

BASS' NK GROUPS AND cdh -FIBRANT HOCHSCHILD HOMOLOGY.

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ABSTRACT. The K -theory of a polynomial ring $R[t]$ contains the K -theory of R as a summand. For R commutative and containing \mathbb{Q} , we describe $K_*(R[t])/K_*(R)$ in terms of Hochschild homology and the cohomology of Kähler differentials for the cdh topology.

We use this to address Bass' question, whether $K_n(R) = K_n(R[t])$ implies $K_n(R) = K_n(R[t_1, t_2])$. The answer to this question is affirmative when R is essentially of finite type over the complex numbers, but negative in general.

In 1972, H. Bass posed the following question (see [2], question (VI)_n):

Does $K_n(R) = K_n(R[t])$ imply that $K_n(R) = K_n(R[t_1, t_2])$?

One can rephrase the question in terms of Bass' groups NK_n , introduced in [1]:

Does $NK_n(R) = 0$ imply that $N^2K_n(R) = 0$?

More generally, for any functor F from rings to an abelian category, Bass defines $NF(R)$ as the kernel of the map $F(R[t]) \rightarrow F(R)$ induced by evaluation at $t = 0$, and $N^2F = N(NF)$. Bass' question was inspired by Traverso's theorem [24], from which it follows that $NPic(R) = 0$ implies $N^2Pic(R) = 0$.

In this paper, we give a new interpretation of the groups $NK_n(R)$ in terms of Hochschild homology and the cohomology of Kähler differentials for the cdh topology, for commutative \mathbb{Q} -algebras. This allows us to give a counterexample to Bass' question in the companion paper [6] (see Theorem 0.2 below).

To state our main structural theorem, recall from [28] that each $NK_n(R)$ has the structure of a module over the ring of big Witt vectors $W(R)$. It is convenient to use the countably infinite-dimensional \mathbb{Q} -vector spaces $t\mathbb{Q}[t]$ and $\Omega_{\mathbb{Q}[t]}^1$. If M is any R -module, then $M \otimes t\mathbb{Q}[t]$ and $M \otimes \Omega_{\mathbb{Q}[t]}^1$ are naturally $W(R)$ -modules by [10].

Theorem 0.1. *Let R be a commutative ring containing \mathbb{Q} . Then there is a $W(R)$ -module isomorphism*

$$N^2K_n(R) \cong (NK_n(R) \otimes t\mathbb{Q}[t]) \oplus (NK_{n-1}(R) \otimes \Omega_{\mathbb{Q}[t]}^1).$$

Thus $K_n(R) = K_n(R[t_1, t_2])$ iff $NK_n(R) = NK_{n-1}(R) = 0$ iff $N^2K_n(R) = 0$.

In addition, the following are equivalent for all $p > 0$:

- a) $K_n(R) = K_n(R[t_1, \dots, t_p])$
- b) $NK_n(R) = 0$ and $K_{n-1}(R) = K_{n-1}(R[t_1, \dots, t_{p-1}])$.
- c) $NK_q(R) = 0$ for all q such that $n - p < q \leq n$.

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The equivalence of (a), (b) and (c) is immediate by induction, using the formula for N^2K_n , and is included for its historical importance; see [25]. Theorem 0.1 also holds for the K -theory of schemes of finite type over a field; see 4.2 below.

Theorem 0.1 allows us to reformulate Bass' question as follows:

Does $NK_n(R) = 0$ imply that $NK_{n-1}(R) = 0$?

Theorem 0.2. *a) For any field F algebraic over \mathbb{Q} , the 2-dimensional normal algebra*

$$R = F[x, y, z]/(z^2 + y^3 + x^{10} + x^7y)$$

has $K_0(R) = K_0(R[t])$ but $K_0(R) \neq K_0(R[t_1, t_2])$.

b) Suppose R is essentially of finite type over a field of infinite transcendence degree over \mathbb{Q} . Then $NK_n(R) = 0$ implies that R is K_n -regular and, in particular, that $K_n(R) = K_n(R[t_1, t_2])$.

Part (a) is proven in the companion paper [6], using Theorem 0.1, while part (b) is proven below as Corollary 6.7.

The proof of Theorem 0.1 relies on methods developed in [5] and [7], which allow us to compute the groups NK_n and N^pK_n in terms of the Hochschild homology of R , and of the *cdh*-cohomology of the higher Kähler differentials Ω^p , both relative to \mathbb{Q} . The groups $NK_n(R)$ have a natural bigraded structure when $\mathbb{Q} \subset R$, and it is convenient to take advantage of this bigrading in stating our results. The bigrading comes from the eigenspaces $NK_n^{(i)}(R)$ of the Adams operations ψ^k (arising from the λ -filtration) and the eigenspaces of the homothety operations $[r]$ (*i.e.*, base change for $t \mapsto rt$). This bigrading will be explained in Sections 1 and 5; the general decomposition for Adams weight i has the form:

$$(0.3) \quad NK_n^{(i)}(R) \cong TK_n^{(i)}(R) \otimes_{\mathbb{Q}} t\mathbb{Q}[t].$$

Here $TK_n^{(i)}$ denotes the *typical piece* of $NK_n^{(i)}(R)$, defined as the simultaneous eigenspace $\{x \in NK_n^{(i)}(R) : [r]x = rx, r \in R\}$. (See Example 1.6.) We provide a concrete description of the typical pieces in Theorem 5.1, reproduced here:

Theorem 0.4. *If R is a commutative \mathbb{Q} -algebra, then $NK_n^{(i)}(R)$ is determined by its typical pieces $TK_n^{(i)}(R)$ and (0.3). For $i \neq n, n+1$ we have:*

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R) & \text{if } i < n, \\ H_{\text{cdh}}^{i-n-1}(R, \Omega^{i-1}) & \text{if } i \geq n+2. \end{cases}$$

For $i = n, n+1$, we have an exact sequence:

$$0 \rightarrow TK_{n+1}^{(n+1)}(R) \rightarrow \Omega_R^n \rightarrow H_{\text{cdh}}^0(R, \Omega^n) \rightarrow TK_n^{(n+1)}(R) \rightarrow 0.$$

The special case $NK_0 = \bigoplus NK_0^{(i)}$ of Theorem 0.4 is that for R essentially of finite type over a field of characteristic zero, with $d = \dim(R)$,

$$(0.5) \quad NK_0(R) \cong \left((R^+/R_{\text{red}}) \oplus \bigoplus_{p=1}^{d-1} H_{\text{cdh}}^p(R, \Omega^p) \right) \otimes_{\mathbb{Q}} t\mathbb{Q}[t].$$

Here R^+ is the seminormalization of R_{red} ; we show in Proposition 2.5 that $R^+ = H_{\text{cdh}}^0(R, \mathcal{O})$. The dimension zero case of Theorem 0.4 is also revealing:

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$TK_3^{(i)}(R)$	0	$HH_2^{(1)}(R)$	$\text{tors } \Omega_R^2$	$\Omega_{cdh}^3(R)/\Omega_R^3$	$H_{cdh}^1 \Omega^4$	0
$TK_2^{(i)}(R)$	0	$\text{tors } \Omega_R^1$	$\Omega_{cdh}^2(R)/\Omega_R^2$	$H_{cdh}^1 \Omega^3$	0	
$TK_1^{(i)}(R)$	$\text{nil}(R)$	$\Omega_{cdh}^1(R)/\Omega_R^1$	$H_{cdh}^1 \Omega^2$	0		
$TK_0^{(i)}(R)$	R^+/R	$H_{cdh}^1 \Omega^1$	0			
$TK_{-1}^{(i)}(R)$	$H_{cdh}^1 \mathcal{O}$	0				
$TK_{-2}(R)$	0					

Table 1. The groups $TK_n^{(i)}(R)$ for $n \leq 3$, $\dim(R) = 2$.

Example 0.6. If $\dim(R) = 0$ then we get $NK_n(R) \cong HH_{n-1}(R, I) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ for all n , where I is the nilradical of R . It is illuminating to compare this with Goodwillie's Theorem [12], which implies that $NK_n(R) \cong NK_n(R, I) \cong NHC_{n-1}(R, I)$. The identification comes from the standard observation (1.2) that the map $HH_* \rightarrow HC_*$ induces $NHC_*(R, I) \cong HH_*(R, I) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$.

The calculations of Theorem 0.4 for small n are summarized in Table 1 when $\dim(R) = 2$. We will need the following cases of 0.4 in [6], to prove Theorem 0.2(a).

Theorem 0.7. *Let R be normal domain of dimension 2 which is essentially of finite type over an algebraic extension of \mathbb{Q} . Then*

- a) $NK_0(R) = NK_0^{(2)}(R) \cong H_{cdh}^1(R, \Omega^1) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ and
- b) $NK_{-1}(R) = NK_{-1}^{(1)}(R) \cong H_{cdh}^1(R, \mathcal{O}) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$.

Here is an overview of this paper: Section 1 reviews the bigrading on the Hochschild and cyclic homology of $R[t]$ (and $X \times \mathbb{A}^1$), and Section 2 reviews the cdh -fibrant analogue. Section 3 describes the sheaf cohomology of the fibers $\mathcal{F}_{HH}(X)$, $\mathcal{F}_{HC}(X)$, etc. of $HH(X) \rightarrow \mathbb{H}_{cdh}(X, HH)$, etc. In Section 4 we use these fibers to prove Theorem 0.1, by relating $NK_{n+1}(X)$ to $H^{-n}\mathcal{F}_{HH}(X)$. We also show that Bass' question is negative for schemes in Lemma 4.5.

In Section 5, we give the detailed computations of the typical pieces $TK_n^{(i)}(R)$ needed to establish (0.5) and Table 1; these computations employ the main result of [8]. In Section 6, we prove Theorem 0.2(b), that the answer to Bass' question is positive provided we are working over a sufficiently large base field. Finally, Section 7 describes how Theorem 0.7 changes if R is of finite type over an arbitrary field of characteristic 0: the map $NK_0(R) \rightarrow H_{cdh}^1(R, \Omega_{/F}^1) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ is onto, and an isomorphism if $NK_{-1}(R) = 0$.

Notation. All rings considered in this paper should be assumed to be commutative and noetherian, unless otherwise stated. Throughout this paper, k denotes a field of characteristic 0 and F is a field containing k as a subfield. We write Sch/k for the category of separated schemes essentially of finite type over k . If \mathcal{F} is a presheaf on Sch/k , we write \mathcal{F}_{cdh} for the associated cdh sheaf, and often simply write $H_{cdh}^*(X, \mathcal{F})$ in place of the more formal $H_{cdh}^*(X, \mathcal{F}_{cdh})$.

If H is a functor on Sch/k and R is an algebra essentially of finite type, we occasionally write $H(R)$ for $H(\text{Spec } R)$. For example, $H_{cdh}^*(R, \Omega^i)$ is used for $H_{cdh}^*(\text{Spec } R, \Omega^i)$. Note that, because the cdh site is noetherian (every cover has

a finite subcovering) H_{cdh}^* sends inverse limits of schemes over diagrams with affine transition morphisms to direct limits.

If H is a contravariant functor from Sch/k to spectra, (co)chain complexes, or abelian groups that takes filtered inverse limits of schemes over diagrams with affine transition morphisms to colimits (as for example K , HH , $\mathbb{H}_{\text{cdh}}(-, HH)$, and \mathcal{F}_{HH}), then for any k -algebra R , we abuse notation and write $H(R)$ for the direct limit of the $H(R_\alpha)$ taken over all subrings R_α of R of finite type over k . (If R is essentially of finite type, the two definitions of $H(R)$ agree up to canonical isomorphism.) In particular, we will use expressions like $\mathbb{H}_{\text{cdh}}(R, HH)$ for general commutative \mathbb{Q} -algebras even though we do not define the *cdh*-topology for arbitrary \mathbb{Q} -schemes.

We use cohomological indexing for all chain complexes in this paper; for a complex C , $C[p]^q = C^{p+q}$. For example, the Hochschild, cyclic, periodic, and negative cyclic homology of schemes over a field k can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [32] and [5, 2.7] for precise definitions. We shall write these presheaves as $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$, respectively, omitting k from the notation if it is clear from the context.

It is well known (see [31, 10.9.19]) that there is an Eilenberg-Mac Lane functor $C \mapsto |C|$ from chain complexes of abelian groups to spectra, and from presheaves of chain complexes of abelian groups to presheaves of spectra. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra, and satisfies $\pi_n(|C|) = H^{-n}(C)$. For example, applying π_n to the Chern character $K \rightarrow |HN|$ yields maps $K_n(R) \rightarrow H^{-n}HN(R) = HN_n(R)$. In this spirit, we will use descent terminology for presheaves of complexes.

1. THE BIGRADING ON NHH AND NHC

Recall that k denotes a field of characteristic 0. In this section, we consider the Hochschild and cyclic homology of polynomial extensions of commutative k -algebras. No great originality is claimed. Throughout, we will use the chain level Hodge decompositions $HH = \prod_{i \geq 0} HH^{(i)}$ and $HC = \prod_{i \geq 0} HC^{(i)}$.

The Künneth formula for Hochschild homology yields

$$(1.1) \quad NHH_n^{(i)}(R) \cong \left(HH_n^{(i)}(R) \otimes t\mathbb{Q}[t] \right) \oplus \left(HH_{n-1}^{(i-1)}(R) \otimes \Omega_{\mathbb{Q}[t]}^1 \right).$$

From the exact SBI sequence $0 \rightarrow NHC_{n-1} \xrightarrow{B} NHH_n \xrightarrow{I} NHC_n \rightarrow 0$ (see [31, 9.9.1]), and induction on n , the map I induces canonical isomorphisms for each i :

$$(1.2) \quad NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes t\mathbb{Q}[t].$$

Remark 1.3. Both (1.1) and (1.2) generalize to non-affine quasi-compact schemes X over k . Indeed, NHH and NHC satisfy Zariski descent because HH and HC do and because, for any open cover $\{U_i \rightarrow X\}$, the collection $\{U_i \times \mathbb{A}^1 \rightarrow X \times \mathbb{A}^1\}$ is also a cover. Thus we have

$$\begin{aligned} NHH^{(i)}(X) &\cong \mathbb{H}_{\text{Zar}}(X, NHH^{(i)}) \\ &\cong \mathbb{H}_{\text{Zar}}(X, HH^{(i)}) \otimes t\mathbb{Q}[t] \oplus \mathbb{H}_{\text{Zar}}(X, HH^{(i-1)})[1] \otimes \Omega_{\mathbb{Q}[t]}^1 \\ &\cong HH^{(i)}(X) \otimes t\mathbb{Q}[t] \oplus HH^{(i-1)}(X)[1] \otimes \Omega_{\mathbb{Q}[t]}^1, \end{aligned}$$

and $NHC^{(i)}(X) = \mathbb{H}_{\text{Zar}}(X, NHC^{(i)}) \cong \mathbb{H}_{\text{Zar}}(X, HH^{(i)}) \otimes t\mathbb{Q}[t] \cong HH^{(i)}(X) \otimes t\mathbb{Q}[t]$.

It is easy to iterate the construction $F \mapsto NF$. For example, we see from (1.1) and (1.2) that

$$(1.4) \quad N^2 HC_n^{(i)}(R) \cong \left(HH_n^{(i)}(R) \otimes t\mathbb{Q}[t] \otimes t\mathbb{Q}[t] \right) \oplus \left(HH_{n-1}^{(i-1)}(R) \otimes t\mathbb{Q}[t] \otimes \Omega_{\mathbb{Q}[t]}^1 \right).$$

By induction, we see that $HH_{n-j}^{(i-j)}(R) \otimes (t\mathbb{Q}[t])^{\otimes(p-j)} \otimes (\Omega_{\mathbb{Q}[t]}^1)^{\otimes j}$ will occur $\binom{p-1}{j}$ times as a summand of $N^p HC_n^{(i)}(R)$ for all $j \geq 0$. We may write this as the formula:

$$(1.5) \quad N^p HC_n^{(i)}(R) \cong \bigoplus_{j=0}^{p-1} HH_{n-j}^{(i-j)}(R) \otimes_k \wedge^j k^{p-1} \otimes (t\mathbb{Q}[t])^{\otimes(p-j)} \otimes (\Omega_{\mathbb{Q}[t]}^1)^{\otimes j}.$$

Cartier operations on NHH and NHC . Let $W(R)$ denote the ring of big Witt vectors over R ; it is well known that in characteristic 0 we have $W(R) \cong \prod_1^\infty R$. (See [28, p. 468] for example.) Cartier showed in [3] that the endomorphism ring $\text{Carf}(R)$ of the additive functor underlying W consists of column-finite sums $\sum V_m[r_{mn}]F_n$, using the *homotheties* $[r]$ (for $r \in R$), and the Verschiebung and Frobenius operators V_m and F_m . Restricting the sum to $m \geq m_0$ yields a descending sequence of ideals of $\text{Carf}(R)$, making it complete as a topological ring; $W(R)$ is the complete topological subring of all sums $\sum V_m[r_m]F_m$; see [3].

We will be interested in the intermediate (topological) subring $\text{Carf}(R)$ of all row and column-finite sums $\sum V_m[r_{mn}]F_n$. As observed in [10, 2.14], there is an equivalence between the category of R -modules and the category of continuous $\text{Carf}(R)$ -modules given by the constructions in the following example. (A left module M is *continuous* if the annihilator ideal of each element is an open left ideal.)

Example 1.6. If M is any R -module, $N = M \otimes t\mathbb{Q}[t]$ is a continuous $\text{Carf}(R)$ -module (and hence a $W(R)$ -module) via the formulas:

$$[r]t^i = r^i t^i, \quad V_m(t^i) = t^{mi}, \quad F_m(t^i) = \begin{cases} mt^{i/m} & \text{if } m|i, \\ 0 & \text{else.} \end{cases}$$

The ring $W(R) = \prod_1^\infty R$ acts on $M \otimes t\mathbb{Q}[t]$ by $(r_1, \dots, r_n, \dots) * \sum m_i t^i = \sum (r_i m_i) t^i$. Conversely, every continuous $\text{Carf}(R)$ -module N has a ‘‘typical piece’’ M , defined as the simultaneous eigenspace $\{x \in N : [r]x = rx, r \in R\}$, and $N \cong M \otimes t\mathbb{Q}[t]$.

Recall that we can define operators $[r]$ on $NHH_n(R)$ and $NHC_n(R)$, associated to the endomorphisms $t \mapsto rt$ of $R[t]$. There are also operators V_m and F_m , defined via the ring inclusions $R[t^m] \subset R[t]$ and their transfers. These operations commute with the Hodge decomposition. The following result follows immediately from [10, 4.11] using the observation that everything commutes with Adams operations.

Proposition 1.7. *The operators $[r]$, V_m and F_m make each $NHC_n^{(i)}(R)$ into a continuous $\text{Carf}(R)$ -module, and hence a $W(R)$ -module. The R -module $HH_n^{(i)}(R)$ is its typical piece, and the canonical isomorphism $NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes t\mathbb{Q}[t]$ of (1.2) is an isomorphism of $\text{Carf}(R)$ -modules, the module structure on the right being given in Example 1.6.*

A similar structure theorem holds for $NHH_n(R)$ and its Hodge components, using (1.1). However, it uses a non-standard R -module structure on the typical piece $HH_n(R) \oplus HH_{n-1}(R)$; see [10, 3.3] for details.

Remark 1.7.1. The conclusions of Proposition 1.7 still hold for $NHC_n^{(i)}(X)$ and $HH_n^{(i)}(X)$ when X is any scheme, where $W(R)$ and $\text{Carf}(R)$ refer to the ring $R = H^0(X, \mathcal{O})$. That is, $HH_n^{(i)}(X)$ is an R -module and $NHC_n^{(i)}(X)$ is a continuous $\text{Carf}(R)$ -module, isomorphic to $HH_n^{(i)}(X) \otimes t\mathbb{Q}[t]$.

This scheme version of 1.7 is not stated in [10], which was written before the cyclic homology of schemes was developed in [32]. However, the proof in [10] is easily adapted. Since the operators V_m , F_m and $[r]$ are defined on the underlying chain complexes in [10, 4.1], they extend to operations on the Hochschild and cyclic homology of schemes. The identities required to obtain continuous $\text{Carf}(R)$ -module structures all come from the Künneth formula for the shuffle product on the chain complexes (see [10, 4.3]), so they also hold for the homology of schemes.

2. cdh -FIBRANT HH AND NHC

Now fix a field F containing k ; all schemes will lie in the category Sch/F (essentially of finite type over F), in order to use the cdh topology on Sch/F of [22]. All rings will be commutative F -algebras; because they are filtered direct limits of finitely generated F -algebras, we can consider their cdh -cohomology.

If C is any (pre-)sheaf of cochain complexes on Sch/F , we can form the cdh -fibrant replacement $X \mapsto \mathbb{H}_{cdh}(X, C)$ and write $\mathbb{H}_{cdh}^n(X, C)$ for the n th cohomology of this complex. (The fibrant replacement is taken with respect to the local injective model structure, as in [5, 3.3].) For example, the cdh -fibrant replacement of a cdh sheaf C (concentrated in degree zero) is just an injective resolution, and $\mathbb{H}_{cdh}^n(X, C)$ is the usual cohomology of the cdh sheaf associated to C .

Hochschild and cyclic homology, as well as differential forms, will be taken relative to k . For $C = HH^{(i)}$, it was shown in [7, Theorem 2.4] that

$$(2.1) \quad \mathbb{H}_{cdh}(X, HH^{(i)}) \cong \mathbb{H}_{cdh}(X, \Omega^i)[i].$$

This has the following consequence for $C = NHH^{(i)}$ and $NHC^{(i)}$.

Lemma 2.2. *Let $H^{(i)}$ denote either $HH^{(i)}$ or $HC^{(i)}$, taken relative to a subfield k of F . Then $\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) = \mathbb{H}_{cdh}(X, H^{(i)}) \oplus \mathbb{H}_{cdh}(X, NH^{(i)})$, and:*

$$\mathbb{H}_{cdh}(X, NHH^{(i)}) \cong (\mathbb{H}_{cdh}(X, \Omega^i)[i] \otimes t\mathbb{Q}[t]) \oplus (\mathbb{H}_{cdh}(X, \Omega^{i-1})[i] \otimes \Omega_{\mathbb{Q}[t]}^1);$$

$$\mathbb{H}_{cdh}(X, NHC^{(i)}) \cong \mathbb{H}_{cdh}(X, \Omega^i)[i] \otimes t\mathbb{Q}[t].$$

Proof. The displayed formulas follow from (1.1), (1.2) and (2.1), using the fact that $-\otimes t\mathbb{Q}[t]$ commutes with \mathbb{H}_{cdh} . Thus it suffices to verify the first assertion. By resolution of singularities, we may assume that X is smooth.

Recall from [5, 3.2.2] that the restriction of the cdh topology to Sm/k is called the $scdh$ -topology. The product of any $scdh$ cover of X with \mathbb{A}^1 is an $scdh$ cover of $X \times \mathbb{A}^1$, and both $HH^{(i)}$ and $HC^{(i)}$ satisfy $scdh$ -descent by [7, Thm. 2.4]. Now by Thomason's Cartan-Leray Theorem [23, 1.56] we have

$$\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) \cong \mathbb{H}_{cdh}(X, H^{(i)}(- \times \mathbb{A}^1)) \cong \mathbb{H}_{cdh}(X, H^{(i)}) \oplus \mathbb{H}_{cdh}(X, NH^{(i)}).$$

This gives the first assertion. Alternatively, we may prove the first assertion by induction on $\dim(X)$, using the definition of $scdh$ descent to see that for smooth X we have $H^{(i)}(X) = \mathbb{H}_{cdh}(X, H^{(i)})$ and

$$\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) = H^{(i)}(X \times \mathbb{A}^1) = H^{(i)}(X) \oplus NH^{(i)}(X).$$

In particular, the first assertion holds when $\dim(X) = 0$. \square

Remark 2.2.1. If R is any commutative F -algebra, the formulas of Lemma 2.2 hold for $X = \text{Spec}(R)$ by naturality. This is because we may write $R = \varinjlim R_\alpha$, where R_α ranges over subrings of finite type over F , and $\mathbb{H}_{cdh}(X, -) = \varinjlim \mathbb{H}_{cdh}(\text{Spec}(R_\alpha), -)$.

Corollary 2.3. *If $X = \text{Spec}(R)$ is in Sch/F , the modules $\mathbb{H}_{cdh}^n(X, HH^{(i)})$ and $\mathbb{H}_{cdh}^n(X, NHC^{(i)})$ are zero unless $0 \leq n + i < \dim(X)$ and $i \geq 0$.*

If $n \geq \dim(X)$ and $n > 0$ then $\mathbb{H}_{cdh}^n(X, HH) = 0$.

Proof. Because $\mathbb{H}_{cdh}^n(X, \Omega^i)[i] = H_{cdh}^{i+n}(X, \Omega^i)$, this follows from (2.1), Lemma 2.2 and the fact that $H_{cdh}^n(X, \Omega^i) = 0$ for $n \geq \dim(X)$, $n > 0$. This bound is given in [5, 6.1] for $i = 0$, and in [7, 2.6] for general i . \square

Here is a useful bound on the cohomology groups appearing in Lemma 2.2. Given X , let Q denote the total ring of fractions of X_{red} ; it is a finite product of fields Q_j , and we let e denote the maximum of the transcendence degrees $\text{tr. deg}(Q_j/k)$.

Lemma 2.4. *Let X be in Sch/F . If $i > e$ then $H_{cdh}^n(X, \Omega^i) = 0$ for all n .*

Proof. By [19, 12.24], we may assume X reduced. Since we may write X as an inverse limit of a sequence of affine morphisms of schemes of finite type with the same ring of total fractions Q , and cdh -cohomology sends such an inverse limit to a direct limit, we may also assume that X is of finite type over F . This implies that $e = \dim(X) + \text{tr. deg}(F/k)$.

The result is clear if $\dim(X) = 0$, since $H_{cdh}^n(X, -) = H_{Zar}^n(X, -)$ in that case. Proceeding by induction on $\dim(X)$, choose a resolution of singularities $X' \rightarrow X$ and observe that the singular locus Y and $Y \times_X X'$ have smaller dimension. The hypothesis implies that $\Omega^i = 0$ on X'_{Zar} , so $H_{cdh}^n(X', \Omega^i) = 0$ by [7, 2.5]. The result now follows by induction from the Mayer-Vietoris sequence of [22, 12.1]. \square

If R is a commutative ring, we write R_{red} and R^+ for the associated reduced ring and the seminormalization of R_{red} , respectively. These constructions are natural with respect to localization, so that we may form the seminormalization X^+ of X_{red} for any scheme X . Because $X^+ \rightarrow X$ is a universal homeomorphism, we have $H_{cdh}^*(X, -) \cong H_{cdh}^*(X^+, -)$ for every X in Sch/k , for any field k of arbitrary characteristic. The case $n = 0$ with coefficients \mathcal{O}_{cdh} is of special interest; recall our convention that $H_{cdh}^0(X, \mathcal{O})$ denotes $H_{cdh}^0(X, \mathcal{O}_{cdh})$.

Proposition 2.5. *For any algebra R , we have $H_{cdh}^0(\text{Spec } R, \mathcal{O}) = R^+$. Moreover, for every X in Sch/F we have $H_{cdh}^0(X, \mathcal{O}) = \mathcal{O}(X^+)$.*

Proof. We may assume R and X are reduced. Writing $R = \varinjlim R_\alpha$ as in Remark 2.2.1, we have $R^+ = \varinjlim R_\alpha^+$ and $H_{cdh}^0(R, \mathcal{O}) = \varinjlim H_{cdh}^0(R_\alpha, \mathcal{O})$, so we may assume that R is of finite type. Thus the second assertion implies the first. Since $H_{cdh}^0(-, \mathcal{O})$ and $\mathcal{O}(-^+)$ are Zariski sheaves, it suffices to consider the case when X is affine.

Let $X = \text{Spec } R$ be in Sch/F , with R reduced. There is an injection $R \rightarrow Q$ with Q regular (for example, Q could be the total quotient ring of R). By [5, 6.3], $H_{\text{cdh}}^0(\text{Spec } Q, \mathcal{O}) = Q$, so R injects into $H_{\text{cdh}}^0(\text{Spec } R, \mathcal{O})$. This implies that \mathcal{O}_{red} is a separated presheaf for the cdh topology on Sch/F . Thus, the ring $H_{\text{cdh}}^0(X, \mathcal{O})$ is the direct limit over all cdh -covers $p : U \rightarrow X$ of the Čech H^0 . (See [SGA4, 3.2.3].)

Fix an element $b \in H_{\text{cdh}}^0(\text{Spec } R, \mathcal{O})$ and represent it by $b \in \mathcal{O}(U)$ for some cdh cover $U \rightarrow X$. Now recall from [19, 12.28] or [22, 5.9] that we may assume, by refining the cdh cover $U \rightarrow X$, that it factors as $U \rightarrow X' \rightarrow X$ where $X' \rightarrow X$ is proper birational cdh cover and $U \rightarrow X'$ is a Nisnevich cover. If the images of $b \in \mathcal{O}(U)$ agree in $U \times_X U$, *i.e.*, b is a Čech cycle for U/X , then its images agree in $U \times_{X'} U$, *i.e.*, it is a Čech cycle for U/X' . But by faithfully flat descent, b descends to an element of $\mathcal{O}(X')$. Thus we can assume that U is proper and birational over X .

Next, we can assume that the Nisnevich cover $p : U \rightarrow X$ is finite, surjective and birational. Indeed, since p is proper and birational we may consider the Stein factorization $U \xrightarrow{q} Y \xrightarrow{r} X$. By [EGAIII, 4.3] or [16, III.11.5 & proof], $q_*(\mathcal{O}_U) = \mathcal{O}_Y$ and r is finite surjective and birational. By [22, 5.8], r is also a cdh cover. Because $q_*(\mathcal{O}_U) = \mathcal{O}_Y$, the canonical map $\mathcal{O}_Y(Y) \rightarrow q_*(\mathcal{O}_U)(Y) = \mathcal{O}_U(U)$ is an isomorphism. Hence b descends to an element of $\mathcal{O}(Y)$. By Lemma 2.6, b lies in the seminormalization of R . \square

Lemma 2.6. *Let A be a seminormal ring and B a ring between A and its normalization. Then the Čech complex $A \rightarrow B \rightarrow B \otimes_A B$ is exact.*

Proof. We use Traverso's description of the seminormalization (see [24, p. 585]): the seminormalization of a ring A inside a ring B is

$$A^+ = \{b \in B \mid (\forall P \in \text{Spec } A) b \in A_P + \text{rad}(B_P)\}.$$

Let $b \in B$ such that $1 \otimes b = b \otimes 1$. We have to show that $b \in A_P + \text{rad}(B_P)$, for all primes P of A . Let $J = \text{rad}(B_P)$; since B_P/J is faithfully flat over the field A_P/P , the image of b in B_P/J lies in A_P/P by flat descent. That is, $b \in A_P + J$, as required. \square

Remark 2.7. Even if X is affine seminormal, it can happen that $H_{\text{cdh}}^i(X, \mathcal{O}) \neq 0$ for some $i > 0$. For example, if R denotes the subring $F[x, g, yg]$ of $F[x, y]$ for $g = x^3 - y^2$ then it is easy to show that R is seminormal and that $H_{\text{cdh}}^1(\text{Spec}(R), \mathcal{O}) = F$, because the normalization of R is $F[x, y]$ and the conductor ideal is $gF[x, y]$. For another example, the normal ring of Theorem 0.2 has $H_{\text{cdh}}^1(X, \mathcal{O}) \neq 0$, by Theorems 0.1 and 0.7(b).

3. THE FIBERS \mathcal{F}_{HH} AND \mathcal{F}_{HC}

If C is a presheaf of complexes on Sch/F , we write \mathcal{F}_C for the shifted mapping cone of $C \rightarrow \mathbb{H}_{\text{cdh}}(-, C)$, so that we have a distinguished triangle:

$$(3.1) \quad \mathbb{H}_{\text{cdh}}(X, C)[-1] \rightarrow \mathcal{F}_C(X) \rightarrow C(X) \rightarrow \mathbb{H}_{\text{cdh}}(X, C)$$

Example 3.1.1. When C is concentrated in degree 0 we have $H^n \mathcal{F}_C = 0$ for all $n < 0$. For $C = \mathcal{O}$ and $X = \text{Spec}(R)$, we see from Proposition 2.5 that $H^0 \mathcal{F}_{\mathcal{O}}(X) = \text{nil}(R)$, $H^1 \mathcal{F}_{\mathcal{O}}(X) = R^+/R$, and $H^n \mathcal{F}_{\mathcal{O}}(X) = H_{\text{cdh}}^{n-1}(X, \mathcal{O})$ for $n \geq 2$. Note that, if $X = \text{Spec } R \in \text{Sch}/F$, then $H^n \mathcal{F}_{\mathcal{O}}(X) = 0$ for $n > \dim(X)$ by [5, 6.1].

We now consider the Hochschild and cyclic homology complexes, taken relative to a subfield k of F . For legibility, we write $\mathcal{F}_{HH}^{(i)}$ for $\mathcal{F}_{HH^{(i)}}$, etc. By the usual homological yoga, \mathcal{F}_{HH} is the direct sum of the $\mathcal{F}_{HH}^{(i)}$, $i \geq 0$, and similarly for \mathcal{F}_{HC} .

Example 3.1.2. If X is smooth over F then $\mathcal{F}_{HH}(X) \simeq 0$ by [7, 2.4].

Lemma 2.2 and Remarks 2.2.1 and 1.3 imply the following analogue for $N\mathcal{F}$.

Lemma 3.2. *If X is in Sch/F , or if $X = \text{Spec}(R)$ for an F -algebra R , we have quasi-isomorphisms:*

$$\begin{aligned} N\mathcal{F}_{HH}^{(i)}(X) &\cong \left(\mathcal{F}_{HH}^{(i)}(X) \otimes t\mathbb{Q}[t] \right) \oplus \left(\mathcal{F}_{HH}^{(i-1)}(X)[1] \otimes \Omega_{\mathbb{Q}[t]}^1 \right); \\ N\mathcal{F}_{HC}^{(i)}(X) &\cong \mathcal{F}_{HH}^{(i)}(X) \otimes t\mathbb{Q}[t]. \end{aligned}$$

Mimicking the argument that establishes (1.4) and (1.5) yields:

Corollary 3.3. *If X is in Sch/F , or if $X = \text{Spec}(R)$ for an F -algebra R ,*

$$N^2\mathcal{F}_{HC}^{(i)}(X) \cong \left(\mathcal{F}_{HH}^{(i)}(X) \otimes t\mathbb{Q}[t] \otimes t\mathbb{Q}[t] \right) \oplus \left(\mathcal{F}_{HH}^{(i-1)}(X)[1] \otimes t\mathbb{Q}[t] \otimes \Omega_{\mathbb{Q}[t]}^1 \right)$$

and

$$N^p\mathcal{F}_{HC}^{(i)}(X) \cong \bigoplus_{j=0}^{p-1} \mathcal{F}_{HH}^{(i-j)}(X)[j] \otimes_k \wedge^j k^{p-1} \otimes t\mathbb{Q}[t]^{\otimes(p-j)} \otimes \left(\Omega_{\mathbb{Q}[t]}^1 \right)^{\otimes j}.$$

The cohomology of the typical pieces $\mathcal{F}_{HH}^{(i)}(R)$ is given as follows.

Lemma 3.4. *If R is an F -algebra and $i \geq 0$, then there is an exact sequence:*

$$0 \rightarrow H^{-i}\mathcal{F}_{HH}^{(i)}(R) \rightarrow \Omega_R^i \rightarrow H_{\text{cdh}}^0(R, \Omega^i) \rightarrow H^{1-i}\mathcal{F}_{HH}^{(i)}(R) \rightarrow 0.$$

For $n \neq i, i-1$ we have:

$$H^{-n}\mathcal{F}_{HH}^{(i)}(R) \cong \begin{cases} HH_n^{(i)}(R) & \text{if } i < n, \\ H_{\text{cdh}}^{i-n-1}(R, \Omega^i) & \text{if } i \geq n+2. \end{cases}$$

Proof. As in Remark 2.2.1, we may assume R is of finite type. Since $HH_i^{(i)}(R) = \Omega_R^i$ for all $i \geq 0$, and $HH_n^{(i)}(R) = 0$ when $i > n$ (see [31, 9.4.15] or [17, 4.5.10]), it suffices to use (2.1) and to observe that $\mathbb{H}_{\text{cdh}}^{-n}(R, HH^{(i)}) = H_{\text{cdh}}^{i-n}(R, \Omega^i)$ vanishes when $n > i$. \square

Example 3.5. Let $X = \text{Spec}(R)$ be in Sch/F . Since $HH^{(0)} = \mathcal{O}$, $\mathcal{F}_{HH}^{(0)}(R)$ is described in Example 3.1.1. Applying Corollary 2.3 and Lemma 3.4 for $i > 0$, and using [7, 2.6] to bound the terms, we see that if $d = \dim(R)$ then $H^n\mathcal{F}_{HH}(X) = 0$ for $n > d$. If $d = 1$, then the only nonzero positive cohomology of \mathcal{F}_{HH} is $H^1\mathcal{F}_{HH}(R) = R^+/R$; if $d > 1$, we have:

$$\begin{aligned} H^1\mathcal{F}_{HH}(R) &\cong (R^+/R) \oplus H_{\text{cdh}}^1(X, \Omega^1) \oplus \cdots \oplus H_{\text{cdh}}^{d-1}(X, \Omega^{d-1}), \\ H^2\mathcal{F}_{HH}(R) &\cong H_{\text{cdh}}^1(X, \mathcal{O}) \oplus H_{\text{cdh}}^2(X, \Omega^1) \oplus \cdots \oplus H_{\text{cdh}}^{d-1}(X, \Omega^{d-2}), \\ &\vdots \\ H^d\mathcal{F}_{HH}(R) &\cong H_{\text{cdh}}^{d-1}(X, \mathcal{O}). \end{aligned}$$

Example 3.6. When R is essentially of finite type over F and $\text{tr. deg}(F/k) < \infty$, $H^m \mathcal{F}_{HH}(R)$ is Hochschild homology for large negative m . To see this, observe that $e = \text{tr. deg}(R/k)$, the maximum transcendence degree of the residue fields of R at its minimal primes, is finite. Using Lemmas 2.4 and 3.4, we get $H^{-n} \mathcal{F}_{HH}^{(i)}(R) = 0$ and $H^{-n} \mathcal{F}_{HH}^{(n)}(R) = \Omega_R^n$ for $i > n > e$, and hence

$$H^{-n} \mathcal{F}_{HH}(R) \cong HH_n(R) \text{ for all } n > e.$$

If $R = k \oplus R_1 \oplus R_2 \oplus \dots$ is graded, and $\widetilde{HC}_*(R) = HC_*(R)/HC_*(k)$, it is well known that the map $\widetilde{HC}_*(R) \xrightarrow{S} \widetilde{HC}_{*-2}(R)$ is zero. (See [31, 9.9.1] for example.) In Lemma 3.8 below, we prove a similar property for \mathcal{F}_{HH} and \mathcal{F}_{HC} , which we derive from Lemma 3.2 using the following trick.

Standard Trick 3.7. If R is a non-negatively graded algebra, there is an algebra map $\nu : R \rightarrow R[t]$ sending $r \in R_n$ to rt^n . The composition of ν with evaluation at $t = 0$ factors as $R \rightarrow R_0 \rightarrow R$, and so if H is a functor on algebras taking values in abelian groups, then the composition $H(R) \xrightarrow{\nu} H(R[t]) \xrightarrow{t=0} H(R)$ is zero on the kernel $\widetilde{H}(R)$ of $H(R) \rightarrow H(R_0)$. Similarly, the composition of ν with evaluation at $t = 1$ is the identity. That is, ν maps $\widetilde{H}(R)$ isomorphically onto a summand of $NH(R)$, and $\widetilde{H}(R)$ is in the image of $(t = 1) : NH(R) \rightarrow H(R)$.

Lemma 3.8. *If $R = k \oplus R_1 \oplus \dots$ is a graded algebra, then for each m the map $\pi_m \mathcal{F}_{HC}(R) \xrightarrow{S} \pi_{m-2} \mathcal{F}_{HC}(R)$ is zero and there is a split short exact sequence:*

$$0 \rightarrow \pi_{m-1} \mathcal{F}_{HC}(R) \xrightarrow{B} \pi_m \mathcal{F}_{HH}(R) \xrightarrow{I} \pi_m \mathcal{F}_{HC}(R) \rightarrow 0.$$

Similarly, there are split short exact sequences:

$$0 \rightarrow \widetilde{\mathbb{H}}_{\text{cdh}}^{m+1}(R, HC) \xrightarrow{B} \widetilde{\mathbb{H}}_{\text{cdh}}^m(R, HH) \xrightarrow{I} \widetilde{\mathbb{H}}_{\text{cdh}}^m(R, HC) \rightarrow 0$$

and

$$0 \rightarrow \widetilde{\mathbb{H}}_{\text{cdh}}^{m-1}(R, \Omega^{<i}) \xrightarrow{B} \widetilde{H}_{\text{cdh}}^{m-i}(R, \Omega^i) \xrightarrow{I} \widetilde{\mathbb{H}}_{\text{cdh}}^m(R, \Omega^{\leq i}) \rightarrow 0.$$

Proof. It suffices to show that I is onto and split. By [7, 2.4], $\mathcal{F}_{HH}(k) = \mathcal{F}_{HC}(k) = 0$, so $\widetilde{\mathcal{F}}_{HH} = \mathcal{F}_{HH}$ and $\widetilde{\mathcal{F}}_{HC} = \mathcal{F}_{HC}$. By the standard trick 3.7, it suffices to show that the maps $N\pi_m \mathcal{F}_{HH}(R) \rightarrow N\pi_m \mathcal{F}_{HC}(R)$ and $N\widetilde{\mathbb{H}}_{\text{cdh}}^m(R, HH) \rightarrow N\widetilde{\mathbb{H}}_{\text{cdh}}^m(R, HC)$ are split surjections. But this is evident from the decompositions of $N\mathcal{F}_{HC}^{(i)}(R)$ and $\mathbb{H}_{\text{cdh}}(R, NHC^{(i)})$ in Lemmas 3.2 and 2.2.

The third sequence is obtained from the second one by taking the i^{th} component in the Hodge decomposition, described in Lemma 2.2. \square

Example 3.9. Splicing the final sequences of Lemma 3.8 together, we see that the de Rham complexes are exact:

$$(3.9a) \quad 0 \rightarrow k \rightarrow R \xrightarrow{d} \widetilde{H}_{\text{cdh}}^0(R, \Omega^1) \xrightarrow{d} \widetilde{H}_{\text{cdh}}^0(R, \Omega^2) \rightarrow \dots$$

$$(3.9b) \quad 0 \rightarrow H_{\text{cdh}}^n(R, \mathcal{O}) \xrightarrow{d} H_{\text{cdh}}^n(R, \Omega^1) \xrightarrow{d} H_{\text{cdh}}^n(R, \Omega^2) \rightarrow \dots, \quad n > 0.$$

An analogous exact sequence

$$\dots \rightarrow \pi_{m-1} \mathcal{F}_{HH}(R) \xrightarrow{d} \pi_m \mathcal{F}_{HH}(R) \xrightarrow{d} \pi_{m+1} \mathcal{F}_{HH}(R) \rightarrow \dots$$

is obtained by splicing the other sequences in Lemma 3.8. Using the interpretation of their Hodge components, described in Lemma 3.4, produces two more exact sequences:

$$(3.9c) \quad 0 \rightarrow \text{nil}(R) \rightarrow \text{tors } \Omega_R^1 \rightarrow \text{tors } \Omega_R^2 \rightarrow \text{tors } \Omega_R^3 \rightarrow \cdots$$

$$(3.9d) \quad 0 \rightarrow (R^+/R) \rightarrow \Omega_{\text{cdh}}^1(R)/\Omega_R^1 \rightarrow \Omega_{\text{cdh}}^2(R)/\Omega_R^2 \rightarrow \cdots$$

Here we have written $\Omega_{\text{cdh}}^i(R)$ for $H_{\text{cdh}}^0(R, \Omega^i)$, and $\text{tors } \Omega_R^i$ is defined as the kernel of $\Omega_R^i \rightarrow \Omega_{\text{cdh}}^i(R)$; the notation reflects the fact that if R is reduced then $\text{tors } \Omega_R^i$ is the torsion submodule of Ω_R^i (see Remark 5.3.1 below).

4. BASS' GROUPS $NK_*(X)$

In this section, we relate algebraic K -theory to our Hochschild and cyclic homology calculations relative to the ground field $k = \mathbb{Q}$. Consider the trace map

$$NK_{n+1}(X) \rightarrow NHC_n(X) = NHC_n(X/\mathbb{Q})$$

induced by the Chern character. In the affine case, it is defined in [27]; for schemes it is defined using Zariski descent. As explained in [27], it arises from the Chern character from the spectrum $NK(X)$ to the Eilenberg-Mac Lane spectrum $|NHC(X)[1]|$ associated to the cochain complex $NHC(X)[1]$. Note that our indexing conventions are such that $\pi_{n+1}|NHC(X)[1]| = H^{-n}NHC(X) = NHC_n(X)$.

Proposition 4.1. *Suppose that $R = \Gamma(X, \mathcal{O})$ for X in Sch/F , or that $X = \text{Spec}(R)$ for an F -algebra R . Then for all n , the Chern character induces a natural isomorphism*

$$NK_{n+1}(X) \cong H^{-n}\mathcal{F}_{HH}(X) \otimes t\mathbb{Q}[t].$$

This is an isomorphism of graded R -modules, and even $\text{Carf}(R)$ -modules, identifying the operations $[r]$, V_m and F_m on $NK_(X)$ with the operations on the right side described in Example 1.6.*

Proof. By Remark 2.2.1, we may suppose $X \in \text{Sch}/F$. By [7, 1.6], the Chern character $K \rightarrow HN$ induces weak equivalences $\mathcal{F}_K(X) \simeq |\mathcal{F}_{HC}(X)[1]|$ and $\mathcal{F}_K(X \times \mathbb{A}^1) \simeq |\mathcal{F}_{HC}(X \times \mathbb{A}^1)[1]|$. Since for any presheaf of spectra E we have a natural objectwise equivalence $E(- \times \mathbb{A}^1) \simeq E \times NE$, we obtain a natural weak equivalence from $NK(X)$ to $|N\mathcal{F}_{HC}(X)[1]|$. Now take homotopy groups and apply Lemma 3.2.

As observed in [10, 4.12], the Chern character also commutes with the ring maps used to define the operators $[r]$, V_m , and with the transfer for $R[t^n] \rightarrow R[t]$ defining F_m . That is, it is a homomorphism of $\text{Carf}(R)$ -modules. Since the transfer is defined via the ring map $R[t] \rightarrow M_n(R[t^n])$, followed by Morita invariance, there is no trouble in passing to schemes. \square

We now come to one of our main results, which implies Corollary 0.1.

Theorem 4.2. *For all n , $N^2K_n(X) \cong (NK_n(X) \otimes t\mathbb{Q}[t]) \oplus (NK_{n-1}(X) \otimes \Omega_{\mathbb{Q}[t]}^1)$, and*

$$N^{p+1}K_n(X) \cong \bigoplus_{j=0}^p NK_{n-j}(X) \otimes \wedge^j \mathbb{Q}^p \otimes (t\mathbb{Q}[t])^{\otimes(p-j)} \otimes (\Omega_{\mathbb{Q}[t]}^1)^{\otimes j}.$$

This holds for every X in Sch/F , as well as for $\text{Spec}(R)$ where R is an arbitrary commutative F -algebra.

Proof. As in Proposition 4.1 it follows that the Chern character induces a natural weak equivalence $N^2K(X) \simeq |N^2\mathcal{F}_{HC}(X)[1]|$. Now take homotopy groups and apply Corollary 3.3. \square

Remark 4.2.1. Jim Davis has pointed out (see [9]) that a computation equivalent to 4.2 can also be derived — for arbitrary rings R — from the Farrell-Jones conjecture for the groups \mathbb{Z}^r . This particular case is covered by F. Quinn’s proof of hyperelementary assembly for virtually abelian groups; see [20].

As an immediate consequence of 4.2 and [1, XII(7.3)], we deduce:

Corollary 4.3. *Suppose that X is in Sch/F , or that $X = \text{Spec}(R)$ for an F -algebra R . Then:*

- a) *If $NK_n(X) = NK_{n-1}(X) = 0$ then $N^2K_n(X) = 0$.*
- b) *If $NK_n(X) = 0$ and $K_{n-1}(X) = K_{n-1}(X \times \mathbb{A}^p)$ then $K_n(X) = K_n(X \times \mathbb{A}^{p+1})$.*
- c) *$K_n(X) = K_n(X \times \mathbb{A}^p)$ if and only if $NK_q(X) = 0$ for all q such that $n-p < q \leq n$.*

Recall that X is called K_n -regular if $K_n(X) = K_n(X \times \mathbb{A}^p)$ for all p .

Corollary 4.4. *Suppose that X is in Sch/F , or that $X = \text{Spec}(R)$ for an F -algebra R . Then the following conditions are equivalent:*

- a) *X is K_n -regular;*
- b) *$NK_n(X) = 0$ and X is K_{n-1} -regular;*
- c) *$NK_q(X) = 0$ for all $q \leq n$.*

Remark 4.4.1. This gives another proof of Vorst’s Theorem [25, 2.1] (in characteristic 0) that K_n -regularity implies K_{n-1} -regularity, and extends it to schemes.

The assumption that the scheme be affine is essential in Bass’ question — here is a non-affine example where the answer is negative.

Negative answer to Bass’ question for nonaffine curves. Let X be a smooth projective elliptic curve over a number field k and let L be a nontrivial degree zero line bundle with $L^{\otimes 3}$ trivial. For example, if X is the Fermat cubic $x^3 + y^3 = z^3$, we may take the line bundle associated to the divisor $P - Q$, where $P = (1 : 0 : 1)$ and $Q = (0 : 1 : 1)$.

Lemma 4.5. *Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O}_X \oplus L = \text{Sym}(L)/(L^2)$, that is, L is regarded as a square-zero ideal.*

Then $NK_7(Y) = 0$ but $N^2K_7(Y) \cong NK_6(Y) \otimes \Omega_{\mathbb{Q}[t]}^1$ is nonzero.

Proof. In this setting, the relative Hochschild homology presheaf $HH_n(Y, L)$ is the kernel of $HH_n(Y) \rightarrow HH_n(X)$; sheafifying, $\mathcal{H}\mathcal{H}_n(Y, L)$ is the kernel of $\mathcal{H}\mathcal{H}_n(Y) \rightarrow \mathcal{H}\mathcal{H}_n(X)$. Since $\Omega_X^1 \cong \mathcal{O}_X$ we see from Lemma 5.3 of [7] that $\mathcal{H}\mathcal{H}_n(Y, L)$ is: $L^{\otimes 3} \oplus L^{\otimes 5}$ if $n = 4$; $L^{\otimes 5} \oplus L^{\otimes 5}$ if $n = 5$; and $L^{\otimes 5} \oplus L^{\otimes 7}$ if $n = 6$. By Serre duality, $H^*(X, L^{\otimes i}) = 0$ if $3 \nmid i$ (cf. [7, 5.1]). By Zariski descent, this implies that $HH_5(Y, L) \cong H^1(X, \mathcal{H}\mathcal{H}_4) \cong H^1(X, L^{\otimes 3}) \cong k$ and $HH_6(Y, L) = 0$. Since $\mathcal{F}_{HH}(Y) \cong HH(Y, L)$, it follows from 4.1 and 4.2 that $NK_7(Y) = 0$ but $NK_6(Y) \cong t\mathbb{Q}[t]$ and $N^2K_7(Y) \cong NK_6(Y) \otimes \Omega_{\mathbb{Q}[t]}^1 \cong t\mathbb{Q}[t] \otimes \Omega_{\mathbb{Q}[t]}^1$. \square

We conclude this section by refining Proposition 4.1 and Corollary 4.3 to take account of the Adams/Hodge/ λ -decompositions on K-theory and Hochschild homology, and by establishing the triviality of $K_*^{(i)}(X)$ for $i \leq 0$.

Recall that by definition, $K_n^{(i)}(X) = \{x \in K_n(X) \otimes \mathbb{Q} : \psi^k(x) = k^i x\}$. For $n < 0$, the Adams operations cannot be defined integrally. However, it is possible to define the operations ψ^k on $K_n(X) \otimes \mathbb{Q}$ for $n < 0$ using descending induction on n and the formula $\psi^k\{x, t\} = k\{\psi^k(x), t\}$ in $K_{n+1}(X \times (\mathbb{A}^1 - 0))$ for $x \in K_n(X)$ and $\mathcal{O}(\mathbb{A}^1 - 0) = F[t, 1/t]$. This definition was pointed out in [30, 8.4].

By [11, 2.3] or [8, 7.2], the Chern character $NK_{n+1}(X) \rightarrow NHC_n(X)$ commutes with the Adams operations ψ^k in the sense that it sends $NK_{n+1}^{(i+1)}(X)$ to $NHC_n^{(i)}(X)$ for all $i \leq n$ (and to 0 if $i > n$). Here is the λ -decomposition of the isomorphism in Proposition 4.1:

Proposition 4.6. *Suppose that $X \in \text{Sch}/F$, or that $X = \text{Spec}(R)$ for an F -algebra R . Then for all n and i , the Chern character induces a natural isomorphism:*

$$NK_{n+1}^{(i)}(X) \cong H^{-n} \mathcal{F}_{HH}^{(i-1)}(X) \otimes t\mathbb{Q}[t].$$

In particular, if $i \leq 0$ then $NK_n^{(i)}(X) = 0$ for all n .

Proof. By [8], the Chern character $K \rightarrow HN$ sends $K^{(i)}(X)$ to $HN^{(i)}(X)$. The proof in [8] shows that the lift $\mathcal{F}_K(X) \rightarrow \mathcal{F}_{HN}(X)$, shown to be a weak equivalence in [7, 1.6], may be taken to send $\mathcal{F}_K^{(i)}(X)$ to $\mathcal{F}_{HN}^{(i)}(X)$. Since $HC \rightarrow HN$ sends $HC^{(i-1)}$ to $HN^{(i)}$, the weak equivalence $\mathcal{F}_{HC}[1] \simeq \mathcal{F}_{HN}$ identifies $\mathcal{F}_{HC}^{(i-1)}[1]$ and $\mathcal{F}_{HN}^{(i)}$. Finally $\mathcal{F}_{HH}^{(i-1)} = 0$ for $i \leq 0$. \square

Corollary 4.7. *Suppose that R is essentially of finite type over F and has dimension d . If $n < 0$ then $NK_n^{(i)}(R) = 0$ unless $1 \leq i \leq d + n$, in which case*

$$NK_n^{(i)}(R) = H_{\text{cdh}}^{i-n-1}(R, \Omega^{i-1}) \otimes t\mathbb{Q}[t].$$

In particular, $NK_n(R) = 0$ for all $n \leq -d$.

If $d \geq 2$ then:

$$\begin{aligned} NK_0(R) &\cong [(R^+/R) \oplus H_{\text{cdh}}^1(R, \Omega^1) \oplus \cdots \oplus H_{\text{cdh}}^{d-1}(R, \Omega^{d-1})] \otimes t\mathbb{Q}[t], \\ NK_{-1}(R) &\cong [H_{\text{cdh}}^1(R, \mathcal{O}) \oplus H_{\text{cdh}}^2(R, \Omega^1) \oplus \cdots \oplus H_{\text{cdh}}^{d-1}(R, \Omega^{d-2})] \otimes t\mathbb{Q}[t], \\ &\vdots \\ NK_{1-d}(R) &\cong H_{\text{cdh}}^{d-1}(R, \mathcal{O}) \otimes t\mathbb{Q}[t]. \end{aligned}$$

If $d = 1$ then $NK_0(R) = (R^+/R) \otimes t\mathbb{Q}[t]$ and $NK_n(R) = 0$ for $n < 0$.

Proof. This is straightforward from Proposition 4.6 and Lemma 3.4. \square

Remark 4.7.1. The $d = 1$ part of Corollary 4.7 holds for any 1-dimensional noetherian ring by [26, 2.8].

Corollary 4.8. $K_n^{(i)}(X) \cong K_n^{(i)}(X \times \mathbb{A}^p)$ if and only if $NK_{n-j}^{(i-j)}(X) = 0$ for all $j = 0, \dots, p-1$.

Theorem 4.9. *For X in Sch/F or $X = \text{Spec}(R)$, and all integers n , we have:*

- (1) For $i < 0$, $K_n^{(i)}(X) = 0$.
- (2) For $i = 0$, $K_n^{(0)}(X) \cong KH_n^{(0)}(X) \cong H_{\text{cdh}}^{-n}(X, \mathbb{Q})$.

Here KH denotes the homotopy K -theory of [29]. Theorem 4.9 answers Question 8.2 of [30].

Proof. We first show that $K_n^{(i)}(X) \cong KH_n^{(i)}(X)$ when $i \leq 0$. Covering X with affine opens and using the Mayer-Vietoris sequences of [29, 5.1], it suffices to consider the case $X = \text{Spec}(R)$.

Since $K(R)_{\mathbb{Q}}$ is the product of the eigen-components, the descent spectral sequence $E_{p,q}^1 = N^p K_q(R)_{\mathbb{Q}} \Rightarrow KH_{p+q}(R)_{\mathbb{Q}}$ (see [29, 1.3]) breaks up into one for each eigen-component. If $i \leq 0$, the spectral sequence collapses by Proposition 4.6 to yield $K_n^{(i)}(R) \cong KH_n^{(i)}(R)$ for all n .

To determine the groups $KH_n^{(i)}(R)$ when $i \leq 0$, we use the *cdh* descent spectral sequence of [15, 1.1]. If $i < 0$, then the *cdh* sheaf $K_{\text{cdh}}^{(i)}$ is trivial as X is locally smooth, so we have $KH_n^{(i)}(R) = 0$ for all n . If $i = 0$ then the *cdh* sheaf $K_{\text{cdh}}^{(0)}$ is the sheaf \mathbb{Q}_{cdh} ; see [21, 2.8]. Hence we have $K_n^{(0)}(R) = KH_n^{(0)}(R) = H_{\text{cdh}}^{-n}(X, \mathbb{Q})$. \square

5. THE TYPICAL PIECES $TK_n^{(i)}(R)$

In this section, R will be a commutative F -algebra. The default ground field k for Kähler differentials and Hochschild homology will be \mathbb{Q} .

As stated in (0.3), the Adams summands $NK_n^{(i)}(R)$ of $NK_n(R)$ decompose as $NK_n^{(i)}(R) = TK_n^{(i)}(R) \otimes t\mathbb{Q}[t]$ for each n and i ; the decomposition is obtained from an action of finite Cartier operators precisely as the corresponding one for NHC and NHH , explained in Section 1. The typical pieces $TK_n^{(i)}(R)$ are described by the following formulas.

Theorem 5.1. *Let R be a commutative F -algebra. For $i \neq n, n+1$ we have:*

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R), & \text{if } i < n, \\ H_{\text{cdh}}^{i-n-1}(R, \Omega^{i-1}) & \text{if } i \geq n+2. \end{cases}$$

For $i = n, n+1$, the typical piece $TK_n^{(i)}(R)$ is given by the exact sequence:

$$0 \rightarrow TK_{n+1}^{(n+1)}(R) \rightarrow \Omega_R^n \rightarrow H_{\text{cdh}}^0(R, \Omega^n) \rightarrow TK_n^{(n+1)}(R) \rightarrow 0.$$

Proof. By Proposition 4.6, $TK_n^{(i)} = H^{1-n} \mathcal{F}_{HH}^{(i-1)}$. The rest is a restatement of Lemma 3.4. \square

Remark 5.1.1. If R is essentially of finite type over a field F whose transcendence degree is finite over \mathbb{Q} , then the $TK_n^{(i)}(R)$ are finitely generated R -modules. This fails if $\text{tr. deg}(F/\mathbb{Q}) = \infty$ because then $\Omega_{F/\mathbb{Q}}^i$ is infinite dimensional. For instance, Example 0.6 implies that, for $R = F[x]/(x^2)$, we have $TK_2^{(2)}(R) = HH_1(R, x) = F \oplus \Omega_{F/\mathbb{Q}}^1$.

Remark 5.1.2. Observe that 4.7 and 4.4 imply that R is K_{-d} -regular. This recovers the affine case of one of the main results in [5].

Here is a special case of the calculations in Theorem 5.1, which proves Theorem 0.7. We will use it to construct the counterexample to Bass' question in the companion paper [6].

Theorem 5.2. *Let F be a field of characteristic 0 and R a normal domain of dimension 2, essentially of finite type over F . Then*

$$\text{a) } H^1 \mathcal{F}_{HH}(R/F) \cong H_{\text{cdh}}^1(R, \Omega_F^1),$$

- b) $H^2\mathcal{F}_{HH}(R/F) \cong H_{\text{cdh}}^1(R, \mathcal{O})$,
- c) $NK_0(R) \cong H_{\text{cdh}}^1(R, \Omega^1) \otimes t\mathbb{Q}[t]$, and
- d) $NK_{-1}(R) \cong H_{\text{cdh}}^1(R, \mathcal{O}) \otimes t\mathbb{Q}[t]$.

Proof. Parts (a) and (b) are immediate from Example 3.5 and the fact that R is reduced and seminormal. Parts (c) and (d) follow from (a) and (b) using Proposition 4.1; cf. Corollary 4.7. \square

In order to compare the torsion submodules $\text{tors}\Omega_R^*$ with the typical pieces of $NK_*(R)$, we need the affine case of the following lemma. Following tradition, we write $F(X)$ for the total ring of fractions of X_{red} . That is, $F(X)$ is the product of the function fields of the irreducible components of X_{red} . When $X = \text{Spec}(R)$ is affine, we write Q instead of $F(X)$.

Lemma 5.3. *Let $X \in \text{Sch}/F$; for $F(X)$ as above, the map $\Omega_{\text{cdh}}^i(X) \rightarrow \Omega_{F(X)}^i$ is an injection.*

Proof. We may assume X reduced, and proceed by induction on $d = \dim(X)$, the case $d = 0$ being trivial. Choose a resolution of singularities $X' \rightarrow X$ and let Y be the singular locus of X , with $Y' = Y \times_X X'$. By [22, 12.1], there is a Mayer-Vietoris exact sequence

$$0 \rightarrow \Omega_{\text{cdh}}^i(X) \rightarrow \Omega_{\text{cdh}}^i(X') \oplus \Omega_{\text{cdh}}^i(Y) \rightarrow \Omega_{\text{cdh}}^i(Y') \xrightarrow{\partial} H_{\text{cdh}}^1(X, \Omega^i) \rightarrow \dots$$

Since $F(Y) \subseteq F(Y')$, $\Omega_{F(Y)}^i \subseteq \Omega_{F(Y')}^i$. Because $\dim(Y') < d$, the inductive hypothesis implies that $\Omega_{\text{cdh}}^i(Y) \rightarrow \Omega_{\text{cdh}}^i(Y')$ is an injection. Hence $\Omega_{\text{cdh}}^i(X) \rightarrow \Omega_{\text{cdh}}^i(X')$ is an injection. But X' is smooth, so by *scdh* descent for Ω^i (see [7, 2.5]) we have $\Omega_{\text{cdh}}^i(X') \cong \Omega^i(X') \subset \Omega_{F(X')}^i = \Omega_{F(X)}^i$. \square

Remark 5.3.1. Lemma 5.3 remains true if, instead of Ω^i , we use $\Omega_{/k}^i$ for $k \subseteq F$. In particular, if $X = \text{Spec}(R)$ is reduced affine, then $\Omega_{\text{cdh}}^i(R/k) = H_{\text{cdh}}^0(R, \Omega_{/k}^i)$ injects into $\Omega_{Q/k}^i$. Thus $\text{tors}(\Omega_{R/k}^i)$, defined as the kernel of $\Omega_{R/k}^i \rightarrow \Omega_{\text{cdh}}^i(R/k)$ in (3.9c), is the torsion submodule of $\Omega_{R/k}^i$.

Corollary 5.4. *For all $n \geq 1$, $TK_n^{(n)}(R) \cong \ker(\Omega_R^{n-1} \rightarrow \Omega_Q^{n-1})$.*

In particular if R is reduced, then $TK_n^{(n)}(R)$ is the torsion submodule of Ω_R^{n-1} .

Proof. By 5.1, $TK_n^{(n)}(R)$ is the kernel of $\Omega_R^{n-1} \rightarrow \Omega_{\text{cdh}}^{n-1}(R)$, so 5.3 applies. \square

We introduce some notation to make the statement of the next theorem more readable. The letter e denotes the maximum transcendence degree of the component fields in the total ring of fractions Q of R_{red} . For simplicity, we write $\Omega_{\text{cdh}}^i(X)$ for $H_{\text{cdh}}^0(X, \Omega^i)$, and we have written $\Omega_{\text{cdh}}^i(R)/\Omega_R^i$ for the cokernel of $\Omega_R^i \rightarrow \Omega_{\text{cdh}}^i(R)$.

Definition 5.5. For any commutative ring R containing \mathbb{Q} , we define:

$$E_n(R) = \Omega_{\text{cdh}}^n(R)/\Omega_R^n \oplus \bigoplus_{p=1}^{\infty} H_{\text{cdh}}^p(R, \Omega^{n+p});$$

$$\widetilde{HH}_n(R) = \ker(HH_n(R) \rightarrow \Omega_Q^n) = \ker(\Omega_R^n \rightarrow \Omega_Q^n) \oplus \bigoplus_{i=1}^{n-1} HH_n^{(i)}(R).$$

Theorem 5.6. *Let R be a commutative ring containing \mathbb{Q} . Then for all n :*

$$NK_n(R) \cong [\widetilde{HH}_{n-1}(R) \oplus E_n(R)] \otimes t\mathbb{Q}[t].$$

If furthermore R is essentially of finite type over a field, and $n \geq e + 2$, then $NK_n(R) \cong HH_{n-1}(R) \otimes t\mathbb{Q}[t]$.

Proof. Assembling the descriptions of the $TK_n^{(i)}(R)$ in Theorem 5.1 yields the first assertion. The second part is immediate from this and 3.6. \square

Remark 5.6.1. The Chern character $NK_n(R) \rightarrow NHC_{n-1}(R) \cong HH_{n-1}(R) \otimes t\mathbb{Q}[t]$ is an isomorphism for $n \geq e+2$. If $n \leq e+1$, neither it nor the map $H^{1-n}\mathcal{F}_{HH}(R) \rightarrow HH_{n-1}(R)$ of 4.1 need be a surjection.

The typical pieces of $NK_1^{(2)}(R)$ and $NK_2^{(2)}(R)$ of 5.1 and 5.4 may be described as follows.

Proposition 5.7. *For all reduced F -algebras R , the typical pieces $TK_1^{(2)}(R) = \Omega_{\text{cdh}}^1(R)/\Omega_R^1$ and $TK_2^{(2)}(R) = \text{tors}(\Omega_R^1)$ fit into an exact sequence:*

$$0 \rightarrow \text{tors}(\Omega_R^1) \rightarrow \text{tors}(\Omega_{R/F}^1) \rightarrow \Omega_F^1 \otimes (R^+/R) \rightarrow \frac{\Omega_{\text{cdh}}^1(R)}{\Omega_R^1} \rightarrow \frac{\Omega_{\text{cdh}}^1(R/F)}{\Omega_{R/F}^1} \rightarrow 0.$$

Proof. We may assume $\text{Spec } R \in \text{Sch}/F$. Recall from [7, 4.2] that there is a bounded second quadrant homological spectral sequence for all p ($0 \leq i < p, j \geq 0$):

$${}_p E_{-i, i+j}^1 = \Omega_{F/k}^i \otimes_F HH_{p-i+j}^{(p-i)}(R/F) \Rightarrow HH_{p+j}^{(p)}(R/k).$$

When $p = 1$, this spectral sequence degenerates to yield exactness of the bottom row in the following commutative diagram; the top row is the First Fundamental Exact Sequence for Ω^1 [31, 9.2.6].

$$\begin{array}{ccccccc} \Omega_F^1 \otimes R & \longrightarrow & \Omega_R^1 & \longrightarrow & \Omega_{R/F}^1 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_F^1 \otimes R^+ & \longrightarrow & \Omega_{\text{cdh}}^1(R) & \longrightarrow & \Omega_{\text{cdh}}^1(R/F) \longrightarrow 0. \end{array}$$

The upper left horizontal map is an injection because the left vertical map is an injection. Now apply the snake lemma, using Remark 5.3.1. \square

6. BASS' QUESTION FOR ALGEBRAS OVER LARGE FIELDS.

We will now show that the answer to Bass' question is positive for algebras R essentially of finite type over a field F of infinite transcendence degree over \mathbb{Q} .

Recall from Proposition 4.1 that $NK_{n+1}(R) \cong H^{-n}\mathcal{F}_{HH}(R/\mathbb{Q}) \otimes t\mathbb{Q}[t]$. In light of this identification, the version of Bass' question stated before Theorem 0.2 becomes the case $k = \mathbb{Q}$ of the following question:

$$(6.1) \quad \text{Does } H^m \mathcal{F}_{HH}(R/k) = 0 \text{ imply that } H^{m+1} \mathcal{F}_{HH}(R/k) = 0?$$

In Theorem 6.6, we show that the answer to question (6.1) is positive provided R is of finite type over a field F that has infinite transcendence degree over k . The proof is essentially a formal consequence of the Künneth formula in Lemma 6.3.

Lemma 6.2. *Let R be a commutative F -algebra, and suppose k is a subfield of F . Then $H^{-*}\mathcal{F}_{HH}(R/k)$ and $\mathbb{H}_{\text{cdh}}^{-*}(R, HH(/k))$ are graded modules over the graded ring $\Omega_{F/k}^\bullet$.*

Proof. As in 2.2.1, we may suppose that R is of finite type over F . Consider the functor on F -algebras that associates to an F -algebra A the Hochschild complex $HH(A/k)$. The shuffle product makes this into a functor to dg - $HH(F/k)$ -modules. Since the *cdh*-site has a set of points (corresponding to valuations by [13, 2.1]), we can use a Godement resolution to find a model for the *cdh*-hypercohomology $\mathbb{H}_{\text{cdh}}(-, HH(/k))$ which is also a functor to dg - $HH(F/k)$ -modules. It follows that there is a model for $\mathcal{F}_{HH}(R/k)$ that is a dg - $HH(F/k)$ -module, functorially in R . This implies the assertion, since $\Omega_{F/k}^\bullet = H^{-\bullet} HH(F/k)$. \square

Lemma 6.3. (*Künneth Formula*) *Suppose that $\mathbb{Q} \subseteq k \subseteq F_0 \subseteq F$ are fields. Let R_0 be an F_0 -algebra, and set $R = F \otimes_{F_0} R_0$.*

- i) *Let $T = \{t_i\}$ be transcendence basis of F/F_0 ; writing $F[dT]$ for the exterior algebra on the set $\{dt_i\}$, we have $\Omega_{F/F_0}^\bullet = F[dT]$ and:*

$$\Omega_{F/k}^\bullet \cong F[dT] \otimes_{F_0} \Omega_{F_0/k}^\bullet$$

In particular, the graded algebra homomorphism $\Omega_{F_0/k}^\bullet \rightarrow \Omega_{F/k}^\bullet$ is flat.

- ii) $HH_*(R/k) \cong \Omega_{F/k}^\bullet \otimes_{\Omega_{F_0/k}^\bullet} HH_*(R_0/k) \cong F[dT] \otimes_{F_0} HH_*(R_0/k)$.

Proof. It is classical that $F[dT] = \Omega_{F/F_0}^\bullet$. The tensor product decomposition of part i) follows from the fact that the fundamental sequence

$$0 \rightarrow F \otimes_{F_0} \Omega_{F_0/k}^1 \rightarrow \Omega_F^1 \rightarrow \Omega_{F/F_0}^1 \rightarrow 0$$

is split exact. This proves i). To prove ii), choose a free chain dg - F_0 -algebra Λ and a surjective quasi-isomorphism of dg -algebras $\Lambda \xrightarrow{\sim} R_0$. Then $\Lambda' = F \otimes_{F_0} \Lambda \rightarrow F \otimes_{F_0} R_0 = R$ is a free chain model of R as a k -algebra. Write $\Omega_{\Lambda/k}^\bullet$ for differential forms; consider $\Omega_{\Lambda'/k}^\bullet$ as a chain dg -algebra with the differential δ induced by that of Λ . Note Λ and Λ' are homologically regular in the sense of [4], so that Theorem 2.6 of [4] applies. Combining this with part i), we obtain

$$\begin{aligned} HH_*(R/k) &= HH_*(\Lambda'/k) = H_*(\Omega_{\Lambda'/k}^\bullet) \\ &= H_*(\Omega_{F/k}^\bullet \otimes_{\Omega_{F_0/k}^\bullet} \Omega_{\Lambda/k}^\bullet) = \Omega_{F/k}^\bullet \otimes_{\Omega_{F_0/k}^\bullet} H_*(\Omega_{\Lambda/k}^\bullet) \\ &= \Omega_{F/k}^\bullet \otimes_{\Omega_{F_0/k}^\bullet} HH_*(R_0/k). \end{aligned} \quad \square$$

Here is an easy consequence of Lemmas 6.2 and 6.3.

Proposition 6.4. *Suppose $\mathbb{Q} \subseteq k \subseteq F_0 \subseteq F$ are field extensions, that R_0 is an F_0 -algebra and $R = F \otimes_{F_0} R_0$. Then there is an isomorphism of graded $\Omega_{F/k}^\bullet$ -modules*

$$F[dT] \otimes_{F_0} H^{-*} \mathcal{F}_{HH}(R_0/k) \cong H^{-*}(\mathcal{F}_{HH}(R/k)).$$

We also need the following lemma to prove the main result of this section.

Lemma 6.5. *Let R be essentially of finite type over $F \supset \mathbb{Q}$, and let $H_n(R)$ denote either $HH_n(R)$ or $H^{-n} \mathcal{F}_{HH}(R)$. Assume that $H_{n_i}(R) = 0$ for some finite set $\{n_1, \dots, n_r\}$ of positive integers. Then there exist an F -algebra of finite type R' , and a multiplicatively closed set S such that $R \cong S^{-1}R'$ and $H_{n_i}(R') = 0$ for $1 \leq i \leq r$.*

Proof. Because R is essentially of finite type, it is the localization $R = S^{-1}R''$ of some finite type F -algebra R'' . It is well known that $HH_n(S^{-1}R'') \cong S^{-1}HH_n(R'')$ (see [31, 9.1.8]), and $H^{-n}\mathcal{F}_{HH}(S^{-1}R'') \cong S^{-1}H^{-n}\mathcal{F}_{HH}(R'')$ by [7, 2.8–9].

Because R'' is of finite type over F , we may write $R'' = F \otimes_{F_0} R_0$ for some finitely generated field extension F_0 of \mathbb{Q} and some finite type F_0 -algebra R_0 . Note R_0 is essentially of finite type over \mathbb{Q} , whence $H_p(R_0)$ is a finitely generated R_0 -module ($p \geq 0$). By 6.3 and/or 6.4, $H_p(R'')$ is isomorphic, as an R'' -module, to a direct sum of copies of $R'' \otimes_{R_0} H_q(R_0)$ with $q \leq p$. In particular, $M = \bigoplus_{i=1}^r H_{n_i}(R'')$ is a finite sum of R'' -modules, each of which is a — possibly infinite — direct sum of copies of one finitely generated module.

Given that M has this form, the hypothesis that $S^{-1}M = 0$ implies that there exists a nonzero element $s \in \text{Ann}(M) \cap S$. Consider the finite type F -algebra $R' = R''[1/s]$. Then $R \cong S^{-1}R'$ and we have $\bigoplus_i H_{n_i}(R') = M[1/s] = 0$. \square

Theorem 6.6. *Suppose $k \subset F$ is an extension with $\text{tr. deg}(F/k) = \infty$, and R is essentially of finite type over F . If $H^n(\mathcal{F}_{HH}(R/k)) = 0$, then $H^m(\mathcal{F}_{HH}(R/k)) = 0$ for all $m \geq n$.*

Proof. By Lemma 6.5, we may assume that R is of finite type over F . There is a finitely generated field extension $F_0 \subset F$ of k and a finite type F_0 -algebra R_0 such that $R = R_0 \otimes_{F_0} F$. Note that $\text{tr. deg}(F/F_0) = \infty$. By Lemma 6.3 and Proposition 6.4, $\Omega_{F/F_0}^i \otimes_{F_0} H^{n+i}(\mathcal{F}_{HH}(R_0/k))$ is a direct summand of $H^n(\mathcal{F}_{HH}(R/k))$ for each $i \geq 0$. Since $\Omega_{F/F_0}^i \neq 0$ for all i , all the $H^{n+i}(\mathcal{F}_{HH}(R_0/k))$ vanish as well. Similarly, $H^m(\mathcal{F}_{HH}(R/k))$ is a direct sum of copies of the groups $\Omega_{F/F_0}^j \otimes_{F_0} H^{m+j}(\mathcal{F}_{HH}(R_0/k))$ for $j \geq 0$, all of which vanish when $m \geq n$, as we just observed. \square

Corollary 6.7. *Let $\mathbb{Q} \subset F$ be a field extension of infinite transcendence degree, and suppose R is essentially of finite type over F . Then $NK_n(R) = 0$ implies that R is K_n -regular.*

Proof. Combine Theorem 6.6 with Proposition 4.1 and Corollary 4.4. \square

Here is another proof of Corollary 6.7, which is essentially due to Murthy and Pedrini and given in their 1972 paper [18]; they stated the result only for $n \leq 1$ because transfer maps for higher K -theory and the $W(R)$ -module structure had not yet been discovered. We are grateful to Joseph Gubeladze [14] for pointing this out to the authors.

Lemma 6.8. *If R is an algebra over a field k of characteristic 0, $N^p K_n(R[t]) \rightarrow N^p K_n(R \otimes_k k(t))$ is injective.*

Proof. The proof in [18, 1.3–1.6] goes through, taking into account that the norm map and localization sequences used there for K_0, K_1 are now known for all K_n . \square

Lemma 6.9. *Suppose that k is an algebraically closed field of infinite transcendence degree over \mathbb{Q} , and that R is a finitely generated k -algebra. If $NK_n(R)$ is zero, then $K_n(R) \xrightarrow{\cong} K_n(R[x_1, \dots, x_p])$ for all $p > 0$.*

Proof. Murthy and Pedrini prove this in [18, 2.1.]; although their result is only stated for $i \leq 1$, their proof works in general. Note that since $NK_n(R)$ has the

form $TK_n(R) \otimes t\mathbb{Q}[t]$ by (0.3) (a result which was not known in 1972), $NK_n(R)$ is torsionfree, and has finite rank if and only if it is zero. \square

Proof of Corollary 6.7. Let Φ denote the functor $N^p K_n$. If $k \subset k_1$ is a finite algebraic field extension and R is a k -algebra, then $\Phi(R) \rightarrow \Phi(R \otimes_k k_1)$ is an injection because its composition with the transfer $\Phi(R \otimes_k k_1) \rightarrow \Phi(R)$ is multiplication by $[k_1 : k]$, and $\Phi(R)$ is a torsionfree group. Since Φ commutes with filtered colimits of rings, $\Phi(R) \rightarrow \Phi(R \otimes_k \bar{k})$ is an injection. Thus Lemma 6.9 suffices to prove Corollary 6.7 when R is of finite type. \square

7. NK_0 OF SURFACES

We conclude with a general description for affine surfaces of the canonical map $\Omega_F^1 \otimes_F NK_{-1} \rightarrow NK_0$. This sheds light on the difference between the cases of small and large base fields, and also explains some results of [33].

If R is a 2-dimensional noetherian ring then $NK_0(R)$ is the direct sum of $NK_0^{(1)}(R) = N \text{Pic}(R)$ and $NK_0^{(2)}(R)$.

Theorem 7.1. *Let R be a 2-dimensional normal domain of finite type over a field F of characteristic 0. There is an exact sequence:*

$$\begin{aligned} 0 \rightarrow NK_1^{(2)}(R) &\rightarrow \left(H^0(R, \Omega_{R/F}^1) / \Omega_{R/F}^1 \right) \otimes t\mathbb{Q}[t] \\ &\rightarrow \Omega_F^1 \otimes_F NK_{-1}(R) \rightarrow NK_0(R) \rightarrow H_{\text{cdh}}^1(R, \Omega_{R/F}^1) \otimes t\mathbb{Q}[t] \rightarrow 0. \end{aligned}$$

Proof. Consider the following short exact sequence of sheaves in $(\text{Sch}/F)_{\text{cdh}}$:

$$0 \rightarrow \Omega_F^1 \otimes_F \mathcal{O} \rightarrow \Omega^1 \rightarrow \Omega_{R/F}^1 \rightarrow 0$$

Applying H_{cdh} yields

$$\begin{aligned} 0 \rightarrow \Omega_F^1 \otimes_F R \xrightarrow{\iota} H^0(R, \Omega^1) \rightarrow H^0(R, \Omega_{R/F}^1) \xrightarrow{\partial} \\ \Omega_F^1 \otimes_F H_{\text{cdh}}^1(R, \mathcal{O}) \rightarrow H_{\text{cdh}}^1(R, \Omega^1) \rightarrow H_{\text{cdh}}^1(R, \Omega_{R/F}^1) \rightarrow 0 \end{aligned}$$

Note that, because $\Omega_R^1 \rightarrow \Omega_{R/F}^1$ is onto, the map ∂ kills the image of $\Omega_{R/F}^1$. Similarly, the image of ι is contained in that of Ω_R^1 . Thus we obtain

$$\begin{aligned} 0 \rightarrow H^0(R, \Omega^1) / \Omega_R^1 \rightarrow H^0(R, \Omega_{R/F}^1) / \Omega_{R/F}^1 \rightarrow \\ \Omega_F^1 \otimes_F H_{\text{cdh}}^1(R, \mathcal{O}) \rightarrow H_{\text{cdh}}^1(R, \Omega^1) \rightarrow H_{\text{cdh}}^1(R, \Omega_{R/F}^1) \rightarrow 0 \end{aligned}$$

Now apply $\otimes t\mathbb{Q}[t]$ and use 5.1 and parts c) and d) of 5.2. \square

Corollary 7.2. *Let R be a 2-dimensional normal domain of finite type over a field F of characteristic 0. If $NK_{-1}(R) = 0$ then $NK_0(R) \cong H_{\text{cdh}}^1(R, \Omega_{R/F}^1) \otimes t\mathbb{Q}[t]$.*

Example 7.3. Let R be a 2-dimensional normal domain of finite type over \mathbb{Q} , and put $R_F = R \otimes F$. By 4.1 and 6.4,

$$(7.4) \quad NK_*(R_F) \cong NK_*(R) \otimes \Omega_{F/\mathbb{Q}}^*$$

Keeping track of the λ -decomposition, as in 5.1, we see from Theorem 0.7 that

$$TK_1^{(2)}(R_F) \cong TK_1^{(2)}(R) \otimes F \cong H^0(R, \Omega^1) \otimes F / \Omega_R^1 \otimes F \cong H^0(R_F, \Omega_{R/F}^1) / \Omega_{R_F/F}^1.$$

From Theorem 7.1 we get an exact sequence

$$(7.5) \quad 0 \rightarrow \Omega_{F/\mathbb{Q}}^1 \otimes_F NK_{-1}(R_F) \rightarrow NK_0(R_F) \rightarrow H_{\text{cdh}}^1(R_F, \Omega_{R/F}^1) \otimes t\mathbb{Q}[t] \rightarrow 0$$

Using (7.4) and 0.7 again, we see that the sequence (7.5) is isomorphic to the sum

$$\begin{aligned} (0 \rightarrow \Omega_{F/\mathbb{Q}}^1 \otimes H_{\text{cdh}}^1(R, \mathcal{O}) \otimes t\mathbb{Q}[t] \xrightarrow{\cong} \Omega_{F/\mathbb{Q}}^1 \otimes H_{\text{cdh}}^1(R, \mathcal{O}) \otimes t\mathbb{Q}[t] \rightarrow 0 \rightarrow 0) \\ \oplus \\ (0 \rightarrow 0 \rightarrow F \otimes H_{\text{cdh}}^1(R, \Omega^1) \otimes t\mathbb{Q}[t] \xrightarrow{\cong} F \otimes H_{\text{cdh}}^1(R, \Omega^1) \otimes t\mathbb{Q}[t] \rightarrow 0) \end{aligned}$$

For example, for $R_F := F[x, y, z]/(z^2 + y^3 + x^{10} + x^7y)$ the results of [6] show that:

$$\begin{aligned} NK_{-1}(R_F) &= F \otimes t\mathbb{Q}[t] \\ NK_0(R_F) &= \Omega_{F/\mathbb{Q}}^1 \otimes t\mathbb{Q}[t] \cong \bigoplus_{p=1}^{\text{tr. deg}(F)} F \otimes t\mathbb{Q}[t]. \end{aligned}$$

In other words, both typical pieces $TK_{-1}(R_F)$ and $TK_0(R_F)$ are F -vectorspaces, but while $\dim_F TK_{-1}(R_F) = 1$ for all F , any cardinal number κ can be realized as $\dim_F TK_0(R_F)$ for an appropriate F .

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