = a_2 - a_1$, so $x = a + y$. Then $1 = \frac{a}{x} + \frac{y}{x}$ yields the relation $1 = \{\frac{a}{x}, \frac{y}{x}\}$. Using $\{x, x\} = \{-1, x\}$, this expands to the desired expression: $\{x, y\} = \{a, y\}\{-1, x\}\{a^{-1}, x\}$.

Together with the Projection Formula (Ex. 5.6), this yields:

COROLLARY 6.1.5. *If $E = F(u)$ is a quadratic field extension of $F$, then $K_2(E)$ is generated by elements coming from $K_2(F)$, together with elements of the form $\{c, u - d\}$. Thus the transfer map $N_{E/F}: K_2(E) \to K_2(F)$ is completely determined by the formulas $N_{E/F}\{c, d\} = \{c, d\}^2$, $N_{E/F}\{c, u - d\} = \{c, N(u - d)\}$ $(c, d \in F)$*

EXAMPLE 6.1.6. Since $\mathbb{C}$ is a quadratic extension of $\mathbb{R}$, every element of $K_2(\mathbb{C})$ is a product of symbols $\{r, s\}$ and $\{r, e^{i\theta}\}$ with $r, s, \theta \in \mathbb{R}$. Moreover, $N\{r, e^{i\theta}\} = 1$ in $K_2(\mathbb{R})$. Under the automorphism of $K_2(\mathbb{C})$ induced by complex conjugation, the symbols of the first kind are fixed and the symbols of the second kind are sent to their inverses. We will see in Theorem 6.4 below that $K_2(\mathbb{C})$ is uniquely divisible, *i.e.*, a vector space over $\mathbb{Q}$, and the decomposition of $K_2(\mathbb{C})$ into eigenspaces for $\pm 1$ corresponds to symbols of the first and second kind.

EXAMPLE 6.1.7. Let $F$ be an algebraically closed field. By Lemma 6.1.4, $K_2F(t)$ is generated by linear symbols of the form $\{a, b\}$ and $\{t - a, b\}$. It will follow from 6.5.2 below that every element $u$ of $K_2F(t)$ uniquely determines finitely many elements $a_i \in F$, $b_i \in F^\times$ so that $u = \lambda(u) \prod \{t - a_i, b_i\}$, where $\lambda(u) \in K_2(F)$ was described in Example 6.1.2.

*Steinberg symbols*

DEFINITION 6.2. A *Steinberg symbol* on a field $F$ with values in a multiplicative abelian group $A$ is a bilinear map $c: F^\times \otimes F^\times \to A$ satisfying $c(r, 1 - r) = 1$. By Matsumoto's Theorem, these are in 1–1 correspondence with homomorphisms $K_2(F) \xrightarrow{c} A$.

EXAMPLE 6.2.1. There is a Steinberg symbol $(x, y)_\infty$ on the field $\mathbb{R}$ with values in the group $\{\pm 1\}$. Define $(x, y)_\infty$ to be: $-1$ if both $x$ and $y$ are negative, and $+1$ otherwise. The Steinberg relation $(x, 1 - x)_\infty = +1$ holds because $x$ and $1 - x$ cannot be negative at the same time. The resulting map $K_2(\mathbb{R}) \to \{\pm 1\}$ is onto because $(-1, -1)_\infty = -1$. This shows that the symbol $\{-1, -1\}$ in $K_2(\mathbb{Z})$ is nontrivial, as promised in 5.2.2, and even shows that $K_2(\mathbb{Z})$ is a direct summand in $K_2(\mathbb{R})$.

For our next two examples, recall that a *local field* is a field $F$ which is complete under a discrete valuation $v$, and whose residue field $k_v$ is finite. Classically, every local field is either a finite extension of the $p$-adic rationals $\hat{\mathbb{Q}}_p$ or of $\mathbb{F}_p((t))$.

EXAMPLE 6.2.2 (HILBERT SYMBOLS). Let $F$ be a local field containing $\frac{1}{2}$. The Hilbert (quadratic residue) symbol on $F$ is defined by setting $c_F(r, s) \in \{\pm 1\}$ equal to $+1$ or $-1$, depending on whether or not the equation $rx^2 + sy^2 = 1$ has a solution in $F$. Bilinearity is classical when $F$ is local; see [OMeara, p.164]. The Steinberg relation is trivial, because $x = y = 1$ is always a solution when $r + s = 1$.

Of course, the definition of $c_F(r, s)$ makes sense for any field of characteristic $\neq 2$, but it will not always be a Steinberg symbol because it can fail to be bilinear in $r$. It is a Steinberg symbol when $F = \mathbb{R}$, because the Hilbert symbol $c_{\mathbb{R}}(r, s)$ is the same as the symbol $(r, s)_\infty$ of the previous example.

Example 6.2.3 (norm residue symbols). The roots of unity in a local field $F$ form a finite cyclic group $\mu$, equal to the group $\mu_m$ of all $m^{th}$ roots of unity for some integer $m$ with $\frac{1}{m} \in F$. The classical $m^{th}$ *power norm residue symbol* is a map $K_2(F) \to \mu_m$ defined as follows (see [S-LF] for more details).

Because $F^{\times m}$ has finite index in $F^\times$, there is a finite "Kummer" extension $K$ containing the $m^{th}$ roots of every element of $F$. The Galois group $G_F = Gal(K/F)$ is canonically isomorphic to $\mathrm{Hom}(F^\times, \mu_m)$, with the automorphism $g$ of $K$ corresponding to the homomorphism $\zeta \colon F^\times \to \mu_m$ sending $a \in F^\times$ to $\zeta(a) = g(x)/x$, where $x^m = a$. In addition, the cokernel of the norm map $K^\times \xrightarrow{N} F^\times$ is isomorphic to $G_F$ by the "norm residue" isomorphism of local class field theory. The composite $F^\times \to F^\times/NK^\times \cong G_F \cong \mathrm{Hom}(F^\times, \mu_m)$, written as $x \mapsto (x, -)_F$, is adjoint to a nondegenerate bilinear map $(\ ,\ )_F \colon F^\times \otimes F^\times \to \mu_m$.

The Steinberg identity $(a, 1-a)_F = 1$ is proven by noting that $(1-a)$ is a norm from the intermediate field $E = F(x)$, $x^m = a$. Since $G_E \subset G_F$ corresponds to the norm map $E^\times/N_{K/E}K^\times \hookrightarrow F^\times/N_{K/F}K^\times$, the element $g$ of $G_F = Gal(K/F)$ corresponding to the map $\zeta(a) = (a, 1-a)_F$ from $F^\times$ to $\mu_m$ must belong to $G_E$, *i.e.*, $\zeta$ must extend to a map $E^\times \to \mu_m$. But then $(a, 1-a)_F = \zeta(a) = \zeta(x)^m = 1$.

The name "norm residue" comes from the fact that for each $x$, the map $y \mapsto \{x, y\}$ is trivial if and only if $x \in NK^\times$. Since a primitive $m^{th}$ root of unity $\zeta$ is not a norm from $K$, it follows that there is an $x \in F$ such that $(\zeta, x)_F \neq 1$. Therefore the norm residue symbol is a split surjection with inverse $\zeta^i \mapsto \{\zeta^i, x\}$.

The role of the norm residue symbol is explained by the following structural result, whose proof we cite from the literature.

Moore's Theorem 6.2.4. *If $F$ is a local field, then $K_2(F)$ is the direct sum of a uniquely divisible abelian group $U$ and a finite cyclic group, isomorphic under the norm residue symbol to the group $\mu = \mu_m$ of roots of unity in $F$.*

Proof. We have seen that the norm residue symbol is a split surjection. A proof that its kernel $U$ is divisible, due to C. Moore, is given in the Appendix to [Milnor]. The fact that $U$ is torsionfree (hence uniquely divisible) was proven by Tate [Tate] when $\mathrm{char}(F) = p$, and by Merkurjev [Merk] when $\mathrm{char}(F) = 0$.

Example 6.2.5 (2-adic rationals). The group $K_2(\hat{\mathbb{Q}}_2)$ is the direct sum of the cyclic group of order 2 generated by $\{-1, -1\}$ and a uniquely divisible group. Since $x^2 + y^2 = -1$ has no solution in $F = \hat{\mathbb{Q}}_2$ we see from definition (6.2.2) that the Hilbert symbol $c_F(-1, -1) = -1$.

*Tame symbols*

Every discrete valuation $v$ on a field $F$ provides a Steinberg symbol. Recall that $v$ is a homomorphism $F^\times \to \mathbb{Z}$ such that $v(r+s) \geq \min\{v(r), v(s)\}$. By convention,

$v(0) = \infty$, so that the ring $R$ of all $r$ with $v(r) \geq 0$ is a discrete valuation ring (DVR). The units $R^\times$ form the set $v^{-1}(0)$, and the maximal ideal of $R$ is generated by any $\pi \in R$ with $v(\pi) = 1$. The residue field $k_v$ is defined to be $R/(\pi)$. If $u \in R$, we write $\bar{u}$ for the image of $u$ under $R \to k_v$.

LEMMA 6.3. *For every discrete valuation $v$ on $F$ there is a Steinberg symbol $K_2(F) \xrightarrow{\partial_v} k_v^\times$, defined by*

$$\partial_v(\{r, s\}) = (-1)^{v(r)v(s)} \overline{\left(\frac{s^{v(r)}}{r^{v(s)}}\right)}.$$

*This symbol is called the tame symbol of the valuation $v$. The tame symbol is onto, because if $u \in R^\times$ then $v(u) = 0$ and $\partial_v(\pi, u) = \bar{u}$.*

PROOF. Writing $r = u_1\pi^{v_1}$ and $s = u_2\pi^{v_2}$ with $u_1, u_2 \in R^\times$, we must show that $\partial_v(r, s) = (-1)^{v_1 v_2} \frac{\bar{u}_2^{v_1}}{\bar{u}_1^{v_2}}$ is a Steinberg symbol. By inspection, $\partial_v(r, s)$ is an element of $k_v^\times$, and $\partial_v$ is bilinear. To see that $\partial_v(r, s) = 1$ when $r + s = 1$ we consider several cases. If $v_1 > 0$ then $r$ is in the maximal ideal, so $s = 1 - r$ is a unit and $\partial_v(r, s) = \bar{s}^{v_1} = 1$. The proof when $v_2 > 0$ is the same, and the case $v_1 = v_2 = 0$ is trivial. If $v_1 < 0$ then $v(\frac{1}{r}) > 0$ and $\frac{1-r}{r} = -1 + \frac{1}{r}$ is congruent to $-1 \pmod{\pi}$. Since $v(r) = v(1 - r)$, we have

$$\partial_v(r, 1 - r) = (-1)^{v_1} \left(\frac{1 - r}{r}\right)^{v_1} = (-1)^{v_1}(-1)^{v_1} = 1.$$

RAMIFICATION 6.3.1. Suppose that $E$ is a finite extension of $F$, and that $w$ is a valuation on $E$ over the valuation $v$ on $F$. Then there is an integer $e$, called the *ramification index*, such that $w(r) = e \cdot v(r)$ for every $r \in F$. The natural map $K_2(F) \to K_2(E)$ is compatible with the tame symbols in the sense that for every $r_1, r_2 \in F^\times$ we have $\partial_w(r_1, r_2) = \partial_v(r_1, r_2)^e$ in $k_w^\times$.

$$\begin{array}{ccc} K_2(F) & \xrightarrow{\partial_v} & k_v^\times \\ \downarrow & & e\downarrow x\mapsto x^e \\ K_2(E) & \xrightarrow{\partial_w} & k_w^\times \end{array}$$

Let $S$ denote the integral closure of $R$ in $E$. Then $S$ has finitely many prime ideals $\mathfrak{p}_1, ..., \mathfrak{p}_n$ lying over $\mathfrak{p}$, with corresponding valuations $w_1, ..., w_n$ on $E$. We say that $S$ is *unramified* over $R$ if the ramification indices $e_1, ..., e_n$ are all 1; in this case the diagonal inclusion $\Delta\colon k_v^\times \hookrightarrow \prod_i k_{w_i}^\times$ is compatible with the tame symbols in the sense that $\Delta\partial_v(r_1, r_2)$ is the product of the $\partial_{w_i}(r_1, r_2)$.

COROLLARY 6.3.2. *If $F$ contains the rational function field $\mathbb{Q}(t)$ or $\mathbb{F}_p(t_1, t_2)$, then $K_2(F)$ has the same cardinality as $F$. In particular, if $F$ is uncountable then so is $K_2(F)$.*

PROOF. By hypothesis, $F$ contains a transcendental element $t$. Choose a subset $X = \{x_\alpha\}$ of $F$ so that $X \cup \{t\}$ is a transcendence basis for $F$ over its ground field $F_0$, and set $k = F_0(X)$. Then the subfield $k(t)$ of $F$ has a $t$-adic valuation with residue class field $k$. Hence $K_2(k(t))$ contains a subgroup $\{t, k^\times\}$ mapped isomorphically under the tame symbol to $k^\times$. By Lemma 6.1.3, the kernel of $k^\times \to K_2(k(t)) \to K_2(F)$ is contained in the torsion subgroup $\mu(k)$ of roots of unity in $k$. Thus the cardinality of $K_2(F)$ is bounded below by the cardinality of $k^\times/\mu(k)$. Since $F$ is an algebraic extension of $k(t)$, and $k$ contains either $\mathbb{Q}$ or $\mathbb{F}_p(t_2)$, we have the inequality $|F| = |k| = |k^\times/\mu(k)| \le |K_2(F)|$. The other inequality $|K_2(F)| \le |F|$ is immediate from Matsumoto's Theorem, since $F$ is infinite.

Theorem 6.4 (Bass-Tate). *When $F$ is an algebraically closed field, $K_2(F)$ is a uniquely divisible abelian group.*

Theorem 6.4 is an immediate consequence of proposition 6.4.1 below. To see this, recall that an abelian group is uniquely divisible when it is uniquely $p$-divisible for each prime $p$; a group is said to be *uniquely $p$-divisible* if it is $p$-divisible and has no $p$-torsion.

Proposition 6.4.1 (Bass-Tate). *Let $p$ be a prime number such that each polynomial $t^p - a$ ($a \in F$) splits in $F[t]$ into linear factors. Then $K_2(F)$ is uniquely $p$-divisible.*

Proof. The hypothesis implies that $F^\times$ is $p$-divisible. Since the tensor product of $p$-divisible abelian groups is always uniquely $p$-divisible, $F^\times \otimes F^\times$ is uniquely $p$-divisible. Let $R$ denote the kernel of the natural surjection $F^\times \otimes F^\times \to K_2(F)$. By inspection (or by the Snake Lemma), $K_2(F)$ is $p$-divisible and the $p$-torsion subgroup of $K_2(F)$ is isomorphic to $R/pR$.

Therefore it suffices to prove that $R$ is $p$-divisible. Now $R$ is generated by the elements $\psi(a) = (a)\otimes(1-a)$ of $F^\times \otimes F^\times$ ($a \in F - \{0,1\}$), so it suffices to show that each $\psi(a)$ is in $pR$. By hypothesis, there are $b_i \in F$ such that $t^p - a = \prod(t - b_i)$ in $F[t]$, so $1 - a = \prod(1 - b_i)$ and $b_i^p = a$ for each $i$. But then we compute in $F^\times \otimes F^\times$:

$$\psi(a) = (a) \otimes (1-a) = \sum (a) \otimes (1 - b_i) = \sum (b_i)^p \otimes (1 - b_i) = p \sum \psi(b_i).$$

Corollary 6.4.2. *If $F$ is a perfect field of characteristic $p$, then $K_2(F)$ is uniquely $p$-divisible.*

*The Localization Sequence for $K_2$*

The following result will be proven in chapter V, but we find it useful to quote this result now. If $\mathfrak{p}$ is a nonzero prime ideal of a Dedekind domain $R$, the local ring $R_\mathfrak{p}$ is a discrete valuation ring, and hence determines a tame symbol.

Localization Theorem 6.5. *Let $R$ be a Dedekind domain with field of fractions $F$. Then the tame symbols $K_2(F) \xrightarrow{\partial_\mathfrak{p}} (R/\mathfrak{p})^\times$ associated to the prime ideals of $R$ fit into a long exact sequence*

$$\coprod_\mathfrak{p} K_2(R/\mathfrak{p}) \to K_2(R) \to K_2(F) \xrightarrow{\partial = \coprod \partial_\mathfrak{p}} \coprod_\mathfrak{p} (R/\mathfrak{p})^\times \to SK_1(R) \to 1$$

*where the coproducts are over all nonzero prime ideals $\mathfrak{p}$ of $R$, and the maps from $(R/\mathfrak{p})^\times = K_1(R/\mathfrak{p})$ to $SK_1(R)$ are the transfer maps of Ex. 1.11. The transfer maps $K_2(R/\mathfrak{p}) \to K_2(R)$ will be defined in chapter V.*

APPLICATION 6.5.1 ($K_2\mathbb{Q}$). If $R = \mathbb{Z}$ then, since $K_2(\mathbb{Z}/p) = 1$ and $SK_1(\mathbb{Z}) = 1$, we have an exact sequence $1 \to K_2(\mathbb{Z}) \to K_2(\mathbb{Q}) \xrightarrow{\partial} \coprod \mathbb{F}_p^\times \to 1$. As noted in Example 6.2.1, this sequence is split by the symbol $(r,s)_\infty$, so we have $K_2(\mathbb{Q}) \cong K_2(\mathbb{Z}) \oplus \coprod \mathbb{F}_p^\times$.

APPLICATION 6.5.2 (FUNCTION FIELDS). If $R$ is the polynomial ring $F[t]$ for some field $F$, we know that $K_2(F[t]) = K_2(F)$ (see 5.2.3). Moreover, the natural map $K_2(F) \to K_2F(t)$ is split by the leading coefficient symbol $\lambda$ of Example 6.1.2. Therefore we have a split exact sequence

$$1 \to K_2(F) \to K_2F(t) \xrightarrow{\partial} \coprod_{\mathfrak{p}} (F[t]/\mathfrak{p})^\times \to 1.$$

WEIL'S RECIPROCITY FORMULA 6.5.3. Just as in the case $R = \mathbb{Z}$, there is a valuation on $F(t)$ not arising from a prime ideal of $F[t]$. In this case, it is the valuation $v_\infty(f) = -deg(f)$ associated with the point at infinity, *i.e.*, with parameter $t^{-1}$. Since the symbol $(f,g)_\infty$ vanishes on $K_2(F)$, it must be expressable in terms of the tame symbols $\partial_{\mathfrak{p}}(f,g) = (f,g)_{\mathfrak{p}}$. The appropriate reciprocity formula first appeared in Weil's 1940 paper on the Riemann Hypothesis for curves:

$$(f,g)_\infty \cdot \prod_{\mathfrak{p}} N_{\mathfrak{p}}(f,g)_{\mathfrak{p}} = 1 \qquad \text{in } F^\times.$$

In Weil's formula "$N_{\mathfrak{p}}$" denotes the usual norm map $(F[t]/\mathfrak{p})^\times \to F^\times$. To establish this reciprocity formula, we observe that $K_2F(t)/K_2F = \coprod (F[t]/\mathfrak{p})^\times$ injects into $K_2\bar{F}(t)/K_2\bar{F}$, where $\bar{F}$ is the algebraic closure of $F$. Thus we may assume that $F$ is algebraically closed. By Example 6.1.7, $K_2F(t)$ is generated by linear symbols of the form $\{a, t-b\}$. But $(a, t-b)_\infty = a$ and $\partial_{t-b}(a, t-b) = a^{-1}$, so the formula is clear.

Our next structural result was discovered by Merkurjev and Suslin in 1981, and published in their landmark paper [MS]; see [GSz, 8.4]. Recall that an automorphism $\sigma$ of a field $E$ induces an automorphism of $K_2(E)$ sending $\{x,y\}$ to $\{\sigma x, \sigma y\}$.

THEOREM 6.6 (HILBERT'S THEOREM 90 FOR $K_2$). *Let $E/F$ be a cyclic Galois field extension of prime degree $p$, and let $\sigma$ be a generator of $Gal(E/F)$. Then the following sequence is exact, where $N$ denotes the transfer map on $K_2$:*

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N} K_2(F).$$

Merkurjev and Suslin gave this result the suggestive name "Hilbert's Theorem 90 for $K_2$," because of its formal similarity to the following result, which is universally called "Hilbert's Theorem 90 (for units)" because it was the $90^{th}$ theorem in Hibert's classical 1897 survey of algebraic number theory, *Theorie der Algebraische Zahlkörper.*

THEOREM 6.6.1 (HILBERT'S THEOREM 90 FOR UNITS). *Let $E/F$ be a cyclic Galois field extension, and let $\sigma$ be a generator of $Gal(E/F)$. If $1-\sigma$ denotes the map $a \mapsto a/\sigma(a)$, then the following sequence is exact:*

$$1 \to F^\times \to E^\times \xrightarrow{1-\sigma} E^\times \xrightarrow{N} F^\times.$$

We omit the proof of Hilbert's Theorem 90 for $K_2$ (and for $K_n^M$; see 7.8.4 below), since the proof does not involve $K$-theory, contenting ourselves with two special cases: when $n = \mathrm{char}(F)$ (7.8.3) and the following special case.

PROPOSITION 6.6.2. *Let $F$ be a field containing a primitive $n^{th}$ root of unity $\zeta$, and let $E$ be a cyclic field extension of degree $n$, with $\sigma$ a generator of $Gal(E/F)$.*

*Suppose in addition that the norm map $E^\times \xrightarrow{N} F^\times$ is onto, and that $F$ has no extension fields of degree $< n$. Then the following sequence is exact:*

$$K_2(E) \xrightarrow{1-\sigma} K_2(E) \xrightarrow{N} K_2(F) \to 1.$$

PROOF. Since $N\zeta = 1$, Hilbert's Theorem 90 gives an $r \in E$ with $\sigma(r) = \zeta r$. Setting $c = N(r) \in F$, it is well-known and easy to see that $E = F(r)$, $r^n = c$.

Again by Hilbert's Theorem 90 for units and our assumption about norms, $E^\times \xrightarrow{1-\sigma} E^\times \xrightarrow{N} F^\times \to 1$ is an exact sequence of abelian groups. Applying the right exact functor $\otimes F^\times$ retains exactness. Therefore we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
E^\times \otimes F^\times & \xrightarrow{(1-\sigma)\otimes 1} & E^\times \otimes F^\times & \xrightarrow{N\otimes 1} & F^\times \otimes F^\times & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\
K_2(E) & \xrightarrow{1-\sigma} & K_2(E) & \longrightarrow & C & \longrightarrow & 1
\end{array}$$

in which $C$ denotes the cokernel of $1 - \sigma$.

Now every element of $E$ is a polynomial $f(r)$ in $r$ of degree $< n$, and $f(t)$ is a product of linear terms $b_i t - a_i$ by our assumption. By Lemma 6.1.4, every element of $K_2(E)$ is a product of symbols of the form $\{a, b\}$ and $\{a, r - b\}$. Therefore the vertical maps $F^\times \otimes E^\times \to K_2(E)$ are onto in the above diagram. Hence $\gamma$ is onto.

If $a \in F^\times$ and $x \in E^\times$ then the projection formula (Ex. 5.6) yields

$$N(1-\sigma)\{a, x\} = N\{a, x/(\sigma x)\} = \{a, Nx/N(\sigma x)\} = 1.$$

Hence the transfer map $K_2(E) \to K_2(F)$ factors through $C$. A diagram chase shows that it suffices to show that $\gamma$ is a Steinberg symbol, so that it factors through $K_2(F)$. For this we must show that for all $y \in E$ we have $\gamma(Ny \otimes (1 - Ny)) = 1$, i.e., that $\{y, 1 - Ny\} \in (1 - \sigma)K_2(E)$.

Fix $y \in E$ and set $z = N_{E/F}(y) \in F$. Factor $t^n - z = \prod f_i$ in $F[t]$, with the $f_i$ irreducible, and let $F_i$ denote the field $F(x_i)$, where $f_i(x_i) = 0$ and $x_i^n = z$. Setting $t = 1$, $1 - z = \prod f_i(1) = \prod N_{F_i/F}(1 - x_i)$. Setting $E_i = E \otimes_F F_i$, so that $N_{F_i/F}(1 - x_i) = N_{E_i/E}(1 - x_i)$ and $\sigma(x_i) = x_i$, the projection formula (Ex. 5.6) gives

$$\{y, 1 - z\} = \prod N_{E_i/E}\{y, 1 - x_i\} = \prod N_{E_i/E}\{y/x_i, 1 - x_i\}.$$

Thus it suffices to show that each $N_{E_i/E}\{y/x_i, 1-x_i\}$ is in $(1-\sigma)K_2(E)$. Now $E_i/F_i$ is a cyclic extension whose norm $N = N_{E_i/F_i}$ satisfies $N(y/x_i) = N(y)/x_i^n = 1$. By Hilbert's Theorem 90 for units, $y/x_i = v_i/\sigma v_i$ for some $v_i \in E_i$. We now compute:

$$N_{E_i/E}\{y/x_i, 1 - x_i\} = N_{E_i/E}\{v_i/\sigma v_i, 1 - x_i\} = (1 - \sigma)N_{E_i/E}\{v_i, 1 - x_i\}. \qquad \square$$

Here are three pretty applications of Hilbert's Theorem 90 for $K_2$. When $F$ is a perfect field, the first of these has already been proven in Proposition 6.4.1.

THEOREM 6.7. *If $char(F) = p \neq 0$, then the group $K_2(F)$ has no $p$-torsion.*

PROOF. Let $x$ be an indeterminate and $y = x^p - x$; the field extension $F(x)/F(y)$ is an Artin-Schrier extension, and its Galois group is generated by an automorphism $\sigma$ satisfying $\sigma(x) = x + 1$. By 6.5.2, $K_2(F)$ is a subgroup of both $K_2F(x)$ and $K_2F(y)$, and the projection formula shows that the norm $N\colon K_2F(x) \to K_2F(y)$ sends $u \in K_2(F)$ to $u^p$.

Now fix $u \in K_2(F)$ satisfying $u^p = 1$; we shall prove that $u = 1$. By Hilbert's Theorem 90 for $K_2$, $u = (1 - \sigma)v = v(\sigma v)^{-1}$ for some $v \in K_2F(x)$.

Every prime ideal $\mathfrak{p}$ of $F[x]$ is unramified over $\mathfrak{p}_y = \mathfrak{p} \cap F[y]$, because $F[x]/\mathfrak{p}$ is either equal to, or an Artin-Schrier extension of, $F[y]/\mathfrak{p}_y$. By 6.3.1 and 6.5.2, we have a commutative diagram in which the vertical maps $\partial$ are surjective:

$$\begin{array}{ccccc}
K_2F(y) & \xrightarrow{i^*} & K_2F(x) & \xrightarrow{1-\sigma} & K_2F(x) \\
\partial\big\downarrow & & \partial\big\downarrow & & \partial\big\downarrow \\
\coprod_{\mathfrak{p}_y}(F[y]/\mathfrak{p}_y)^\times & \xrightarrow{\Delta} & \coprod_{\mathfrak{p}}(F[x]/\mathfrak{p})^\times & \xrightarrow{1-\sigma} & \coprod_{\mathfrak{p}}(F[x]/\mathfrak{p})^\times
\end{array}$$

We claim that the bottom row is exact. By decomposing the row into subsequences invariant under $\sigma$, we see that there are two cases to consider. If a prime $\mathfrak{p}$ is not fixed by $\sigma$, then the fields $F[x]/\sigma^i\mathfrak{p}$ are all isomorphic to $E = F[y]/\mathfrak{p}_y$, and for $a_i \in E^\times$ we have

$$(1 - \sigma)(a_0, a_1, \dots, a_{p-1}) = (a_0 a_{p-1}^{-1}, a_1 a_0^{-1}, \dots, a_{p-1} a_{p-2}^{-1})$$

in $\prod_{i=0}^{p-1}(F[x]/\sigma^i\mathfrak{p})^\times$. This vanishes if and only if the $a_i$ agree, in which case $(a_0, \dots, a_{p-1})$ is the image of $a \in E^\times$. On the other hand, if $\sigma$ fixes $\mathfrak{p}$ then $F[x]/\mathfrak{p}$ is a cyclic Galois extension of $E = F[y]/\mathfrak{p}_y$. Therefore if $a \in F[x]/\mathfrak{p}$ and $(1 - \sigma)a = a/(\sigma a)^{-1}$ equals 1, then $a = \sigma(a)$, *i.e.*, $a \in E$. This establishes the claim.

A diagram chase shows that since $1 = \partial u = \partial(1 - \sigma)v$, there is a $v_0$ in $K_2F(y)$ with $\partial(v) = \partial(i^* v_0)$. Since $i^* = \sigma i^*$, we have $(1 - \sigma)i^* v_0 = 1$. Replacing $v$ by $v(i^* v_0)^{-1}$, we may assume that $\partial(v) = 1$, *i.e.*, that $v$ is in the subgroup $K_2(F)$ of $K_2F(x)$. Therefore we have $u = v(\sigma v)^{-1} = 1$. As $u$ was any element of $K_2(F)$ satisfying $u^p = 1$, $K_2(F)$ has no $p$-torsion.

EXAMPLE 6.7.1. If $F = \mathbb{F}_q(t)$, $q = p^r$, we have $K_2(F) = \coprod(\mathbb{F}_q[t]/\mathfrak{p})^\times$. Since the units of each finite field $\mathbb{F}_q[t]/\mathfrak{p}$ form a cyclic group, and its order can be arbitrarily large (yet prime to $p$), $K_2\mathbb{F}_q(t)$ is a very large torsion group.

THEOREM 6.8. *If $F$ contains a primitive $n^{th}$ root of unity $\zeta$, then every element of $K_2(F)$ of exponent $n$ has the form $\{\zeta, x\}$ for some $x \in F^\times$.*

PROOF. We first suppose that $n$ is a prime number $p$. Let $x$ be an indeterminate and $y = x^p$; the Galois group of the field extension $F(x)/F(y)$ is generated by an automorphism $\sigma$ satisfying $\sigma(x) = \zeta x$. By Application 6.5.2, $K_2(F)$ is a subgroup of $K_2F(x)$, and by the projection formula the norm $N\colon K_2F(x) \to K_2F(y)$ sends $u \in K_2(F)$ to $u^p$.

Fix $u \in K_2(F)$ satisfying $u^p = 1$. By Hilbert's Theorem 90 for $K_2$, if $u^p = 1$ then $u = (1-\sigma)v = v(\sigma v)^{-1}$ for some $v \in K_2F(x)$.

Now the extension $F[y] \subset F[x]$ is unramified at every prime ideal except $\mathfrak{p} = (x)$. As in the proof of Theorem 6.7, we have a commutative diagram whose bottom row is exact:

$$\begin{array}{ccccc} K_2F(y) & \xrightarrow{i^*} & K_2F(x) & \xrightarrow{1-\sigma} & K_2F(x) \\ \partial'\downarrow & & \partial'\downarrow & & \partial'\downarrow \\ \coprod_{\mathfrak{p}_y\neq(y)}(F[y]/\mathfrak{p}_y)^\times & \xrightarrow{\Delta} & \coprod_{\mathfrak{p}\neq(x)}(F[x]/\mathfrak{p})^\times & \xrightarrow{1-\sigma} & \coprod_{\mathfrak{p}\neq(x)}(F[x]/\mathfrak{p})^\times \end{array}$$

As before, we may modify $v$ by an element from $K_2F(y)$ to arrange that $\partial_\mathfrak{p}(v) = 1$ for all $\mathfrak{p} \neq (x)$. For $\mathfrak{p} = (x)$, let $a \in F = F[x]/(x)$ be such that $\partial_{(x)}(v) = a$ and set $v' = v\{a, x\}$. Then $\partial_{(x)}(v') = 1$ and $\partial_\mathfrak{p}(v') = \partial_\mathfrak{p}(v) = 1$ for every other $\mathfrak{p}$. It follows from 6.5.2 that $v'$ is in $K_2(F)$. Therefore $(1-\sigma)v' = 1$; since $v = v'\{a,x\}^{-1}$ this implies that $u$ has the asserted form:

$$u = (1-\sigma)\{a,x\}^{-1} = \{a,x\}^{-1}\{a,\zeta x\} = \{a,\zeta\}.$$

Now we proceed inductively, supposing that $n = mp$ and that the theorem has been proven for $m$ (and $p$). If $u \in K_2(F)$ has exponent $n$ then $u^p$ has exponent $m$, so there is an $x \in F^\times$ so that $u^p = \{\zeta^p, x\}$. The element $u\{\zeta^p, x\}^{-1}$ has exponent $p$, so it equals $\{\zeta^m, y\} = \{\zeta, y^m\}$ for some $y \in F^\times$. Hence $u = \{\zeta, xy^m\}$, as required.

Remark 6.8.1. Suslin also proved the following result in [Su87]. Let $F$ be a field containing a primitive $p^{th}$ root of unity $\zeta$, and let $F_0 \subset F$ be the subfield of constants. If $x \in F_0^\times$ and $\{\zeta, x\} = 1$ in $K_2(F)$ then $\{\zeta, x\} = 1$ in $K_2(F_0)$. If $\{\zeta, y\} = 1$ in $K_2(F)$ for some $y \in F^\times$ then $y = xz^p$ for some $x \in F_0^\times$ and $z \in F^\times$.

Application 6.8.2. We can use Theorem 6.8 to give another proof of Theorem 6.4, that when $F$ is an algebraically closed field, the group $K_2(F)$ is uniquely divisible. Fix a prime $p$. For each $a \in F^\times$ there is an $\alpha$ with $\alpha^p = a$. Hence $\{a,b\} = \{\alpha,b\}^p$, so $K_2(F)$ is $p$-divisible. If $p \neq \mathrm{char}(F)$ then there is no $p$-torsion because $\{\zeta, a\} = \{\zeta, \alpha\}^p = 1$. Finally, if $\mathrm{char}(F) = p$, there is no $p$-torsion either by Theorem 6.7.

Application 6.8.3 ($K_2\mathbb{R}$). Theorem 6.8 states that $\{-1,-1\}$ is the only element of order 2 in $K_2\mathbb{R}$. Indeed, if $r$ is a positive real number then:

$$\{-1, r\} = \{-1, \sqrt{r}\}^2 = 1, \quad \text{and} \quad \{-1,-r\} = \{-1,-1\}\{-1,r\} = \{-1,-1\}.$$

Note that $\{-1,-1\}$ is in the image of $K_2(\mathbb{Z})$, which is a summand by either Example 6.2.1 or Example 5.9.1. Recall from Example 6.1.6 that the image of $K_2\mathbb{R}$ in the uniquely divisible group $K_2\mathbb{C}$ is the eigenspace $K_2\mathbb{C}^+$, and that the composition $K_2\mathbb{R} \to K_2\mathbb{C} \xrightarrow{N} K_2\mathbb{R}$ is multiplication by 2, so its kernel is $K_2(\mathbb{Z})$. It follows that

$$K_2\mathbb{R} \cong K_2(\mathbb{Z}) \oplus K_2\mathbb{C}^+.$$

*$K_2$ and the Brauer group*

Let $F$ be a field. Recall from II.5.4.3 that the Brauer group $\mathrm{Br}(F)$ is generated by the classes of central simple algebras with two relations: $[A \otimes_F B] = [A] \cdot [B]$ and $[M_n(F)] = 0$. Here is one classical construction of elements in the Brauer group; it is a special case of the construction of crossed product algebras.

Cyclic algebras 6.9. Let $\zeta$ be a primitive $n^{th}$ root of unity in $F$, and $\alpha, \beta \in F^\times$. The *cyclic algebra* $A = A_\zeta(\alpha, \beta)$ is defined to be the associative algebra with unit, which is generated by two elements $x, y$ subject to the relations $x^n = \alpha \cdot 1$, $y^n = \beta \cdot 1$ and $yx = \zeta xy$. Thus $A$ has dimension $n^2$ over $F$, a basis being the monomials $x^i y^j$ with $0 \le i, j < n$. The identity $(x+y)^n = (\alpha + \beta) \cdot 1$ is also easy to check.

When $n = 2$ (so $\zeta = -1$), cyclic algebras are called *quaternion algebras*. The name comes from the fact that the usual quaternions $\mathbb{H}$ are the cyclic algebra $A(-1, -1)$ over $\mathbb{R}$. Quaternion algebras arise in the Hasse invariant of quadratic forms.

It is classical, and not hard to prove, that $A$ is a central simple algebra over $F$; see [BA, §8.5]. Moreover, the $n$-fold tensor product $A \otimes_F A \otimes_F \cdots \otimes_F A$ is a matrix algebra; see [BA, Theorem 8.12]. Thus we can consider $[A] \in \mathrm{Br}(F)$ as an element of exponent $n$. We shall write ${}_n \mathrm{Br}(F)$ for the subgroup of $\mathrm{Br}(F)$ consisting of all elements $x$ with $x^n = 1$, so that $[A] \in {}_n \mathrm{Br}(F)$

For example, the following lemma shows that $A_\zeta(1, \beta)$ must be a matrix ring because $x^n = 1$. Thus $[A_\zeta(1, \beta)] = 1$ in $\mathrm{Br}(F)$.

Lemma 6.9.1. *Let $A$ be a central simple algebra of dimension $n^2$ over a field $F$ containing a primitive $n^{th}$ root of unity $\zeta$. If $A$ contains an element $u \notin F$ such that $u^n = 1$, then $A \cong M_n(F)$.*

Proof. The subalgebra $F[u]$ of $A$ spanned by $u$ is isomorphic to the commutative algebra $F[t]/(t^n - 1)$. Since $t^n - 1 = \prod (t - \zeta^i)$, the Chinese Remainder Theorem yields $F[u] \cong F \times F \times \cdots \times F$. Hence $F[u]$ contains $n$ idempotents $e_i$ with $e_i e_j = 0$ for $i \ne j$. Therefore $A$ splits as the direct sum $e_1 A \oplus \cdots \oplus e_n A$ of right ideals. By the Artin-Wedderburn theorem, if $A = M_d(D)$ then $A$ can be the direct sum of at most $d$ right ideals. Hence $d = n$, and $A$ must be isomorphic to $M_n(F)$.

Proposition 6.9.2 (The $n^{th}$ power norm residue symbol). *If $F$ contains a primitive $n^{th}$ root of unity, there is a homomorphism $K_2(F) \to \mathrm{Br}(F)$ sending $\{\alpha, \beta\}$ to the class of the cyclic algebra $A_\zeta(\alpha, \beta)$.*

*Since the image is a subgroup of exponent $n$, we shall think of the power norm residue symbol as a map $K_2(F)/nK_2(F) \to {}_n \mathrm{Br}(F)$.*

This homomorphism is sometimes also called the *Galois symbol*.

Proof. From Ex. 6.12 we see that in $\mathrm{Br}(F)$ we have $[A_\zeta(\alpha, \beta)] \cdot [A_\zeta(\alpha, \gamma)] = [A_\zeta(\alpha, \beta\gamma)]$. Thus the map $F^\times \times F^\times \to \mathrm{Br}(F)$ sending $(\alpha, \beta)$ to $[A_\zeta(\alpha, \beta)]$ is bilinear. To see that it is a Steinberg symbol we must check that $A = A_\zeta(\alpha, 1 - \alpha)$ is isomorphic to the matrix algebra $M_n(F)$. Since the element $x + y$ of $A$ satisfies $(x + y)^n = 1$, Lemma 6.9.1 implies that $A$ must be isomorphic to $M_n(F)$.

Remark 6.9.3. Merkurjev and Suslin proved in [MS] that $K_2(F)/mK_2(F)$ is isomorphic to the subgroup ${}_m \mathrm{Br}(F)$ of elements of order $m$ in $\mathrm{Br}(F)$ when

$\mu_m \subset F$. By Matsumoto's Theorem, this implies that the $m$-torsion in the Brauer group is generated by cyclic algebras. The general description of $K_2(F)/m$, due to Merkurjev-Suslin, is given in 6.10.4; see VI.3.1.1.

*The Galois symbol*

We can generalize the power norm residue symbol to fields not containing enough roots of unity by introducing Galois cohomology. Here are the essential facts we shall need; see [WHomo] or [Milne].

SKETCH OF GALOIS COHOMOLOGY 6.10. Let $F_{\text{sep}}$ denote the separable closure of a field $F$, and let $G = G_F$ denote the Galois group $\text{Gal}(F_{\text{sep}}/F)$. The family of subgroups $G_E = \text{Gal}(F_{\text{sep}}/E)$, as $E$ runs over all finite extensions of $F$, forms a basis for a topology of $G$. A $G$-module $M$ is called *discrete* if the multiplication $G \times M \to M$ is continuous.

For example, the abelian group $\mathbf{G}_m = F_{\text{sep}}^\times$ of units of $F_{\text{sep}}$ is a discrete module, as is the subgroup $\mu_m$ of all $m^{th}$ roots of unity. We can also make the tensor product of two discrete modules into a discrete module, with $G$ acting diagonally. For example, the tensor product $\mu_m^{\otimes 2} = \mu_m \otimes \mu_m$ is also a $G$-discrete module. Note that the three $G$-modules $\mathbb{Z}/m$, $\mu_m$ and $\mu_m^{\otimes 2}$ have the same underlying abelian group, but are isomorphic $G_F$-modules only when $\mu_m \subset F$.

The $G$-invariant subgroup $M^G$ of a discrete $G$-module $M$ is a left exact functor on the category of discrete $G_F$-modules. The *Galois cohomology groups* $H^i_{et}(F;M)$ are defined to be its right derived functors. In particular, $H^0_{et}(F;M)$ is just $M^G$.

If $E$ is a finite separable extension of $F$ then $G_E \subset G_F$. Thus there is a forgetful functor from $G_F$-modules to $G_E$-modules, inducing maps $H^i_{et}(F;M) \to H^i_{et}(E;M)$. In the other direction, the induced module functor from $G_E$-modules to $G_F$-modules gives rise to cohomological transfer maps $tr_{E/F}\colon H^i_{et}(E;M) \to H^i_{et}(F;M)$; see [WHomo, 6.3.9 and 6.11.11].

EXAMPLE 6.10.1 (KUMMER THEORY). The cohomology of the module $\mathbf{G}_m$ is of fundamental importance. Of course $H^0_{et}(F,\mathbf{G}_m) = F^\times$. By Hilbert's Theorem 90 for units, and a little homological algebra [WHomo, 6.11.16], we also have $H^1_{et}(F;\mathbf{G}_m) = 0$ and $H^2_{et}(F;\mathbf{G}_m) \cong \text{Br}(F)$.

If $m$ is prime to $\text{char}(F)$, the exact sequence of discrete modules

$$1 \to \mu_m \to \mathbf{G}_m \xrightarrow{m} \mathbf{G}_m \to 1$$

is refered to as the *Kummer sequence.* Writing $\mu_m(F)$ for the group $\mu_m^G$ of all $m^{th}$ roots of unity in $F$, the corresponding cohomology sequence is called the *Kummer sequence.*

$$1 \to \mu_m(F) \to F^\times \xrightarrow{m} F^\times \to H^1_{et}(F;\mu_m) \to 1$$
$$1 \to H^2_{et}(F;\mu_m) \to \text{Br}(F) \xrightarrow{m} \text{Br}(F)$$

This yields isomorphisms $H^1_{et}(F;\mu_m) \cong F^\times/F^{\times m}$ and $H^2_{et}(F;\mu_m) \cong {}_m\text{Br}(F)$. If $\mu_m \subset F^\times$, this yields a natural isomorphism $H^2_{et}(F;\mu_m^{\otimes 2}) \cong {}_m\text{Br}(F) \otimes \mu_m(F)$.

There are also natural cup products in cohomology, such as the product

$$F^\times \otimes F^\times \to H^1_{et}(F;\mu_m) \otimes H^1_{et}(F;\mu_m) \xrightarrow{\cup} H^2_{et}(F;\mu_m^{\otimes 2}) \tag{6.10.2}$$

which satisfies the following *projection formula:* if $E/F$ is a finite separable extension, $a \in F^\times$ and $b \in E^\times$, then $tr_{E/F}(a \cup b) = a \cup N_{E/F}(b)$.

PROPOSITION 6.10.3 (GALOIS SYMBOL). *The bilinear pairing (6.10.2) induces a Steinberg symbol* $K_2(F)/mK_2(F) \to H^2_{et}(F;\mu_m^{\otimes 2})$ *for every* $m$ *prime to* $\mathrm{char}(F)$.

PROOF. It suffices to show that $a\cup(1-a)$ vanishes for every $a \in F-\{0,1\}$. Fixing $a$, factor the separable polynomial $t^m - a = \prod f_i$ in $F[t]$ with the $f_i$ irreducible, and let $F_i$ denote the field $F(x_i)$ with $f_i(x_i) = 0$. Setting $t = 1$, $1-a = \prod_i N_{F_i/F}(1-x_i)$. Writing $H^2_{et}$ additively, we have

$$a \cup (1-a) = \sum_i a \cup N_{F_i/F}(1-x_i) = \sum_i tr_{F_i/F}\big(a \cup (1-x_i)\big)$$
$$= m \sum_i tr_{F_i/F}\big(x_i \cup (1-x_i)\big).$$

Since the group $H^2_{et}(F;\mu_m^{\otimes 2})$ has exponent $m$, all these elements vanish, as desired.

REMARK 6.10.4. Suppose that $F$ contains a primitive $m^{th}$ root of unity $\zeta$. If we identify $\mathbb{Z}/m$ with $\mu_m$ via $1 \mapsto \zeta$, we have a natural isomorphism

$${}_m\mathrm{Br}(F) \cong {}_m\mathrm{Br}(F) \otimes \mathbb{Z}/m \cong {}_m\mathrm{Br}(F) \otimes \mu_m \cong H^2_{et}(F;\mu_m^{\otimes 2}).$$

Tate showed in [Tate] that this isomorphism identifies the Galois symbol of Proposition 6.10.3 with the $m^{th}$ power norm residue symbol of Proposition 6.9.2. The Merkurjev-Suslin isomorphism of [MS] cited above in Remark 6.9.3 is a special case of the more general assertion that the Galois symbol is an isomorphism: $K_2(F)/mK_2(F) \cong H^2_{et}(F;\mu_m^{\otimes 2})$ for all fields $F$ of characteristic prime to $m$. See chapter VI, 3.1.1.

EXERCISES

**6.1** Given a discrete valuation on a field $F$, with residue field $k$ and parameter $\pi$, show that there is a surjection $\lambda\colon K_2(F) \to K_2(k)$ given by the formula $\lambda\{u\pi^i, v\pi^j\} = \{\bar{u}, \bar{v}\}$. Example 6.1.2 is a special case of this, in which $\pi = t^{-1}$.

**6.2** (Bass-Tate) If $E = F(u)$ is a field extension of $F$, and $e_1, e_2 \in E$ are monic polynomials in $u$ of some fixed degree $d > 0$, show that $\{e_1, e_2\}$ is a product of symbols $\{e_1, e_2'\}$ and $\{e, e_2''\}$ with $e, e_2', e_2''$ polynomials of degree $< d$. This generalizes Lemma 6.1.4.

**6.3** (Bass-Tate) Let $k$ be a field and set $F = k((t))$.
(a) Show that $K_2(F) \cong K_2(k) \times k^\times \times K_2(k[[t]], t)$.
(b) Show that the group $K_2(k[[t]], t)$ is torsionfree; by Ex. 5.14, it is uniquely divisible if $\mathrm{char}(k) = 0$. *Hint:* Use 6.7 and the proof of 6.4.1.

**6.4** If $F$ is a number field with $r_1$ distinct embeddings $F \hookrightarrow \mathbb{R}$, show that the $r_1$ symbols $(,)_\infty$ on $F$ define a surjection $K_2(F) \to \{\pm 1\}^{r_1}$.

**6.5** If $\bar{F}$ denotes the algebraic closure of a field $F$, show that $K_2(\bar{\mathbb{Q}}) = K_2(\bar{\mathbb{F}}_p) = 1$.

**6.6** *2-adic symbol on* $\mathbb{Q}$. Any nonzero rational number $r$ can be written uniquely as $r = (-1)^i 2^j 5^k u$, where $i, k \in \{0, 1\}$ and $u$ is a quotient of integers congruent to 1 (mod 8). If $s = (-1)^{i'} 2^{j'} 5^{k'} u'$, set $(r, s)_2 = (-1)^{ii'+jj'+kk'}$. Show that this is a Steinberg symbol on $\mathbb{Q}$, with values in $\{\pm 1\}$.

**6.7** Let $((r,s))_p$ denote the Hilbert symbol on $\hat{\mathbb{Q}}_p$ (6.2.2), and $(r,s)_p$ the tame symbol $K_2(\hat{\mathbb{Q}}_p) \to \mathbb{F}_p^\times$. Assume that $p$ is odd, so that there is a unique surjection $\varepsilon\colon \mathbb{F}_p^\times \to \{\pm 1\}$. Show that $((r,s))_p = \varepsilon\ (r,s)_p$ for all $r,s \in \hat{\mathbb{Q}}_p^\times$.

**6.8** *Quadratic Reciprocity.* If $r,s \in \mathbb{Q}^\times$, show that $(r,s)_\infty (r,s)_2 \prod_{p\neq 2} ((r,s))_p = +1$. Here $(r,s)_2$ is the 2-adic symbol of Ex. 6.6.

*Hint:* From 6.5.1 and Ex. 6.7, the 2-adic symbol of Ex. 6.6 must satisfy some relation of the form

$$(r,s)_2 = (r,s)_\infty^{\varepsilon_\infty} \prod_{p\neq 2} ((r,s))_p^{\varepsilon_p},$$

where the exponents $\varepsilon_p$ are either 0 or 1. Since $(-1,-1)_2 = (-1,-1)_\infty$ we have $\varepsilon_\infty = 1$. If $p$ is a prime not congruent to 1 (mod 8), consider $\{2,p\}$ and $\{-1,p\}$. If $p$ is a prime congruent to 1 (mod 8), Gauss proved that there is a prime $q < \sqrt{p}$ such that $p$ is not a quadratic residue modulo $q$. Then $((p,q))_q = -1$, even though $(p,q)_\infty = (p,q)_2 = 1$. Since we may suppose inductively that $\varepsilon_q$ equals 1, this implies that $\varepsilon_p \neq 0$.

**6.9** (Suslin) Suppose that a field $F$ is algebraically closed in a larger field $E$. Use Lemma 6.1.3 and Remark 6.8.1 to show that $K_2(F)$ injects into $K_2(E)$.

**6.10** Let $F$ be a field, and let $\Omega_F^1 = \Omega_{F/\mathbb{Z}}^1$ denote the vector space of absolute Kähler differentials (see Ex. 2.6). The $n^{th}$ exterior power of $\Omega_F^1$ is written as $\Omega_F^n$. Show that there is a homomorphism $K_2(F) \to \Omega_F^2$ sending $\{x,y\}$ to $\frac{dx}{x} \wedge \frac{dy}{y}$. This map is not onto, because the image is in the kernel of the de Rham differential $d : \Omega_F^2 \to \Omega_F^3$.

**6.11** If $F$ is a field of transcendence degree $\kappa$ over the ground field, $\Omega_F$ is a vector space of dimension $\kappa$. Now suppose that $\kappa$ is an infinite cardinal number, so that $\Omega_F^n$ is also a vector space of dimension $\kappa$ for all $n > 1$. Show that the image of the map $K_2(F) \to \Omega_F^2$ in the previous exercise is an abelian group of rank $\kappa$.

In particular, if $F$ is a local field then the uniquely divisible summand $U$ of $K_2(F)$ in Moore's Theorem (6.2.4) is uncountable.

**6.12** Show that $A_\zeta(\alpha,\beta) \otimes A_\zeta(\alpha,\gamma) \cong M_n(A)$, where $A = A_\zeta(\alpha,\beta\gamma)$. *Hint:* Let $x', y'$ generate $A_\zeta(\alpha,\beta)$ and $x'', y''$ generate $A_\zeta(\alpha,\gamma)$, and show that $x', y = y'y''$ generate $A$. Then show that $u = (x')^{-1}x'' + y''$ has $u^n = 1$. (For another proof, see [BA, Ex. 8.5.2].)

## §7. Milnor $K$-theory of fields

Fix a field $F$, and consider the tensor algebra of the group $F^\times$,

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus (F^\times \otimes F^\times \otimes F^\times) \oplus \cdots .$$

To keep notation straight, we write $l(x)$ for the element of degree one in $T(F^\times)$ corresponding to $x \in F^\times$.

DEFINITION 7.1. The graded ring $K_*^M(F)$ is defined to be the quotient of $T(F^\times)$ by the ideal generated by the homogeneous elements $l(x) \otimes l(1-x)$ with $x \neq 0, 1$. The *Milnor $K$-group* $K_n^M(F)$ is defined to be the subgroup of elements of degree $n$. We shall write $\{x_1, \dots, x_n\}$ for the image of $l(x_1) \otimes \cdots \otimes l(x_n)$ in $K_n^M(F)$.

That is, $K_n^M(F)$ is presented as the group generated by symbols $\{x_1, \dots, x_n\}$ subject to two defining relations: $\{x_1, \dots, x_n\}$ is multiplicative in each $x_i$, and equals zero if $x_i + x_{i+1} = 1$ for some $i$.

The name comes from the fact that the ideas in this section first arose in Milnor's 1970 paper [M-QF]. Clearly we have $K_0^M(F) = \mathbb{Z}$, and $K_1^M = F^\times$ (with the group operation written additively). By Matsumoto's Theorem 6.1 we also have $K_2^M(F) = K_2(F)$, the elements $\{x, y\}$ being the usual Steinberg symbols, except that the group operation in $K_2^M(F)$ is written additively.

Since $\{x_i, x_{i+1}\} + \{x_{i+1}, x_i\} = 0$ in $K_2^M(F)$, we see that interchanging two entries in $\{x_1, \dots, x_n\}$ yields the inverse. It follows that these symbols are alternating: for any permutation $\pi$ with sign $(-1)^\pi$ we have

$$\{x_{\pi 1}, \dots, x_{\pi n}\} = (-1)^\pi \{x_1, \dots, x_n\}.$$

EXAMPLES 7.2. (a) If $\mathbb{F}_q$ is a finite field, then $K_n^M(\mathbb{F}_q) = 0$ for all $n \geq 2$, because $K_2^M(\mathbb{F}_q) = 0$ by Cor. 6.1.1. If $F$ has transcendence degree 1 over a finite field (a global field of finite characteristic), Bass and Tate proved in [BT] that $K_n^M(F) = 0$ for all $n \geq 3$.

(b) If $F$ is algebraically closed then $K_n^M(F)$ is uniquely divisible. Divisibility is clear because $F^\times$ is divisible. The proof that there is no $p$-torsion is the same as the proof for $n = 2$ given in Theorem 6.4, and is relegated to Ex. 7.3.

(c) When $F = \mathbb{R}$ we can define a symbol $K_n^M(\mathbb{R}) \to \{\pm 1\}$ by the following formula: $(x_1, \dots, x_n)_\infty$ equals $-1$ if all the $x_i$ are negative, and equals $+1$ otherwise. When $n = 2$ this will be the symbol defined in Example 6.2.1.

To construct it, extend $\mathbb{Z} \to \mathbb{Z}/2$ to a ring homomorphism $T(\mathbb{R}^\times) \to (\mathbb{Z}/2)[t]$ by sending $l(x)$ to $t$ if $x < 0$ and to 0 if $x > 0$. This sends the elements $l(x) \otimes l(1-x)$ to zero (as in 6.2.1), so it induces a graded ring homomorphism $K_*^M(\mathbb{R}) \to (\mathbb{Z}/2)[t]$. The symbol above is just the degree $n$ part of this map.

By induction on $n$, it follows that $K_n^M(\mathbb{R})$ is the direct sum of a cyclic group of order 2 generated by $\{-1, \dots, -1\}$, and a divisible subgroup. In particular, this shows that $K_*^M(\mathbb{R})/2K_*^M(\mathbb{R})$ is the polynomial ring $(\mathbb{Z}/2)[\epsilon]$ on $\epsilon = l(-1)$. Using the norm map we shall see later that the divisible subgroup of $K_n^M(\mathbb{R})$ is in fact uniquely divisible. This gives a complete description of each $K_n^M(\mathbb{R})$ as an abelian group.

(d) When $F$ is a number field, let $r_1$ be the number of embeddings of $F$ into $\mathbb{R}$. Then we have a map from $K_n^M(F)$ to $K_n^M(\mathbb{R})^{r_1} \cong (\mathbb{Z}/2)^{r_1}$. Bass and Tate proved in [BT] that this map is an isomorphism for all $n \geq 3$: $K_n^M(F) \cong (\mathbb{Z}/2)^{r_1}$.

*Tame symbols*

Recall from Lemma 6.3 and Ex. 6.1 that every discrete valuation $v$ on $F$ induces a Steinberg symbol $K_2(F) \xrightarrow{\partial_v} k_v^\times$ and a map $K_2(F) \xrightarrow{\lambda} K_2(k_v)$. These symbols extend to all of Milnor $K$-theory; the $\partial_v$ are called *higher tame symbols*, and the $\lambda$ are called *specialization maps*.

THEOREM 7.3 (HIGHER TAME SYMBOLS AND SPECIALIZATION). *For every discrete valuation $v$ on $F$, there are two surjections*

$$K_n^M(F) \xrightarrow{\partial_v} K_{n-1}^M(k_v) \quad \textit{and} \quad K_n^M(F) \xrightarrow{\lambda} K_n^M(k_v)$$

*satisfying the following conditions. Let $R = \{r \in F : v(r) \geq 0\}$ be the valuation ring, and $\pi$ a parameter for $v$. If $u_i \in R^\times$, and $\bar{u}_i$ denotes the image of $u_i$ in $k_v = R/(\pi)$ then*

$$\lambda\{u_1\pi^{i_1}, \dots, u_n\pi^{i_n}\} = \{\bar{u}_1, \dots, \bar{u}_n\}, \qquad \partial_v\{\pi, u_2, \dots, u_n\} = \{\bar{u}_2, \dots, \bar{u}_n\}.$$

*In particular, $\partial_v : K_2^M(F) \to k_v^\times$ is the tame symbol of Lemma 6.3, and $\lambda : K_2(F) \to K_2(k)$ is the map of Example 6.1.2 and Ex. 6.1.*

PROOF. (Serre) Let $L$ denote the graded $K_*^M(k_v)$-algebra generated by an indeterminate $\Pi$ in $L_1$, with the relation $\{\Pi, \Pi\} = \{-1, \Pi\}$. We claim that the group homomorphism

$$d : F^\times \to L_1 = l(k_v^\times) \oplus \mathbb{Z} \cdot \Pi, \qquad d(u\pi^i) = l(\bar{u}) + i\Pi$$

satisfies the relation: for $r \neq 0, 1$, $d(r)d(1-r) = 0$ in $L_2$. If so, the presentation of $K_*^M(F)$ shows that $d$ extends to a graded ring homomorphism $d : K_*^M(F) \to L$. Since $L_n$ is the direct sum of $K_n^M(k_v)$ and $K_{n-1}^M(k_v)$, we get two maps: $\lambda : K_n^M(F) \to K_n^M(k_v)$ and $\partial_v : K_n^M(F) \to K_{n-1}^M(k_v)$. The verification of the relations is routine, and left to the reader.

If $1 \neq r \in R^\times$, then either $1 - r \in R^\times$ and $d(r)d(1-r) = \{\bar{r}, 1 - \bar{r}\} = 0$, or else $v(1-r) = i > 0$ and $d(r) = l(1) + 0 \cdot \Pi = 0$ so $d(r)d(1-r) = 0 \cdot d(1-r) = 0$. If $v(r) > 0$ then $1 - r \in R^\times$ and the previous argument implies that $d(1-r)d(r) = 0$. If $r \notin R$ then $1/r \in R$, and we see from (5.10.3) and the above that $d(r)d(1-r) = d(1/r)d(-1/r)$. Therefore it suffices to show that $d(r)d(-r) = 0$ for every $r \in R$. If $r = \pi$ this is the given relation upon $L$, and if $r \in R^\times$ then $d(r)d(-r) = \{r, -r\} = 0$ by (5.10.3). Since the product in $L$ is anticommutative, the general case $r = u\pi^i$ follows from this.

COROLLARY 7.3.1 (RIGIDITY). *Suppose that $F$ is complete with respect to the valuation $v$, with residue field $k = k_v$. For every integer $q$ prime to* char$(k)$, *the maps $\lambda \oplus \partial_v : K_n^M(F)/q \to K_n^M(k)/q \oplus K_{n-1}^M(k)/q$ are isomorphisms for every $n$.*

PROOF. Since the valuation ring $R$ is complete, Hensel's Lemma implies that the group $1+\pi R$ is $q$-divisible. It follows that $l(1+\pi R) \cdot K_{n-1}^M(F)$ is also $q$-divisible. But by Ex. 7.2 this is the kernel of the map $d : K_n^M(F) \to L_n \cong K_n^M(k_v) \oplus K_{n-1}^M(k_v)$.

LEADING COEFFICIENTS 7.3.2. As in Example 6.1.2, $K_n^M(F)$ is a direct summand of $K_n^M F(t)$. To see this, we consider the valuation $v_\infty(f) = -deg(f)$ on $F(t)$ of Example 6.5.3. Since $t^{-1}$ is a parameter, each polynomial $f = ut^{-i}$ has $\text{lead}(f) = \bar{u}$. The map $\lambda\colon K_n^M F(t) \to K_n^M(F)$, given by $\lambda\{f_1, \dots, f_n\} = \{\text{lead}(f_1), \dots, \text{lead}(f_n)\}$, is clearly inverse to the natural map $K_n^M(F) \to K_n^M F(t)$.

Except for $v_\infty$, every discrete valuation $v$ on $F(t)$ which is trivial on $F$ is the $\mathfrak{p}$-adic valuation $v_\mathfrak{p}$ associated to a prime ideal $\mathfrak{p}$ of $F[t]$. In this case $k_v$ is the field $F[t]/\mathfrak{p}$, and we write $\partial_\mathfrak{p}$ for $\partial_v$.

THEOREM 7.4 (MILNOR). *There is a split exact sequence for each $n$, natural in the field $F$, and split by the map $\lambda$:*

$$0 \to K_n^M(F) \to K_n^M F(t) \xrightarrow{\partial = \coprod \partial_\mathfrak{p}} \coprod_\mathfrak{p} K_{n-1}^M(F[t]/\mathfrak{p}) \to 0.$$

PROOF. Let $L_d$ denote the subgroup of $K_n^M F(t)$ generated by those symbols $\{f_1, \dots, f_r\}$ such that all the polynomials $f_i$ have degree $\le d$. By Example 7.3.2, $L_0$ is a summand isomorphic to $K_n^M(F)$. Since $K_n^M F(t)$ is the union of the subgroups $L_d$, the theorem will follow from Lemma 7.4.2 below, using induction on $d$.

Let $\pi$ be an irreducible polynomial of degree $d$ and set $k = k_\pi = F[t]/(\pi)$. Then each element $\bar{a}$ of $k$ is represented by a unique polynomial $a \in F[t]$ of degree $< d$.

LEMMA 7.4.1. *There is a unique homomorphism $h = h_\pi\colon K_{n-1}^M(k) \to L_d/L_{d-1}$ carrying $\{\bar{a}_2, \dots, \bar{a}_n\}$ to the class of $\{\pi, a_2, \dots, a_n\}$ modulo $L_{d-1}$.*

PROOF. The formula gives a well-defined set map $h$ from $k^\times \times \cdots \times k^\times$ to $L_d/L_{d-1}$. To see that it is linear in $\bar{a}_2$, suppose that $\bar{a}_2 = \bar{a}_2'\bar{a}_2''$. If $a_2 \ne a_2'a_2''$ then there is a nonzero polynomial $f$ of degree $< d$ with $a_2 = a_2'a_2'' + f\pi$. Since $f\pi/a_2 = 1 - a_2'a_2''/a_2$ we have $\{f\pi/a_2, a_2'a_2''/a_2\} = 0$. Multiplying by $\{a_3, \dots, a_n\}$ gives

$$\{\pi, a_2'a_2''/a_2, a_3, \dots, a_n\} \equiv 0 \qquad \text{modulo } L_{d-1}.$$

Similarly, $h$ is linear in $a_3, \dots, a_n$. To see that the multilinear map $h$ factors through $K_{n-1}^M(k)$, we observe that if $\bar{a}_i + \bar{a}_{i+1} = 1$ in $k$ then $a_i + a_{i+1} = 1$ in $F$.

LEMMA 7.4.2. *The homomorphisms $\partial_{(\pi)}$ and $h_\pi$ induce an isomorphism between $L_d/L_{d-1}$ and the direct sum $\oplus_\pi K_{n-1}^M(k_\pi)$ as $\pi$ ranges over all monic irreducible polynomials of degree $d$ in $F[t]$.*

PROOF. Since $\pi$ cannot divide any polynomial of degree $< d$, the maps $\partial_{(\pi)}$ vanish on $L_{d-1}$ and induce maps $\bar{\partial}_{(\pi)}\colon L_d/L_{d-1} \to K_{n-1}^M(k_\pi)$. By inspection, the composition of $\oplus h_\pi$ with the direct sum of the $\bar{\partial}_{(\pi)}$ is the identity on $\oplus_\pi K_{n-1}^M(k_\pi)$. Thus it suffices to show that $\oplus h_\pi$ maps onto $L_d/L_{d-1}$. By Ex. 6.2, $L_d$ is generated by $L_{d-1}$ and symbols $\{\pi, a_2, \dots, a_n\}$ where $\pi$ has degree $d$ and the $a_i$ have degree $< d$. But each such symbol is $h_\pi$ of an element of $K_{n-1}^M(k_\pi)$, so $\oplus h_\pi$ is onto.

*The Transfer Map*

Let $v_\infty$ be the valuation on $F(t)$ with parameter $t^{-1}$. The formulas in Theorem 7.3 defining $\partial_\infty$ show that it vanishes on $K_*^M(F)$. By Theorem 7.4, there are unique homomorphisms $N_\mathfrak{p}\colon K_n^M(F[t]/\mathfrak{p}) \to K_n^M(F)$ so that $-\partial_\infty = \sum_\mathfrak{p} N_\mathfrak{p}\partial_\mathfrak{p}$.

DEFINITION 7.5. Let $E$ be a finite field extension of $F$ generated by an element $a$. Then the *transfer map*, or *norm map* $N = N_{a/F}: K_*^M(E) \to K_*^M(F)$, is the unique map $N_\mathfrak{p}$ defined above, associated to the kernel $\mathfrak{p}$ of the map $F[t] \to E$ sending $t$ to $a$.

We can calculate the norm of an element $x \in K_n^M(E)$ as $N_\mathfrak{p}(x) = -\partial_{v_\infty}(y)$, where $y \in K_{n+1}^M F(t)$ is such that $\partial_\mathfrak{p}(y) = x$ and $\partial_{\mathfrak{p}'}(y) = 0$ for all $\mathfrak{p}' \neq \mathfrak{p}$.

If $n = 0$, the transfer map $N: \mathbb{Z} \to \mathbb{Z}$ is multiplication by the degree $[E : F]$ of the field extension, while if $n = 1$ the map $N: E^\times \to F^\times$ is the usual norm map; see Ex. 7.5. We will show in 7.6 below that $N$ is independent of the choice of $a \in E$ for all $n$. First we make two elementary observations.

If we let $N_\infty$ denote the identity map on $K_n^M(F)$, and sum over the set of all discrete valuations on $F(t)$ which are trivial on $F$, the definition of the $N_v$ yields:

WEIL RECIPROCITY FORMULA 7.5.1. $\sum_v N_v \partial_v(x) = 0$ for all $x \in K_n^M F(t)$.

PROJECTION FORMULA 7.5.2. *Let $E = F(a)$. Then for $x \in K_*^M(F)$ and $y \in K_*^M(E)$ the map $N = N_{a/F}$ satisfies $N\{x, y\} = \{x, N(y)\}$.*

PROOF. The inclusions of $F$ in $F(t)$ and $F[t]/\mathfrak{p}$ allow us to view $K_*^M F(t)$ and $K_*^M(F[t]/\mathfrak{p})$ as graded modules over the ring $K_*^M(F)$. It follows from Theorem 7.4 that each $\partial_\mathfrak{p}$ is a graded module homomorphism of degree $-1$. This remark also applies to $v_\infty$ and $\partial_\infty$, because $F(t) = F(t^{-1})$. Therefore each $N_\mathfrak{p}$ is a graded module homomorphism of degree 0.

Taking $y = 1$ in $K_0^M(E) = \mathbb{Z}$, so $N(y) = [E : F]$ by Ex. 7.5, this yields

COROLLARY 7.5.3. *If the extension $E/F$ has degree $d$, then the composition $K_*^M(F) \to K_*^M(E) \xrightarrow{N} K_*^M(F)$ is multiplication by $d$. In particular, the kernel of $K_*^M(F) \to K_*^M(E)$ is annihilated by $d$.*

DEFINITION 7.6. Let $E = F(a_1, \dots, a_r)$ be a finite field extension of $F$. The *transfer map* $N_{E/F}: K_*^M(E) \to K_*^M(F)$ is defined to be the composition of the transfer maps defined in 7.5:

$$K_n^M(E) \xrightarrow{N_{a_r}} K_n^M(F(a_1, \dots, a_{r-1})) \xrightarrow{N_{a_{r-1}}} \cdots K_n^M(F(a_1)) \xrightarrow{N_{a_1}} K_n^M(F).$$

The transfer map is well-defined by the following result of K. Kato.

THEOREM 7.6.1 (KATO). *The transfer map $N_{E/F}$ is independent of the choice of elements $a_1, \dots, a_r$ such that $E = F(a_1, \dots, a_r)$. In particular, if $F \subset F' \subset E$ then $N_{E/F} = N_{F'/F} N_{E/F'}$.*

The key trick used in the proof of this theorem is to fix a prime $p$ and pass from $F$ to the union $F'$ of all finite extensions of $F$ of degree prime to $p$. By Corollary 7.5.3 the kernel of $K_n^M(F) \to K_n^M(F')$ has no $p$-torsion, and the degree of every finite extension of $F'$ is a power of $p$.

LEMMA 7.6.2. *(Kato) If $E$ is a normal extension of $F$, and $[E : F]$ is a prime number $p$, then the map $N_{E/F} = N_{a/F}: K_*^M(E) \to K_*^M(F)$ does not depend upon the choice of $a$ such that $E = F(a)$.*

PROOF. If also $E = F(b)$, then from Corollary 7.5.3 and Ex. 7.7 with $F' = E$ we see that $\delta(x) = N_{a/F}(x) - N_{b/F}(x)$ is annihilated by $p$. If $\delta(x) \neq 0$ for some $x \in K_n^M(E)$ then, again by Corollary 7.5.3, $\delta(x)$ must be nonzero in $K_n^M(F')$, where $F'$ is the union of all finite extensions of $F$ of degree prime to $p$. Again by Ex. 7.7, we see that we may replace $F$ by $F'$ and $x$ by its image in $K_n^M(EF')$. Since the degree of every finite extension of $F'$ is a power of $p$, the assertion for $F'$ follows from Ex. 7.6, since the Projection Formula 7.5.2 yields $N_{a/F'}\{y, x_2, \dots, x_n\} = \{N(y), x_2, \dots, x_n\}$.

COROLLARY 7.6.3. *If in addition $F$ is a complete discrete valuation field with residue field $k_v$, and the residue field of $E$ is $k_w$, the following diagram commutes.*

$$\begin{array}{ccc} K_n^M(E) & \xrightarrow{\partial_w} & K_{n-1}^M(k_w) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ K_n^M(F) & \xrightarrow{\partial_v} & K_{n-1}^M(k_v) \end{array}$$

PROOF. Ex. 7.6 implies that for each $u \in K_n^M(E)$ there is a finite field extension $F'$ of $F$ such that $[F' : F]$ is prime to $p$ and the image of $u$ in $K_n(EF')$ is generated by elements of the form $u' = \{y, x_2, \dots, x_n\}$ ($y \in EF'$, $x_i \in F'$). By Ex. 7.7 and Ex. 7.8 it suffices to prove that $N_{k_w/k_v}\partial_w(u) = \partial_v(N_{EF'/F'}u)$ for every element $u$ of this form. But this is an easy computation.

PROPOSITION 7.6.4 (KATO). *Let $E$ and $F' = F(a)$ be extensions of $F$ with $E/F$ normal of prime degree $p$. If $E' = E(a)$ denotes the composite field, the following diagram commutes.*

$$\begin{array}{ccc} K_*^M(E') & \xrightarrow{N_{a/E}} & K_*^M(E) \\ {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\ K_*^M(F') & \xrightarrow{N_{a/F}} & K_*^M(F). \end{array}$$

PROOF. The vertical norm maps are well-defined by Lemma 7.6.2. Let $\pi \in F[t]$ and $\pi' \in E[t]$ be the minimal polynomials of $a$ over $F$ and $E$, respectively. Given $x \in K_n^M(E')$, we have $N_{a/E}(x) = -\partial_\infty(y)$, where $y \in K_{n+1}^M E(t)$ satisfies $\partial_{\pi'}(y) = x$ and $\partial_w(y) = 0$ if $w \neq w_{\pi'}$. If $v$ is a valuation on $F(t)$, Ex. 7.9 gives:

$$\partial_v(N_{E(t)/F(t)}y) = \sum_{w|v} N_{E(w)/F(v)}(\partial_w y) = \begin{cases} N_{E'/F'}(x) & \text{if } v = v_\pi \\ N_{E/F}(\partial_\infty y) & \text{if } v = v_\infty \\ 0 & \text{else} \end{cases}$$

in $K_*^M(F')$. Two applications of Definition 7.5 give the desired calculation:

$$N_{a/F}(N_{E'/F'}x) = -\partial_\infty(N_{E(t)/F(t)}y) = -N_{E/F}(\partial_\infty y) = N_{E/F}(N_{a/F}x).$$

PROOF OF THEOREM 7.6.1. As in the proof of Lemma 7.6.2, we see from Corollary 7.5.3 and Ex. 7.7 with $F' = E$ that the indeterminacy is annihilated by $[E : F]$. Using the key trick of passing to a larger field, we may assume that the degree of every finite extension of $F$ is a power of a fixed prime $p$.

Let us call a tower of intermediate fields $F = F_0 \subset F_1 \subset \cdots \subset F_r = E$ *maximal* if $[F_i : F_{i-1}] = p$ for all $i$. By Lemma 7.6.2, the transfer maps $N\colon K_*^M(F_i) \to K_*^M(F_{i-1})$ are independent of the choice of $a$ such that $F_i = F_{i-1}(a)$. If $F \subset F_1 \subset E$ and $F \subset F' \subset E$ are maximal towers, Proposition 7.6.4 states that $N_{F'/F}N_{E/F'} = N_{F_1/F}N_{E/F_1}$, because if $F' \neq F_1$ then $E = F'F_1$. It follows by induction on $[E : F]$ that if $F = F_0 \subset F_1 \subset \cdots \subset F_r = E$ is a maximal tower then the composition of the norm maps

$$K_n^M(E) \xrightarrow{N} K_n^M(F_{r-1}) \xrightarrow{N} \cdots K_n^M(F_1) \xrightarrow{N} K_n^M(F)$$

is independent of the choice of maximal tower.

Comparing any tower to a maximal tower, we see that it suffices to prove that if $F \subset F_1 \subset F'$ is a maximal tower and $F' = F(a)$ then $N_{a/F} = N_{F_1/F}N_{F'/F_1}$. But this is just Proposition 7.6.4 with $E = F_1$ and $E' = F'$.

*The dlog symbol and $\nu(n)_F$*

For any field $F$, we write $\Omega_F^n$ for the $n$th exterior power of the vector space $\Omega_F = \Omega_{F/\mathbb{Z}}$ of Kähler differentials (Ex. 2.6). The direct sum over $n$ forms a graded-commutative ring $\Omega_F^*$, and the map $dlog\colon F^\times \to \Omega_F$ sending $a$ to $\frac{da}{a}$ extends to a graded ring map from the tensor algebra $T(F^\times)$ to $\Omega_F^*$. By Ex. 6.10, $l(a) \otimes l(1-a)$ maps to zero, so it factors through the quotient ring $K_*^M(F)$ of $T(F^\times)$. We record this observation for later reference.

LEMMA 7.7. *If $F$ is any field, there is a graded ring homomorphism*

$$dlog\colon K_*^M(F) \to \Omega_F^*, \qquad dlog\{a_1, \dots, a_n\} = \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}.$$

Now let $F$ be a field of characteristic $p \neq 0$, so that $d(a^p) = p\,da = 0$. In fact, if $\{x_i\}$ is a $p$-basis of $F$ over $F^p$ then the symbols $dx_i$ form a basis of the $F$-vector space $\Omega_F$. Note that the set $d\Omega_F^{n-1}$ of all symbols $da_1 \wedge \cdots \wedge da_n$ forms an $F^p$-vector subspace of $\Omega_F^n$.

DEFINITION 7.7.1. If $\mathrm{char}(F) = p \neq 0$, let $\nu(n)_F$ denote the kernel of the Artin-Schrier operator $\wp\colon \Omega_F^n \to \Omega_F^n/d\Omega_F^{n-1}$, which is defined by

$$\wp\left(x\,\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}\right) = (x^p - x)\,\frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}.$$

(In the literature, $\wp + 1$ is the inverse of the "Cartier" operator.)

Clearly $\wp(dlog\{a_1, \dots, a_n\}) = 0$, so the image of the *dlog* map lies in $\nu(n)_F$. The following theorem, which implies that these symbols span $\nu(n)_F$, was proven by Kato [K82] for $p = 2$, and for general $p$ by Bloch, Kato and Gabber [BK, 2.1].

THEOREM 7.7.2. *(Bloch-Kato-Gabber) Let $F$ be a field of characteristic $p \neq 0$. Then the dlog map induces an isomorphism $K_n^M(F)/pK_n^M(F) \cong \nu(n)_F$ for every $n \geq 0$.*

Using this result, Bloch and Kato also proved that the $p$-torsion subgroup of $K_n^M(F)$ is divisible [BK, 2.8]. Using this divisibility, Izhboldin found the following generalization of theorem 6.7; see [Izh].

Izhboldin's Theorem 7.8. *If* $\mathrm{char}(F) = p$*, the group* $K_n^M(F)$ *has no* $p$*-torsion.*

Proof. We proceed by induction on $n$, the case $n = 2$ being theorem 6.7. As in the proof of theorem 6.7, let $x$ be an indeterminate and $y = x^p - x$; the field extension $F(x)/F(y)$ is an Artin-Schrier extension, and its Galois group is generated by an automorphism $\sigma$ satisfying $\sigma(x) = x + 1$. By theorem 7.4, we can regard $K_n^M(F)$ as a subgroup of both $K_n^M F(x)$ and $K_n^M F(y)$.

For all field extensions $E$ of $F(y)$ linearly disjoint from $F(x)$, i.e., with no root of $t^p - t - y$, write $E(x)$ for the field $E \otimes_{F(y)} F(x)$. Let $I(E)$ denote the set of all $p$-torsion elements in $K_n^M E(x)$ of the form $v - \sigma(v)$, $v \in K_n^M E(x)$, and let $P(E)$ denote the $p$-torsion subgroup of the kernel of the norm map $N_{x/E}\colon K_n^M E(x) \to K_n^M(E)$. Since $N\sigma(v) = N(v)$, $I(E) \subseteq P(E)$. Both $I(E)$ and $P(E)$ vary naturally with $E$, and are equal by proposition 7.8.2 below.

Fix $u \in K_n^M(F)$ with $pu = 0$. The projection formula 7.5.2 shows that the norm map $K_n^M F(x) \to K_n^M F(y)$ sends $u$ to $pu = 0$. Hence $u \in P(F(y))$. By proposition 7.8.2, $u \in I(F(y))$, i.e., there is a $v \in K_n^M F(x)$ so that $u = v - \sigma(v)$ in $K_n^M F(x)$. Now apply the leading coefficient symbol $\lambda$ of 7.3.2; since $\lambda(\sigma v) = \lambda(v)$ we have: $u = \lambda(u) = \lambda(v) - \lambda(\sigma v) = 0$. This proves Izhboldin's theorem.

Before proceeding to proposition 7.8.2, we need some facts about the group $I(E)$. We first claim that the transcendental extension $E \subset E(t)$ induces an isomorphism $I(E) \cong I(E(t))$. Indeed, since $E(x, t)$ is purely transcendental over $E(x)$, theorem 7.4 and induction on $n$ imply that $K_n^M E(x) \to K_n^M E(x, t)$ is an isomorphism on $p$-torsion subgroups, and the claim follows because the leading coefficient symbol 7.3.2 commutes with $\sigma$.

We next claim that if $E/E'$ is a purely inseparable field extension then $I(E') \to I(E)$ is onto. For this we may assume that $E^p \subseteq E' \subset E$. The composition of the Frobenius map $E \to E^p$ with this inclusion induces the endomorphism of $K_n^M(E)$ sending $\{a_1, \dots, a_n\}$ to $\{a_1^p, \dots, a_n^p\} = p^n\{a_1, \dots, a_n\}$. Hence this claim follows from the following result.

Lemma 7.8.1. *The group* $I(E)$ *is* $p$*-divisible.*

Proof. Pick $v \in K_n^M E(x)$ so that $u = v - \sigma(v)$ is in $I(E)$. Now we invoke the Bloch-Kato result, mentioned above, that the $p$-torsion subgroup of $K_n^M(L)$ is divisible for every field $L$ of characteristic $p$. By theorem 7.7.2, this implies that $u$ vanishes in $K_n^M E(x)/p \cong \nu(n)_{E(x)}$. By Ex. 7.12 and theorem 7.7.2, the class of $v \bmod p$ comes from an element $w \in K_n^M(E)$, i.e., $v - w = pv'$ for some $v' \in K_n^M E(x)$. Then $u = v - \sigma(v) = pv' - p\sigma(v')$, it follows that $u' = v' - \sigma(v')$ is an element of $I(E)$ with $u = pu'$.

Proposition 7.8.2. *For all* $E$ *containing* $F(y)$*, linearly disjoint from* $F(x)$*,* $P(E) = I(E)$*.*

Proof. We shall show that the obstruction $V(E) = P(E)/I(E)$ vanishes. This group has exponent $p$, because if $u \in P(E)$ then

$$\begin{aligned} pu &= pu - N_{EL/E}u = (p - 1 - \sigma - \cdots - \sigma^{p-1})u \\ &= \big((1-\sigma) + (1-\sigma^2) + \cdots + (1-\sigma^{p-1})\big)u \end{aligned}$$

is in $(1-\sigma)K_n^M E(x)$ and hence in $I(E)$. It follows that $V(E)$ injects into $I(E')$ whenever $E'/E$ is an extension of degree prime to $p$.

Now we use the "Brauer-Severi" trick; this trick will be used again in chapter V, 1.6, in connection with Severi-Brauer varieties. For each $b \in E$ we let $E_b$ denote the field $E(t_1, \dots, t_{p-1}, \beta)$ with $t_1, \dots, t_{p-1}$ purely transcendental over $E$ and $\beta^p - \beta - y + \sum b^i t_i^p = 0$. It is known that $b$ is in the image of the norm map $E_b(x)^\times \to E_b^\times$; see [J37]. Since $E \cdot (E_b)^p$ is purely transcendental over $E$ (on $\beta, t_2^p, ..., t_{p-1}^p$), it follows that $I(E) \to I(E_b)$ is onto. Since $E_b(x)$ is purely transcendental over $E(x)$ (why?), $I(E(x)) = I(E_b(x))$ and $K_n^M E(x)$ embeds in $K_n^M E_b(x)$ by theorem 7.4. Hence $K_n^M(E(x))/I(E)$ embeds in $K_n^M E_b(x)/I(E_b)$. Since $V(E) \subset K_n^M E(x)/I(E)$ by definition, we see that $V(E)$ also embeds into $V(E_b)$.

Now if we take the composite of all the fields $E_b$, $b \in E$, and then form its maximal algebraic extension $E'$ of degree prime to $p$, it follows that $V(E)$ embeds into $V(E')$. Repeating this construction a countable number of times yields an extension field $E''$ of $E$ such that $V(E)$ embeds into $V(E'')$ and every element of $E''$ is a norm from $E''(x)$. Hence it suffices to prove that $V(E'') = 0$. The proof in this special case is completely parallel to the proof of proposition 6.6.2, and we leave the details to Ex. 7.13.

This completes the proof of Izhboldin's Theorem 7.8.

Corollary 7.8.3 (Hilbert's Theorem 90 for $K_*^M$). *Let $j\colon F \subset L$ be a degree $p$ field extension, with $\mathrm{char}(F) = p$, and let $\sigma$ be a generator of $G = Gal(L/F)$. Then $K_n^M(F) \cong K_n^M(L)^G$, and the following sequence is exact for all $n > 0$:*

$$0 \to K_n^M(F) \xrightarrow{j^*} K_n^M(L) \xrightarrow{1-\sigma} K_n^M(L) \xrightarrow{N} K_n^M(F).$$

Proof. Since $K_n^M(F)$ has no $p$-torsion, Corollary 7.5.3 implies that $j^*$ is an injection. To prove exactness at the next spot, suppose that $v \in K_n^M(L)$ has $\sigma(v) = v$. By Ex. 7.12 and theorem 7.7.2, the class of $v \bmod p$ comes from an element $w \in K_n^M(E)$, i.e., $v - j^*(w) = pv'$ for some $v' \in K_n^M E(x)$. Hence $p\sigma(v') = \sigma(pv') = pv'$. Since $K_n^M(L)$ has no $p$-torsion, $\sigma(v') = v'$. But then $pv'$ equals $j^*N(v') = \sum \sigma^i(v')$, and hence $v = j^*(w) + j^*(Nv')$. In particular, this proves that $K_n^M(F) \cong K_n^M(L)^G$.

To prove exactness at the final spot, note that $G$ acts on $K_n^M(L)$, and that $\ker(N)/\mathrm{im}(1-\sigma)$ is isomorphic to the cohomology group $H^1(G, K_n^M(L))$; see [WHomo, 6.2.2]. Now consider the exact sequence of $Gal(L/F)$-modules

$$0 \to K_n^M(L) \xrightarrow{p} K_n^M(L) \xrightarrow{7.7.2} \nu(n)_L \to 0.$$

Using Ex. 7.12, the long exact sequence for group cohomology begins

$$0 \to K_n^M(F) \xrightarrow{p} K_n^M(F) \to \nu(n)_F \to H^1(G, K_n^M(L)) \xrightarrow{p} H^1(G, K_n^M(L)).$$

But $K_n^M(F)$ maps onto $\nu(n)_F$ by theorem 7.7.2, and the group $H^1(G, A)$ has exponent $p$ for all $G$-modules $A$ [WHomo, 6.5.8]. It follows that $H^1(G, K_n^M(L)) = 0$, so $\ker(N) = \mathrm{im}(1-\sigma)$, as desired.

REMARK 7.8.4. Hilbert's Theorem 90 for $K_n^M$, which extends Theorem 6.6, states that for any Galois extension $F \subset E$ of degree $p$, with $\sigma$ generating $\mathrm{Gal}(E/F)$, the following sequence is exact:

$$K_n^M(E) \xrightarrow{1-\sigma} K_n^M(E) \xrightarrow{N_{E/F}} K_n^M(F).$$

This is a consequence of the Norm Residue Theorem (Chapter VI, 3.1) and is due to Voevodsky; we refer the reader to [HW, 3.2] for a proof.

*Relation to the Witt ring*

Let $F$ be a field of characteristic $\neq 2$. Recall from §5.6 of chapter II that the *Witt ring* $W(F)$ is the quotient of the Grothendieck group $K_0\mathbf{SBil}(F)$ of symmetric inner product spaces over $F$ by the subgroup $\{nH\}$ generated by the hyperbolic form $\langle 1\rangle \oplus \langle -1\rangle$. The dimension of the underlying vector space induces an augmentation $K_0\mathbf{SBil}(F) \to \mathbb{Z}$, sending $\{nH\}$ isomorphically onto $2\mathbb{Z}$, so it induces an augmentation $\varepsilon\colon W(F) \to \mathbb{Z}/2$.

We shall be interested in the augmentation ideals $I = \ker(\varepsilon)$ of $W(F)$ and $\hat{I}$ of $K_0\mathbf{SBil}(F)$. Since $H \cap \hat{I} = 0$, we have $\hat{I} \cong I$. Now $I$ is generated by the classes $\langle a\rangle - 1$, $a \in F - \{0, 1\}$. The powers $I^n$ of $I$ form a decreasing chain of ideals $W(F) \supset I \supset I^2 \supset \cdots$.

For convenience, we shall write $K_n^M(F)/2$ for $K_n^M(F)/2K_n^M(F)$.

THEOREM 7.9 (MILNOR). *There is a unique surjective homomorphism*

$$s_n\colon K_n^M(F)/2 \to I^n/I^{n+1}$$

*sending each product $\{a_1, \dots, a_n\}$ in $K_n^M(F)$ to the product $\prod_{i=1}^n\big(\langle a_i\rangle - 1\big)$ modulo $I^{n+1}$. The homomorphisms $s_1$ and $s_2$ are isomorphisms.*

PROOF. Because $\big(\langle a\rangle - 1\big) + \big(\langle b\rangle - 1\big) \equiv \langle ab\rangle - 1$ modulo $I^2$ (II.5.6.5), the map $l(a_1)\times\cdots\times l(a_n) \mapsto \prod\big(\langle a_i\rangle - 1\big)$ is a multilinear map from $F^\times$ to $I^n/I^{n+1}$. Moreover, if $a_i + a_{i+1} = 1$ for any $i$, we know from Ex. II.5.12 that $\big(\langle a_i\rangle - 1\big)\big(\langle a_{i+1}\rangle - 1\big) = 0$. By the presentation of $K_*^M(F)$, this gives rise to a group homomorphism from $K_n^M(F)$ to $I^n/I^{n+1}$. It annihilates $2K_*^M(F)$ because $\langle a^2\rangle = 1$:

$$2s_n\{a_1, \dots, a_n\} = s_n\{a_1^2, a_2 \dots\} = \big(\langle a_1^2\rangle - 1\big)\prod_{i=2}^n\big(\langle a_i\rangle - 1\big) = 0.$$

It is surjective because $I$ is generated by the $\big(\langle a\rangle - 1\big)$. When $n = 1$ the map is the isomorphism $F^\times/F^{\times 2} \cong I/I^2$ of chapter II. We will see that $s_2$ is an isomorphism in Corollary 7.10.3 below, using the Hasse invariant $w_2$.

EXAMPLE 7.9.1. For the real numbers $\mathbb{R}$, we have $W(\mathbb{R}) = \mathbb{Z}$ and $I = 2\mathbb{Z}$ on $s_1(-1) = \langle -1\rangle - 1 = 2\langle -1\rangle$. On the other hand, we saw in Example 7.2(c) that $K_n^M(\mathbb{R})/2 \cong \mathbb{Z}/2$ on $\{-1, \dots, -1\}$. In this case each $s_n$ is the isomorphism $\mathbb{Z}/2 \cong 2^n\mathbb{Z}/2^{n+1}\mathbb{Z}$.

At the other extreme, if $F$ is algebraically closed then $W(F) = \mathbb{Z}/2$. Since $K_n^M(F)$ is divisible, $K_n^M(F)/2 = 0$ for all $n \geq 1$. Here $s_n$ is the isomorphism $0 = 0$.

REMARK 7.9.2. In 1970, Milnor asked if the surjection $s_n\colon K_n^M(F)/2 \to I^n/I^{n+1}$ is an isomorphism for all $n$ and $F$, $\mathrm{char}(F) \neq 2$ (on p. 332 of [M-QF]). Milnor proved this was so for local and global fields. This result was proven for all fields and all $n$ by Orlov, Vishik and Voevodsky in [OVV].

DEFINITION 7.10 (STIEFEL-WHITNEY INVARIANT). The (total) *Stiefel-Whitney invariant* $w(M)$ of the symmetric inner product space $M = \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle$ is the element of $\prod_{i=0}^{\infty} K_i^M(F)/2$ defined by the formula

$$w(M) = \prod_{i=1}^{n} \bigl(1 + l(a_i)\bigr) = 1 + l(a_1 \cdots a_n) + \cdots + \{a_1, \dots, a_n\}$$

The lemma below shows that $w(M)$ is independent of the representation of $M$ as a direct sum of 1-dimensional forms. We write $w(M) = 1 + w_1(M) + w_2(M) + \cdots$, where the $i^{th}$ *Stiefel-Whitney invariant* $w_i(M) \in K_i^M(F)/2$ equals the $i^{th}$ elementary symmetric function of $l(a_1), \dots, l(a_n)$. For example, $w_1(M) = a_1 \cdots a_n \in F^\times/F^{\times 2}$ is just the classical "discriminant" of $M$ defined in II.5.6.3, while the second elementary symmetric function $w_2(M) = \sum_{i<j} \{a_i, a_j\}$ lies in $K_2(F)/2$ and is called the *Hasse invariant* of $M$; see [M-SBF].

For $M = \langle a \rangle \oplus \langle b \rangle$ we have $w_1(M) = ab$ and $w_2(M) = \{a, b\}$, with $w_i(M) = 0$ for $i \geq 3$. In particular, the hyperbolic plane $H$ has $w_i(H) = 0$ for all $i \geq 2$.

LEMMA 7.10.1. *$w(M)$ is a well-defined unit in the ring $\prod_{i=1}^{\infty} K_i^M(F)/2$. It satisfies the Whitney sum formula*

$$w(M \oplus N) = w(M)w(N),$$

*so $w$ extends to a function on $K_0\mathbf{SBil}(F)$. Hence each Stiefel-Whitney invariant $w_i$ extends to a function $K_0\mathbf{SBil}(F) \xrightarrow{w_i} K_i^M(F)/2$.*

PROOF. To show that $w(M)$ is well defined, it suffices to consider the rank two case. Suppose that $\langle a \rangle \oplus \langle b \rangle \cong \langle \alpha \rangle \oplus \langle \beta \rangle$. Then the equation $ax^2 + by^2 = \alpha$ must have a solution $x, y$ in $F$. The case $y = 0$ (or $x = 0$) is straightforward, since $\langle \alpha \rangle = \langle ax^2 \rangle = \langle a \rangle$, so we may assume that $x$ and $y$ are nonzero. Since the discriminant $w_1$ is an invariant, we have $ab = \alpha\beta u^2$ for some $u \in F$, and all we must show is that $\{a, b\} = \{\alpha, \beta\}$ in $K_2(F)/2$. The equation $1 = ax^2/\alpha + by^2/\alpha$ yields the equation

$$0 = \{ax^2/\alpha, by^2/\alpha\} \equiv \{a, b\} + \{\alpha, \alpha\} - \{a, \alpha\} - \{b, \alpha\} \equiv \{a, b\} - \{\alpha, ab/\alpha\}$$

in $K_2(F)/2K_2(F)$. Substituting $ab = \alpha\beta u^2$, this implies that $\{a, b\} \equiv \{\alpha, \beta\}$ modulo $2K_2(F)$, as desired.

EXAMPLE 7.10.2. Since $I \cong \hat{I}$, we may consider the $w_i$ as functions on $I \subseteq W(F)$. However, care must be taken as $w_i(M)$ need not equal $w_2(M \oplus H)$. For example, $w_2(M \oplus H) = w_2(M) + \{w_1(M), -1\}$. In particular, $w_2(H \oplus H) = \{-1, -1\}$ can be nontrivial. The *Hasse-Witt invariant* of an element $x \in I \subset W(F)$ is defined to be $h(x) = w_2(V, B)$, where $(V, B)$ is an inner product space representing $x$ so that $\dim(V) \equiv 0 \bmod 8$.

COROLLARY 7.10.3. *The Hasse invariant $w_2 \colon \hat{I} \to K_2(F)/2$ induces an isomorphism from $\hat{I}^2/\hat{I}^3 \cong I^2/I^3$ to $K_2^M(F)/2$, inverse to the map $s_2$ of Theorem 7.9.*

PROOF. By Ex. 7.11, $w_2$ vanishes on the ideal $\hat{I}^3 \cong I^3$, and hence defines a function from $\hat{I}^2/\hat{I}^3$ to $K_2(F)/2$. Since the total Stiefel-Whitney invariant of

$s_2\{a,b\} = \big(\langle a\rangle - 1\big)\big(\langle b\rangle - 1\big)$ is $1 + \{a,b\}$, this function provides an inverse to the function $s_2$ of Theorem 7.9.

If char$(F) = 2$, there is an elegant formula for the filtration quotients of the Witt ring $W(F)$ and the $W(F)$-module $WQ(F)$ (see II.5) due to K. Kato [K82]. Recall from 7.7.2 that $K_n^M(F)/2 \cong \nu(n)_F$, where $\nu(n)_F$ is the kernel of the operator $\wp$. The case $n = 0$ of Kato's result was described in Ex. II.5.13(d).

THEOREM 7.10.4 (KATO [K82]). *Let $F$ be a field of characteristic 2. Then the map $s_n$ of Theorem 7.9 induces an isomorphism $K_n^M(F)/2 \cong \nu(n)_F \cong I^n/I^{n+1}$, and there is a short exact sequence*

$$0 \to I^n/I^{n+1} \to \Omega_F^n \xrightarrow{\wp} \Omega_F^n/d\Omega_F^{n-1} \to I^n\ WQ(F)/I^{n+1}\ WQ(F) \to 0.$$

*The Galois symbol*

For the next result, we need some facts about Galois cohomology, expanding slightly upon the facts mentioned in 6.10. Assuming that $n$ is prime to char$(F)$, there are natural cohomology cup products $H_{et}^i(F;M)\otimes H_{et}^j(F;N) \xrightarrow{\cup} H_{et}^{i+j}(F;M\otimes N)$ which are associative in $M$ and $N$. This makes the direct sum $H_{et}^*(F;M^{\otimes *}) = \oplus_{i=0}^{\infty} H_{et}^i(F;M^{\otimes i})$ into a graded-commutative ring for every $\mathbb{Z}/n$-module $M$ over the Galois group $Gal(F_{\text{sep}}/F)$. (By convention, $M^{\otimes 0}$ is $\mathbb{Z}/n$.) In particular, both $H_{et}^*(F;\mathbb{Z}/n)$ and $H_{et}^*(F;\mu_n^{\otimes *})$ are rings, and are isomorphic only when $F$ contains a primitive $n^{th}$ root of unity.

THEOREM 7.11 (GALOIS SYMBOLS). *(Bass-Tate) Fix a field $F$ and an integer $n$ prime to* char$(F)$.

(1) *If $F$ contains a primitive $n^{th}$ root of unity, the Kummer isomorphism from $F^\times/F^{\times n}$ to $H_{et}^1(F;\mathbb{Z}/n)$ extends uniquely to a graded ring homomorphism*

$$h_F\colon K_*^M(F)/n \to H^*(F;\mathbb{Z}/n).$$

(2) *More generally, the Kummer isomorphism from $F^\times/F^{\times n}$ to $H^1(F;\mu_n)$ extends uniquely to a graded ring homomorphism*

$$h_F\colon K_*^M(F)/n \to H_{et}^*(F;\mu_n^{\otimes *}) = \oplus_{i=0}^{\infty} H_{et}^i(F;\mu_n^{\otimes i}).$$

*The individual maps $K_i^M(F) \to H_{et}^i(F;\mu_n^{\otimes i})$ are called the higher Galois symbols.*

PROOF. The first assertion is just a special case of the second assertion. As in (6.10.2), the Kummer isomorphism induces a map from the tensor algebra $T(F^\times)$ to $H_{et}^*(F;\mu_n^{\otimes *})$, which in degree $i$ is the iterated cup product

$$F^\times \otimes \cdots F^\times = (F^\times)^{\otimes n} \cong \big(H_{et}^1(F;\mu_n)\big)^{\otimes i} \xrightarrow{\cup} H_{et}^i(F;\mu_n^{\otimes i}).$$

By Proposition 6.10.3, the Steinberg Relation is satisfied in $H_{et}^2(F,\mu_n^{\otimes 2})$. Hence the presentation of $K_*^M(F)$ yields a ring homomorphism from $K_*^M(F)$ to $H_{et}^*(F;\mu_n^{\otimes *})$.

REMARK 7.11.1. In his seminal paper [M-QF], Milnor studied the Galois symbol for $K_n^M(F)/2$ and stated (on p.340) that, "I do not know any examples for which the homomorphism $h_F$ fails to be bijective." Voevodsky proved that $h_F$ is an isomorphism for $n = 2^\nu$ in his 2003 paper [V-MC]. The proof that $h_F$ is an isomorphism for all $n$ prime to char$(F)$ was proven a few years later; see VI.3.1.1.

EXERCISES

**7.1** Let $v$ be a discrete valuation on a field $F$. Show that the maps $\lambda\colon K_n^M(F) \to K_n^M(k_v)$ and $\partial_v\colon K_n^M(F) \to K_{n-1}^M(k_v)$ of Theorem 7.3 are independent of the choice of parameter $\pi$, and that they vanish on $l(u) \cdot K_{n-1}^M(F)$ whenever $u \in (1 + \pi R)$. Show that the map $\lambda$ also vanishes on $l(\pi) \cdot K_{n-1}^M(F)$.

**7.2** Continuing Exercise 7.1, show that the kernel of the map $d\colon K_n^M(F) \to L_n$ of Theorem 7.3 is exactly $l(1 + \pi R) \cdot K_{n-1}^M(F)$. Conclude that the kernel of the map $\lambda$ is exactly $l(1 + \pi R) \cdot K_{n-1}^M(F) + l(\pi) \cdot K_{n-1}^M(F)$.

**7.3** (Bass-Tate) Generalize Theorem 6.4 to show that for all $n \geq 2$:

(a) If $F$ is an algebraically closed field, then $K_n^M(F)$ is uniquely divisible.

(b) If $F$ is a perfect field of characteristic $p$ then $K_n^M(F)$ is uniquely $p$-divisible.

**7.4** Let $F$ be a local field with valuation $v$ and finite residue field $k$. Show that $K_n^M(F)$ is divisible for all $n \geq 3$. *Hint:* By Moore's Theorem 6.2.4, $K_n^M(F)$ is $\ell$-divisible unless $F$ has a $\ell^{th}$ root of unity. Moreover, for every $x \notin F^{\times \ell}$ there is a $y \notin F^{\times \ell}$ so that $\{x, y\}$ generates $K_2(F)/\ell$. Given $a, b, c$ with $\{b, c\} \notin \ell K_2(F)$, find $a', b' \notin F^{\times \ell}$ so that $\{b', c\} \equiv 0$ and $\{a', b'\} \equiv \{a, b\}$ modulo $\ell K_2(F)$, and observe that $\{a, b, c\} \equiv \{a', b', c\} \equiv 0$.

In fact, I. Sivitskii has shown that $K_n^M(F)$ is uniquely divisible for $n \geq 3$ when $F$ is a local field. See [Siv]. We will give a proof of this in VI.5.1.

**7.5** Let $E = F(a)$ be a finite extension of $F$, and consider the transfer map $N = N_{a/F}\colon K_n^M(E) \to K_n^M(F)$ in definition 7.5. Use Weil's Formula (7.5.1) to show that when $n = 0$ the transfer map $N\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by $[E : F]$, and that when $n = 1$ the transfer map $N\colon E^\times \to F^\times$ is the usual norm map.

**7.6** Suppose that the degree of every finite extension of a field $F$ is a power of some fixed prime $p$. If $E$ is an extension of degree $p$ and $n > 0$, use Ex. 6.2 to show that $K_n^M(E)$ is generated by elements of the form $\{y, x_2, \dots, x_n\}$, where $y \in E^\times$ and the $x_i$ are in $F^\times$.

**7.7** *Ramification and the transfer.* Let $F'$ and $E = F(a)$ be finite field extensions of $F$, and suppose that the irreducible polynomial $\pi \in F[t]$ of $a$ has a decomposition $\pi = \prod \pi^{e_i}$ in $F'[t]$. Let $E_i$ denote $F'(a_i)$, where each $a_i$ has minimal polynomial $\pi_i$. Show that the following diagram commutes.

$$\begin{array}{ccc}
K_n^M(E) & \xrightarrow{e_1,\dots,e_r} & \oplus K_n^M(E_i) \\
{\scriptstyle N_{a/F}}\big\downarrow & & \big\downarrow{\scriptstyle \sum N_{a_i/F'}} \\
K_n^M(F) & \longrightarrow & K_n^M(F')
\end{array}$$

**7.8** *Ramification and* $\partial_v$. Suppose that $E$ is a finite extension of $F$, and that $w$ is a valuation on $E$ over the valuation $v$ on $F$, with ramification index $e$. (See 6.3.1.) Use the formulas for $\partial_v$ and $\partial_w$ in Theorem 7.3 to show that for every $x \in K_n^M(F)$ we have $\partial_w(x) = e \cdot \partial_v(x)$ in $K_{n-1}^M(k_w)$

**7.9** If $E/F$ is a normal extension of prime degree $p$, and $v$ is a valuation on $F(t)$ trivial on $F$, show that $\partial_v N_{E(t)/F(t)} = \sum_w N_{E(w)/F(v)}\partial_w$, where the sum is over all the valuations $w$ of $E(t)$ over $v$. *Hint:* If $F(t)_v$ and $E(t)_w$ denote the completions of $F(t)$ and $E(t)$ at $v$ and $w$, respectively, use Ex. 7.7 and Lemma 7.6.3 to show that the following diagram commutes.

$$\begin{array}{ccccc}
K^M_{n+1}E(t) & \longrightarrow & \bigoplus_w K^M_{n+1}E(t)_w & \xrightarrow{\partial} & \bigoplus_w K^M_n E(w) \\
{\scriptstyle N_{E(t)/F(t)}}\big\downarrow & & \big\downarrow{\scriptstyle \sum_w N_{E(t)_w/F(t)_v}} & & \big\downarrow{\scriptstyle \sum_w N_{e(w)/F(v)}} \\
K^M_n F(t) & \longrightarrow & K^M_n F(t)_w & \xrightarrow{\partial} & K^M_n F(v)
\end{array}$$

**7.10** If $v$ is a valuation on $F$, and $x \in K^M_i(F)$, $y \in K^M_j(F)$, show that

$$\partial_v(xy) = \lambda(x)\partial_v(y) + (-1)^j \partial_v(x)\rho(y)$$

where $\rho\colon K^M_*(F) \to K^M_*(k_v)$ is a ring homomorphism characterized by the formula $\rho(l(u\pi^i)) = l((-1)^i\bar{u})$.

**7.11** Let $t = 2^{n-1}$ and set $z = \prod_{i=1}^n \big(\langle a_i\rangle - 1\big)$; this is a generator of the ideal $\hat{I}^n$ in $K_0\mathbf{SBil}(F)$. Show that the Stiefel-Whitney invariant $w(z)$ is equal to: $1 + \{a_n, \dots, a_n, -1, -1, \dots, -1\}$ if $n$ is odd, and to $1 + \{a_n, \dots, a_n, -1, -1, \dots, -1\}$ if $n$ is even. This shows that the invariants $w_i$ vanish on the ideal $\hat{I}^n$ if $i < t = 2^{n-1}$, and that $w_t$ induces a homomorphism from $I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$ to $K^M_t(F)/2$.

For example, this implies that $w_1$ vanishes on $\hat{I}^2$, while $w_2$ and $w_3$ vanish on $\hat{I}^3$.

**7.12** (Izhboldin) Let $L/F$ be a field extension of degree $p = \mathrm{char}(F)$, with Galois group $G$. Show that $\Omega^n_F$ is isomorphic to $(\Omega^n_L)^G$, and that $\Omega^n_F/d\Omega^{n-1}_F$ is isomorphic to $(\Omega^n_L/d\Omega^{n-1}_L)^G$. Conclude that $\nu(n)_F \cong \nu(n)^G_L$.

**7.13** In this exercise we complete the proof of proposition 7.8.2, and establish a special case of 7.8.3. Suppose that $E(x)$ is a degree $p$ field extension of $E$, $\mathrm{char}(E) = p$, and that $\sigma$ is a generator of $Gal(E(x)/E)$. Suppose in addition that the norm map $E(x)^\times \to E^\times$ is onto, and that $E$ has no extensions of degree $< p$. Modify the proof of proposition 6.6.2 to show that the following sequence is exact:

$$K^M_n E(x) \xrightarrow{1-\sigma} K^M_n E(x) \xrightarrow{N} K^M_n E \to 0.$$

**7.14** Suppose that $F$ is a field of infinite transcendence degree $\kappa$ over the ground field. Show that the image of the *dlog* symbol of 7.7 lies in the kernel of $\Omega^n_F \xrightarrow{d} \Omega^{n+1}_F$. Using Ex. 6.11, show that $K^M_n(F)$ has cardinality $\kappa$ for all $n > 0$.

If $F$ is a local field, this and Ex. 7.4 implies that $K^M_n(F)$ is an uncountable, uniquely divisible abelian group.