

TORIC VARIETIES, MONOID SCHEMES AND *cdh* DESCENT

G. CORTIÑAS, C. HAESEMEYER, MARK E. WALKER, AND C. WEIBEL

ABSTRACT. We give conditions for the Mayer-Vietoris property to hold for the algebraic K -theory of blow-up squares of toric varieties, using the theory of monoid schemes. These conditions are used to relate algebraic K -theory to topological cyclic homology in characteristic p . To achieve our goals, we develop for monoid schemes many notions from classical algebraic geometry, such as separated and proper maps and resolution of singularities.

The goal of this paper is to prove Haesemeyer's Theorem [15, 3.12] for toric varieties in any characteristic. It is proven below as Corollary 14.4.

Theorem 0.1. *Assume k is an infinite field and let \mathcal{G} be a presheaf of spectra defined on the category of schemes of finite type over k . If \mathcal{G} satisfies the Mayer-Vietoris property for Zariski covers, finite abstract blow-up squares, and blow-ups along regularly embedded closed subschemes, then \mathcal{G} satisfies the Mayer-Vietoris property for all abstract blow-up squares of toric k -varieties.*

The application we have in mind is to understand the relationship between the algebraic K -theory $K_*(X) = \pi_*\mathcal{K}(X)$ and topological cyclic homology $TC_*(X) = \{\pi_*TC^\nu(X, p)\}$ of a toric variety in characteristic p . Thus we consider the presheaf of homotopy fibers $\{\mathcal{F}^\nu(X)\}$ of the map of pro-spectra from $\mathcal{K}(X)$ to $\{TC^\nu(X, p)\}$. Work of Geisser-Hesselholt [9, Thm. B], [10] shows that this homotopy fiber (regarded as a pro-presheaf of spectra) satisfies the hypotheses of Theorem 0.1 and hence a slight modification of the proof of our theorem implies that it satisfies the Mayer-Vietoris property for all abstract blow-up squares of toric varieties. We will give a rigorous proof of this in Corollary 14.7 below.

One major tool in our proof will be a theorem of Bierstone-Milman [1] which says that the singularities of a toric variety can be resolved by a sequence of blow-ups $X_C \rightarrow X$ along a center C that is a smooth, equivariant closed subscheme of X along which X is normally flat. If one only had to consider toric varieties, this would allow one to use Haesemeyer's original argument to prove Theorem 0.1, since toric varieties are normal and Cohen-Macaulay. However, examples show that the blow-up of a toric variety along a smooth center (even a point) can be non-normal. Thus,

Date: May 30, 2011.

2000 Mathematics Subject Classification. 19D55, 14M25, 19D25, 14L32.

Key words and phrases. abelian monoids, monoid schemes toric varieties, algebraic K -theory.

Cortiñas' research was supported by Conicet and partially supported by grants PICT 2006-00836, UBACyT X051, PIP 112-200801-00900, and MTM2007-64704 (Feder funds).

Haesemeyer's research was partially supported by NSF grant DMS-0966821.

Walker's research was partially supported by NSF grant DMS-0601666.

Weibel's research was supported by NSA and NSF grants.

even starting with a toric variety, the tower of blow-ups constructed by Bierstone–Milman will often involve non-normal varieties with a torus action. The proof of our theorem requires us to work with a larger class of varieties, one containing all the varieties in this tower. Beyond this, we need a class of varieties which is closed under passage to (possibly non-reduced) equivariant closed subschemes, pullbacks and blow-ups.

It turns out that all these operations may be lifted to the category of *monoid schemes* of finite type, and that the realizations of monoid schemes over a field k form a class of varieties with the above-mentioned properties. The k -realization of an affine monoid scheme is a scheme of the form $\text{Spec } k[A]$, with A an abelian monoid; the k -realization of a monoid scheme (Definition 5.3) is a scheme over k which is covered by affine open subschemes of this form, with homomorphisms of the underlying monoids inducing the gluing maps between these open subschemes.

To achieve our goals, it is easier to work directly with the category of monoid schemes, and Sections 1–3 of this paper are devoted to an introduction to monoid schemes. Toric monoid schemes are introduced in Section 4 and the relation to toric varieties is carefully described. In Sections 5 and 6, we prove that the k -realization functor preserves limits and show that many monoid scheme-theoretic properties translate well into algebraic geometry. Projective monoid schemes, blow-ups and proper maps are introduced in Sections 7 and 8. After introducing the technical notion of pcf monoid schemes in Section 9, birational maps and resolution of singularities are given in Sections 10 and 11.

The last part of this paper (Sections 12–14) is devoted to the notion of cohomological descent (Definition 12.10), the proof of our Main Theorem 0.1 and its application to algebraic K -theory and topological cyclic homology.

1. MONOIDS

Since we know of no suitable reference for the facts we need concerning monoids and their prime spectra, we begin with a short exposé of this basic material.

Unless otherwise stated, a *monoid* in this paper is a pointed abelian monoid; i.e., an abelian monoid object in the category of pointed sets. More explicitly, a monoid is a pointed set A with basepoint 0 , equipped with a pairing $\mu : A \wedge A \rightarrow A$ (written $\mu(a, b) = ab$) that is associative and commutative and has an identity element 1 . For example, if R is a commutative ring, then forgetting addition gives a monoid (R, \times) of this type. Sometimes $+$ notation is used for μ , for example in applications to toric varieties; in these cases we write 0 for the identity element, and ∞ for the basepoint.

We can convert any unpointed abelian monoid B into a pointed abelian monoid B_* by adjoining a basepoint. Neither the zero monoid $\{0\}$ nor the monoid $\{0, 1, t\}$ with $t^2 = 0$ are of this form.

A morphism of monoids is a map of pointed sets preserving the multiplicative identity and multiplication. The initial monoid is $S^0 = \{0, 1\}$, and the initial map $\iota : S^0 \rightarrow A$ is such that the identity on A equals the composition

$$A \xrightarrow{\cong} S^0 \wedge A \xrightarrow{\iota \wedge id} A \wedge A \xrightarrow{\mu} A.$$

Localization. Given a multiplicatively closed subset S of A , the *localization* $S^{-1}A$ consists of equivalence classes of fractions of the form $\frac{a}{s}$ with $a \in A$ and $s \in S$. As usual, $\frac{a}{s} = \frac{a'}{s'}$ if and only if $as's'' = a's's''$ for some $s'' \in S$. There is a canonical monoid homomorphism $A \rightarrow S^{-1}A$ sending a to $\frac{a}{1}$, and $a, b \in A$ are mapped to the same element of $S^{-1}A$ if and only if $as = bs$ for some $s \in S$.

An *ideal* I in a monoid A is a pointed subset such that $AI \subseteq I$. If $I \subset A$ is an ideal, A/I is the monoid obtained by collapsing I to 0 — i.e., $A/I = (A \setminus I) \cup \{0\}$ and the multiplication rule is the unique one making the canonical surjection $A \rightarrow A/I$ a morphism of monoids.

Every non-zero monoid A has a unique maximal ideal (written \mathfrak{m}_A), namely the complement of the submonoid of units $U(A) := \{a \in A \mid ab = 1 \text{ for some } b\}$. We say that a monoid morphism $g : A \rightarrow B$ is *local* if $g(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ or, equivalently, if $g^{-1}(U(B)) \subseteq U(A)$.

A *prime ideal* is a proper ideal \mathfrak{p} of A whose complement $S = A \setminus \mathfrak{p}$ is closed under multiplication; in this case we write $A_{\mathfrak{p}}$ for the localization $S^{-1}A$. The *dimension* of A is the length of the longest chain of prime ideals, and the *height* of \mathfrak{p} is the dimension of $A_{\mathfrak{p}}$. Since the intersection of a chain of primes is prime, every prime ideal contains a minimal prime ideal.

Lemma 1.1. *For every multiplicatively closed subset S of A with $0 \notin S$, there is a prime ideal \mathfrak{p} of A such that $S^{-1}A = A_{\mathfrak{p}}$.*

Proof. Since $S^{-1}A$ is a non-zero monoid it has a maximal (proper) ideal \mathfrak{m} ; the inverse image of \mathfrak{m} in A is a prime ideal \mathfrak{p} . Let T denote $A \setminus \mathfrak{p}$; then $S \subset T$ and any $t \in T$ is a unit in $S^{-1}A$. Hence there are homomorphisms $S^{-1}A \rightarrow T^{-1}A = A_{\mathfrak{p}}$ and $T^{-1}A \rightarrow S^{-1}A$ covering the identity of A . Hence both composites $S^{-1}A \rightarrow S^{-1}A$ and $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}$ are identity maps, by the universal property of localization. \square

We let $\text{MSpec}(A)$ denote the set of prime ideals of A ; it is a topological space when equipped with its Zariski topology, in which closed subsets are those of the form $V(I) = \{\mathfrak{p} \mid I \subset \mathfrak{p}\}$ for an ideal I of A . The principal open subsets

$$D(s) = \{\mathfrak{p} \in \text{MSpec}(A) \mid s \notin \mathfrak{p}\} = \text{MSpec}(A[1/s])$$

form a basis for the Zariski topology. The space $\text{MSpec}(A)$ is quasi-compact, since $\mathfrak{m} \in D(s)$ implies $D(s) = \text{MSpec}(A)$.

There is a sheaf of monoids \mathcal{A} on $\text{MSpec}(A)$ whose stalk at \mathfrak{p} is $A_{\mathfrak{p}}$; if U is open then $\mathcal{A}(U)$ is the subset of $\prod_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ consisting of elements which locally come from some $S^{-1}A$. In particular, $\mathcal{A} = \mathcal{A}_{\mathfrak{m}_A}$, and $\mathcal{A}(D(s)) = A[1/s]$.

Example 1.2. The free (abelian) pointed monoid on the set $\{t_1, \dots, t_n\}$ is the multiplicative monoid F_n consisting of all monomials in the polynomial ring $k[t_1, \dots, t_n]$ (together with 0). Each of the 2^n subsets of $\{t_1, \dots, t_n\}$ generates a prime ideal \mathfrak{p} , and every prime ideal of F_n has this form. We write \mathbb{A}^n for $\text{MSpec}(F_n)$.

If $A \rightarrow B$ is a morphism of monoids, then the inverse image of a prime ideal is a prime ideal, and we have a continuous map $\text{MSpec}(B) \rightarrow \text{MSpec}(A)$. If I is an ideal of A then $\text{MSpec}(A/I) \rightarrow \text{MSpec}(A)$ is a closed injection onto $V(I)$. If S is multiplicatively closed in A then either $S^{-1}A = 0$ or $S^{-1}A = A_{\mathfrak{p}}$ for some \mathfrak{p} (Lemma 1.1), and in either case $\iota : \text{MSpec}(S^{-1}A) \rightarrow \text{MSpec}(A)$ is an injection onto the primes disjoint from S . The restriction $\iota^{-1}(\mathcal{A})$ to this subset is the sheaf of monoids on $\text{MSpec}(A_{\mathfrak{p}})$.

Lemma 1.3. *Let \mathfrak{p} be a prime ideal in a monoid A . Then $\text{MSpec}(A_{\mathfrak{p}}) \rightarrow \text{MSpec}(A)$ is an injection, closed under generalization, and the following are equivalent:*

- (i) $\text{MSpec}(A_{\mathfrak{p}})$ is open in $\text{MSpec}(A)$.
- (ii) $\text{MSpec}(A_{\mathfrak{p}}) = D(s)$ for some $s \in A$.
- (iii) There is an $s \in A$ such that $A_{\mathfrak{p}} = A[1/s]$.

Proof. The first assertion was observed above. Since $D(s) = \text{MSpec}(A[1/s])$, (iii) is equivalent to (ii), a special case of (i). Conversely, suppose that $U = \text{MSpec}(A_{\mathfrak{p}})$ is the complement of $V(I)$ for some ideal I of A . Then $U = \cup_{s \in I} D(s)$. In particular, there is an s in I such that $\mathfrak{p} \in D(s)$. But then $U \subseteq D(s)$ and hence $U = D(s)$. \square

Example 1.4. Let A be the free pointed monoid generated by an infinite set $\{t_1, t_2, \dots\}$. If \mathfrak{p} is the prime ideal generated by a fixed finite subset of the t_i 's then $\text{MSpec}(A_{\mathfrak{p}})$ cannot be open in $\text{MSpec}(A)$. Indeed, if it were open then by Lemma 1.3 it would have the form $D(s)$ for some element $s \in A$. But any s involves only a finite number of variables, so the prime ideal $t_j A$ belongs to $D(s)$ for infinitely many $t_j \notin \mathfrak{p}$.

Lemma 1.5. *If A is finitely generated as a monoid, then $\text{MSpec}(A)$ is a finite partially ordered set. If S is a multiplicative subset of A , then $S^{-1}A$ is also finitely generated, and $\text{MSpec}(S^{-1}A)$ is open in $\text{MSpec}(A)$.*

Proof. Suppose A is generated by x_1, \dots, x_m . Then for any prime ideal \mathfrak{p} , the multiplicative subset $S = A \setminus \mathfrak{p}$ is generated by $\{x_i \mid x_i \notin \mathfrak{p}\}$. Indeed, if $s \in S$, then $s = \prod_i x_i^{e_i}$ with $e_i = 0$ whenever $x_i \in \mathfrak{p}$. Thus A has at most 2^m prime ideals.

By Lemma 1.1, we may assume $S = A \setminus \mathfrak{p}$ for some prime \mathfrak{p} . If s is the product of the generators of S , then $A_{\mathfrak{p}} = A[1/s]$. By Lemma 1.3, $\text{MSpec}(A_{\mathfrak{p}})$ is open. \square

We say A is *cancellative* if whenever $ab = ac$ and $a \neq 0$ we must have $b = c$. In this case, $A \setminus \{0\}$ is an unpointed submonoid of its group completion and $\{0\}$ is the unique minimal prime ideal. We define the *pointed group completion* of A to be the pointed monoid A^+ obtained by adjoining a basepoint to the usual group completion of the unpointed monoid $A \setminus \{0\}$. Note that A is a pointed submonoid of A^+ , and that A^+ is the localization $A_{\{0\}}$ of A at the minimal prime ideal.

We say A is *torsionfree* if whenever $a^n = b^n$ for $a, b \in A$ and some $n \geq 1$, we have $a = b$. The monoid $\{0, \pm 1\}$ is cancellative but not torsionfree. If A is cancellative and $A^+ \setminus \{0\}$ is a torsionfree abelian group, then A is torsionfree.

An element is *nilpotent* if $a^n = 0$ for some n , and the *nilradical* of A is the set $\text{nil}(A)$ of nilpotent elements. It is easy to prove (using Zorn's lemma as in ring theory), that $\text{nil}(A)$ is the intersection of the minimal prime ideals of A . We say that A is *reduced* if $\text{nil}(A) = 0$, and set $A_{\text{red}} = A/\text{nil}(A)$.

Any closed subset Z of $X = \text{MSpec}(A)$ defines a largest ideal I such that $Z = V(I)$, and A/I is a reduced monoid. Indeed, if $Z = V(I_0)$ then $A/I = (A/I_0)_{\text{red}}$; I is the intersection of the prime ideals containing I_0 . Anticipating Lemma 2.8, we write \bar{Z}^{eq} for $\text{MSpec}(A/I)$ and call it the *equivariant closure* of Z in X . For example, \bar{X}^{eq} is $\text{MSpec}(A_{\text{red}})$.

Definition 1.6. The *normalization* of a cancellative monoid A is defined to be the submonoid

$$A_{\text{nor}} = \{\alpha \in A^+ \mid \alpha^n \in A \text{ for some } n \geq 1\}.$$

We say that A is *normal* if $A = A_{\text{nor}}$. The normalization of $S^{-1}A$ is $S^{-1}A_{\text{nor}}$. If A is torsionfree then so is A_{nor} .

Remark 1.6.1. If A is cancellative then $\mathrm{MSpec}(A_{\mathrm{nor}}) \rightarrow \mathrm{MSpec}(A)$ is a topological homeomorphism. Indeed, if \mathfrak{p} is a prime ideal of A then $\mathfrak{p}_{\mathrm{nor}} := \{b \in A_{\mathrm{nor}} \mid (\exists n)b^n \in \mathfrak{p}\}$ is a prime ideal of A_{nor} and $\mathfrak{p} = \mathfrak{p}_{\mathrm{nor}} \cap A$. It is easily seen that every prime ideal of A_{nor} has the form $\mathfrak{p}_{\mathrm{nor}}$ for some \mathfrak{p} .

Remark 1.6.2. If A is normal and \mathfrak{p} is a prime ideal, an elementary argument shows that A/\mathfrak{p} is also normal.

More generally, let $f : A \rightarrow B$ be a morphism of monoids. We say that f is *integral* if for every $b \in B$ there is an integer $n \geq 1$ such that b^n lies in the image of A , and we say that f is *finite* if there exist $b_1, \dots, b_n \in B$ ($n \geq 1$) such that $B = \bigcup_i Ab_i$. The normalization $A \rightarrow A_{\mathrm{nor}}$ is integral but not always finite.

Lemma 1.7. *Let $A \xrightarrow{f} B$ be a monoid morphism with B finitely generated over A .*

- i) *If f is integral, then f is finite.*
- ii) *If f is finite and B is cancellative, then f is integral.*

Proof. Choose a surjection $A[t_1, \dots, t_n] \twoheadrightarrow B$, with the t_i mapping onto generators b_i of B over A . If f is integral, then there is an m such that b_i^m is in the image of A for all i ; thus every element of B can be written as a product $f(a)c_j$, where $a \in A$ and c_j is a monomial on the b_i with exponents $\leq m$. This proves i).

Next assume that f is finite and that B is cancellative. Let $b_1, \dots, b_n \in B$ be such that $B = \bigcup_i Ab_i$. For each i , we choose an index $\pi(i)$ and $a_i \in A$ such that $b_i^2 = a_i b_{\pi(i)}$; then π is a map from the finite set $\{1, \dots, n\}$ to itself. For each fixed i , the iterates $\pi^r(i)$ cannot all be distinct, so there exist $s \geq 1$ and $r \geq 1$ such that $j = \pi^r(i)$ satisfies $\pi^s(j) = j$. Hence there is an $a \in A$ and $m \geq 1$ such that $b_j^m = ab_j$. Because B is cancellative, this implies that $b_j^{m-1} = f(a)$. Thus b_j and hence b_i is integral over A , as required. \square

Remark 1.7.1. The hypothesis that B be cancellative in part ii) of Lemma 1.7 is necessary. For example, the monoid B generated by x, y subject to $y^2 = xy$ contains the free monoid A generated by x ; the extension $A \subset B$ is finite but not integral.

For a pointed set X and commutative ring k , $k[X]$ denotes the free k -module on X , modulo the summand indexed by the base point of X . If A is a pointed monoid, $k[A]$ is a ring in the usual way, with multiplication given by the product rule for A . If B is an unpointed monoid, $k[B_*]$ coincides with the usual monoid ring for B with k coefficients. If I is an ideal of the monoid A then $k[I]$ is an ideal of the ring $k[A]$, and $k[A/I] = k[A]/k[I]$. If I is prime, $k[I]$ need not be a prime ideal.

The category of pointed monoids has all colimits. The functor $A \mapsto k[A]$ preserves colimits since it has a right adjoint, sending an algebra R to (R, \times) , the underlying multiplicative monoid of R . The coproduct of A_1 and A_2 is the smash product $A_1 \wedge A_2$, so $k[A_1 \wedge A_2] \cong k[A_1] \otimes_k k[A_2]$; the maps from A_1 and A_2 to $A_1 \wedge A_2$ send a_1 to $a_1 \wedge 1$ and a_2 to $1 \wedge a_2$. More generally, the pushout $A_1 \wedge_C A_2$ of a diagram

$$(1.8) \quad \begin{array}{ccc} C & \xrightarrow{f} & A_2 \\ \downarrow g & & \downarrow \text{dotted} \\ A_1 & \dashrightarrow & A_1 \wedge_C A_2 \end{array}$$

is the quotient of $A_1 \wedge A_2$ by the congruence generated by $(a_1 f(c), a_2) \sim (a_1, g(c) a_2)$. Note that $k[A_1 \wedge_C A_2] \cong k[A_1] \otimes_{k[C]} k[A_2]$.

Lemma 1.9. *Every prime ideal of $A_1 \wedge A_2$ has the form $\mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$ for unique prime ideals \mathfrak{p}_1 in A_1 , \mathfrak{p}_2 in A_2 .*

Proof. Given a prime ideal \mathfrak{p} of $A_1 \wedge A_2$, set $\mathfrak{p}_1 = \mathfrak{p} \cap A_1$, $\mathfrak{p}_2 = \mathfrak{p} \cap A_2$ and $\mathfrak{q} = \mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$. Then \mathfrak{q} is prime because its complement is $(A_1 \setminus \mathfrak{p}_1) \times (A_2 \setminus \mathfrak{p}_2)$, which is multiplicatively closed. Clearly $\mathfrak{q} \subseteq \mathfrak{p}$; to see that $\mathfrak{q} = \mathfrak{p}$, consider an element $a_1 \wedge a_2$ of \mathfrak{p} . As \mathfrak{p} is prime, either $a_1 \wedge 1$ or $1 \wedge a_2$ is in \mathfrak{p} . In the first case, $a_1 \in \mathfrak{p}_1$ so $a_1 \wedge a_2$ is in $\mathfrak{p}_1 \wedge A_2 \subseteq \mathfrak{q}$; in the second case, $a_2 \in \mathfrak{p}_2$ so $a_1 \wedge a_2$ is in $A_1 \wedge \mathfrak{p}_2 \subseteq \mathfrak{q}$. \square

Example 1.10. If T is the free monoid on one element t , then $A \wedge T$ is the analogue of a polynomial ring over A , and $k[A \wedge T] = k[A][t]$. For any prime ideal \mathfrak{p} of A there are exactly two primes of $A \wedge T$ over \mathfrak{p} : the extended prime $\mathfrak{p} \wedge T$ and the prime generated by \mathfrak{p} and t (i.e., $\mathfrak{p} \wedge T \cup A \wedge \{t^n : n \geq 1\}$). The map $\text{MSpec}(A \wedge T) \rightarrow \text{MSpec}(A)$ is both open and closed, because the image of $D(at^n)$ is $D(a)$ and the image of $V(I)$ is $V(I \cap A)$.

Proposition 1.11. *Given a pushout diagram (1.8), every prime ideal of $A_1 \wedge_C A_2$ has the form $\mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$ for unique prime ideals \mathfrak{p}_1 in A_1 , \mathfrak{p}_2 in A_2 .*

Moreover, the ideal $\mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$ of $A_1 \wedge_C A_2$ is prime if and only if \mathfrak{p}_1 and \mathfrak{p}_2 have a common inverse image in C .

Proof. If \mathfrak{p} is a prime in $A_1 \wedge_C A_2$, its inverse image in $A_1 \wedge A_2$ is prime; by Lemma 1.9 it has the form $\mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$, where $\mathfrak{p}_i \subset A_i$ are the inverse images of \mathfrak{p} . Since $A_1 \wedge_C A_2$ is a quotient, this proves the first assertion; because (1.8) commutes, \mathfrak{p}_1 and \mathfrak{p}_2 have a common inverse image in C .

Conversely, suppose that \mathfrak{p}_1 and \mathfrak{p}_2 have a common inverse image \mathfrak{q} in C , and set $S_1 = A_1 \setminus \mathfrak{p}_1$, $S_2 = A_2 \setminus \mathfrak{p}_2$ and $I = \mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2 \subset A_1 \wedge_C A_2$. To see that the ideal I is prime, it suffices to show that the image of $S_1 \times S_2$ in $A_1 \wedge_C A_2$ is disjoint from I . Since \mathfrak{p}_1 and \mathfrak{p}_2 are prime, $a_1 f(c) \in S_1$ if and only if $a_1 \in S_1$ and $c \notin \mathfrak{q}$, while $g(c)a_2 \in S_2$ if and only if $a_2 \in S_2$ and $c \notin \mathfrak{q}$. It follows that $(a_1 f(c), a_2)$ is in $S_1 \times S_2$ if and only if $(a_1, g(c)a_2)$ is. Thus $S_1 \times S_2$ is closed under the equivalence relation defining $A_1 \wedge_C A_2$, and its image in $A_1 \wedge_C A_2$ is disjoint from I . \square

2. MONOID SCHEMES

We will need to consider *monoid schemes*, sometimes known as “schemes over the field with one element”. These are the objects which result by gluing together spectra of pointed monoids along open subsets, and will be related to classical schemes in Section 5. The theory of monoid schemes was developed by Kato [16], Deitmar [6], Connes-Consani-Marcolli [4], [2], [3], etc. The survey [17] by López and Lorscheid gives a nice overview of this notion and related ideas.

A *monoid space* is a pair (X, \mathcal{A}_X) consisting of a topological space X and a sheaf \mathcal{A}_X of pointed abelian monoids on X . A *morphism of monoid spaces* from (X, \mathcal{A}_X) to (Y, \mathcal{A}_Y) is given by a continuous map $f : X \rightarrow Y$ together with a morphism of sheaves $f^{-1}\mathcal{A}_Y \rightarrow \mathcal{A}_X$ on X (or, equivalently, a morphism $\mathcal{A}_Y \rightarrow f_*\mathcal{A}_X$ of sheaves on Y) that is *local* in the sense that the maps on stalks $\mathcal{A}_{Y,f(x)} \rightarrow \mathcal{A}_{X,x}$ are local morphisms of monoids, for all $x \in X$. By abuse of notation, we will often write simply X for the monoid scheme (X, \mathcal{A}_X) .

The association $A \mapsto \text{MSpec}(A)$ extends to a fully faithful contravariant functor from monoids to monoid spaces, which we will call MSpec by abuse of notation.

An *affine monoid scheme* is a monoid space isomorphic to $\mathrm{MSpec}(A)$ for some monoid A . A *monoid scheme* is a monoid space such that every point has an open neighborhood isomorphic to an affine monoid scheme. A morphism of monoid schemes is just a morphism of the underlying monoid spaces. The *dimension* of a monoid scheme is the largest dimension of its affine open neighborhoods.

Lemma 2.1. *Let (X, \mathcal{A}) be a monoid scheme. For any open $U \subseteq X$, the monoid space $(U, \mathcal{A}|_U)$ is a monoid scheme. We refer to it as the open subscheme of X associated to U .*

Proof. If $x \in U$ and $V = \mathrm{MSpec}(A)$ is an affine open neighborhood of x in X , $U \cap V$ is also open. Since $U \cap V$ is the union of basic open subschemes $D(s)$ of V , x has a neighborhood of the form $D(s)$, and $D(s) = \mathrm{MSpec}(A[1/s])$ is affine. \square

We say that a monoid scheme is cancellative (resp., reduced, normal, ...) if its stalks are cancellative monoids (resp., reduced, normal, ... monoids).

Given a closed subset Z of a monoid scheme X , there is a reduced closed subscheme Z_{red} associated to Z , defined by patching; if $X = \mathrm{MSpec}(A)$ and $Z = V(I)$ then $Z_{\mathrm{red}} = \mathrm{MSpec}(A/I)_{\mathrm{red}}$.

Example 2.2. The projective line \mathbb{P}^1 is obtained by gluing $\mathrm{MSpec}(\{t^n\}_*)$ and $\mathrm{MSpec}(\{t^{-n}\}_*)$ along $\mathrm{MSpec}(\{t^{\pm n}\}_*)$. This monoid scheme is connected, torsion-free and normal.

Recall that the points of any topological space may be partially ordered by the relation that $x \leq y$ if and only if y is in the closure of $\{x\}$. In this way we can speak of maximal and minimal points. For the topological space $\mathrm{MSpec}(A)$ of a monoid A , we have $\mathfrak{p} \leq \mathfrak{q}$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$. Minimal points exist in any monoid scheme because, as noted before 1.1, every prime ideal contains a minimal prime ideal.

Lemma 2.3. *Each cancellative monoid scheme X decomposes as the disjoint union of (closed and open) monoid subschemes X_η , each the closure of a unique minimal point η of X . In particular, if X is connected then it has a unique minimal point.*

Proof. Let X_η denote the closure of η in X . Given $x \in X$, choose an affine neighborhood $U_x = \mathrm{MSpec}(A)$ of x . If y is the point of X corresponding to \mathfrak{m}_A , then $A = \mathcal{A}_y$. Since A is cancellative, U_x has a unique minimal point η , so $U_x \subseteq X_\eta$. It follows that $X_\eta = \cup U_x$ is open (and closed) in X , and that X is the disjoint union of the X_η . \square

Lemma 2.4. *Let X be a monoid scheme and $U \subseteq X$ an open subscheme. Then the following are equivalent.*

- (i) U is an affine monoid scheme.
- (ii) U has a unique maximal point.

If $X = \mathrm{MSpec}(A)$, every affine open subscheme is $\mathrm{MSpec}(A_{\mathfrak{p}})$ for some \mathfrak{p} .

Proof. Since monoids have unique maximal ideals, (i) implies (ii). Conversely, suppose that U has a unique maximal point x . Note that $U = \{y | y \leq x\}$. If $\mathrm{MSpec}(A)$ is an affine open neighborhood of x , then $U \subseteq \mathrm{MSpec}(A)$, so we may assume that $X = \mathrm{MSpec}(A)$. In this case $U = \mathrm{MSpec}(A_x)$ by Lemma 1.3. \square

Definition 2.5. Let $f : Y \rightarrow X$ be a map of monoid schemes. We say that f is a *closed immersion* if it induces a homeomorphism of Y onto its image (equipped with the subspace topology), and for every affine open subscheme $U = \mathrm{MSpec}(A)$

of X (i) the open subscheme $V = U \cap Y$ of Y is affine (possibly empty) and (ii) the map $\mathcal{A}_X(U) \rightarrow \mathcal{A}_Y(V)$ is surjective. A *closed subscheme* of a monoid scheme X is an isomorphism class of closed immersions into X .

A closed immersion $f : Y \rightarrow X$ is called *equivariant* if in addition each such $\mathcal{A}_X(U) \rightarrow \mathcal{A}_Y(V)$ is the quotient by an ideal.

The terminology “equivariant closed immersion” comes from the theory of toric varieties: the equivariant closed subschemes of a toric variety are precisely those closed subschemes that are equivariant for the action of the underlying torus. We will see in Section 4 that a toric variety has an associated toric monoid scheme, and that the equivariant closed subschemes of the monoid scheme determine equivariant closed subschemes of the toric variety.

Lemma 2.6. *Any surjection of monoids $A \twoheadrightarrow B$ determines a closed immersion $\mathrm{MSpec}(B) \subseteq \mathrm{MSpec}(A)$. If $B = A/I$ then it is an equivariant closed subscheme.*

Proof. Set $Y = \mathrm{MSpec}(B)$ and $X = \mathrm{MSpec}(A)$. The map $\pi^* : Y \rightarrow X$ of underlying spaces is injective, since if $\mathfrak{q}_1 \neq \mathfrak{q}_2$ then $\pi^{-1}(\mathfrak{q}_1) \neq \pi^{-1}(\mathfrak{q}_2)$. If $a \in A$, the image of the basic open $D(\pi(a)) \subseteq Y$ is $D(a) \cap \pi^*(Y)$. Thus Y is homeomorphic to $\pi^*(Y)$.

Let $U \subseteq \mathrm{MSpec}(A)$ be an affine open subscheme. By Lemma 2.4 there is a prime \mathfrak{p} of A such that $U = \mathrm{MSpec}(A_{\mathfrak{p}})$; by Lemma 1.3, $U = D(s)$ for some s . Hence $U \cap Y = D(\pi(s)) = \mathrm{MSpec}(B[1/s])$, which is affine or empty. Since $A[1/s] \rightarrow B[1/s]$ is onto, $Y \rightarrow X$ is a closed immersion. \square

Remark 2.6.1. A closed subscheme $Y \subset X$ need not determine a closed subset of the underlying topological space. For example, the diagonal embedding $\mathbb{A}^1 \rightarrow \mathbb{A}^2$ is a closed immersion by Lemma 2.6, but it is not topologically closed, because it takes the generic point of \mathbb{A}^1 to the generic point of \mathbb{A}^2 and the maximal point to the maximal point; the intermediate points are not in the image.

Definition 2.7. If (X, \mathcal{A}) is a monoid scheme, a sheaf of ideals $\mathcal{I} \subset \mathcal{A}$ is said to be *quasi-coherent* if its restriction to any affine open subscheme $U \subset X$ is the sheaf associated to the ideal $\mathcal{I}(U)$ of $\mathcal{A}(U)$. Given any closed immersion $i : Y \rightarrow X$, the inverse image \mathcal{I} of 0 under $\mathcal{A}_X \rightarrow i_*\mathcal{A}_Y$ is quasi-coherent. Lemma 2.6 shows that conversely any quasi-coherent sheaf \mathcal{I} defines an equivariant closed immersion.

We saw in Section 1 that if Z is any subset of $\mathrm{MSpec}(A)$, there is an equivariant closed subscheme $\bar{Z}^{\mathrm{eq}} = \mathrm{MSpec}(A/I)$ which contains Z (and its closure), and which is minimal with this property. Indeed, if the closure of Z is $V(I_0)$, then $A/I = (A/I_0)_{\mathrm{red}}$. An important special case is when $Z = \{\mathfrak{p}_1, \dots, \mathfrak{p}_l\}$ is a set of prime ideals of A ; in this case $\bar{Z}^{\mathrm{eq}} = \mathrm{MSpec}(A/\cap \mathfrak{p}_i)$. Since $S^{-1}(A/I) = (S^{-1}A/S^{-1}I_0)_{\mathrm{red}}$, this construction patches to give a general construction, formalized in Lemma 2.8.

Lemma 2.8. *For any monoid scheme X and any subset Z of the underlying poset, there is an equivariant closed subscheme \bar{Z}^{eq} of X that contains Z and is contained in every other equivariant closed subscheme of X containing Z . We call \bar{Z}^{eq} the equivariant closure of Z in X .*

If U is an open subscheme of X then $\bar{Z}^{\mathrm{eq}} \cap U$ is $\overline{Z \cap U}^{\mathrm{eq}}$.

Remark 2.8.1. If every point in Z has height at least i in X then every point in \bar{Z}^{eq} has height at least i in X . This follows from the local description of \bar{Z}^{eq} .

Finite type. We say that a monoid scheme has *finite type* if it admits a finite open cover by affine monoid schemes associated to finitely generated monoids. These monoid schemes are the analogues of Noetherian schemes, just as finitely generated

monoids are the analogues of commutative Noetherian rings: if A is a finitely generated monoid then every ideal is finitely generated, and A has the ascending chain condition on ideals. (The usual proof of the Hilbert Basis Theorem works.)

By Lemma 1.5, if (X, \mathcal{A}) is a monoid scheme of finite type, then X is a finite poset, with the poset topology. The sheaf of monoids \mathcal{A} determines a (contravariant) functor $A : x \mapsto \mathcal{A}_x$ from the poset X to monoids, called the *stalk functor* of (X, \mathcal{A}) . The following proposition shows that the stalk functor is always enough to determine a monoid scheme of finite type.

Proposition 2.9. *A monoid scheme of finite type is given uniquely by a finite poset X and a functor A from X to monoids such that for all $x \in X$ the restriction of A to the subset $\langle x \rangle = \{y \in X \mid y \leq x\}$ of X coincides with the poset $\text{MSpec}(A(x))$ equipped with the functor $\mathfrak{p} \mapsto A(x)_{\mathfrak{p}}$.*

A morphism $(X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ between monoid schemes of finite type, with stalk functors A and B , is given by a morphism of posets $f : X \rightarrow Y$ together with a natural transformation $\eta : B \circ f \Rightarrow A$ such that each $\eta_x : B(f(x)) \rightarrow A(x)$ is a local homomorphism.

Proof. Given a finite poset X and functor A as in the statement, define the presheaf \mathcal{A} on X by

$$\mathcal{A}(U) = \varprojlim_{x \in U} A(x).$$

It is easily verified that (X, \mathcal{A}) is a monoid scheme with stalk functor A . The remaining details are left to the reader. \square

By abuse, if a functor A from a finite poset to monoids satisfies the requirement of Proposition 2.9, we will call A a *stalk functor*. A monoid scheme (X, \mathcal{A}) of finite type will often be specified by a poset together with a stalk functor A , *viz.*, (X, A) . To avoid confusion, we shall use roman letters for stalk functors and script letters for sheaves.

Remark 2.10. If (X, \mathcal{A}) is an arbitrary monoid scheme with stalk functor A , and $U \subseteq X$ is affine open, we know by Lemma 2.4 that there is a unique $x \in X$ such that $U = \text{MSpec}(A(x))$ and hence $A(x) = \mathcal{A}(U)$. It follows that A determines the value of \mathcal{A} on all affine open subschemes. Given an open U in X , any point $y \in U$ lies in affine open $V \subset U$, and $V = \text{MSpec}(A(x))$ for some $x \in U$ with $y < x$ by Lemma 2.4. Thus we can recover \mathcal{A} from A and the topological space underlying X , using the formula

$$\mathcal{A}(U) = \varprojlim_{x \in U} A(x).$$

3. BASECHANGE AND SEPARATED MORPHISMS

It is useful to simplify constructions using base-change. For this, we need pullback squares in the category of monoid schemes.

There is a canonical morphism $X \rightarrow \text{MSpec}(\mathcal{A}(X))$ which is universal for maps from X to affine monoid schemes. The universal property shows that the (contravariant) functor $X \mapsto \mathcal{A}_X(X)$ from monoid schemes to monoids is left adjoint to the functor MSpec . It follows that MSpec converts pushouts of diagrams of monoids to pullbacks of diagrams of monoid schemes. In particular, for any pushout diagram of monoids (1.8), the pullback $\text{MSpec}(A) \times_{\text{MSpec}(C)} \text{MSpec}(B)$ exists in the category of monoid schemes and equals $\text{MSpec}(A \wedge_C B)$.

Proposition 3.1. *The pullback $X \times_S Y$ of a diagram of monoid schemes*

$$\begin{array}{ccc} X \times_S Y & \dashrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

exists in the category of all monoid schemes. Its underlying topological space is the pullback $X \times_S Y$ in the category of topological spaces.

Proof. Existence of the pullback $X \times_S Y$ is derived from the existence of pullbacks of affine monoid schemes, just as for usual schemes ([13, Thm. 3.3]).

To prove the assertion about underlying topological spaces, it suffices to consider the affine case. Using the notation of (1.8), write P for the pullback of $\mathrm{MSpec}(A_1)$ and $\mathrm{MSpec}(A_2)$ over $\mathrm{MSpec}(C)$ in Top . The map $f : \mathrm{MSpec}(A_1 \wedge_C A_2) \rightarrow P$ is a continuous bijection by Proposition 1.11. To show that f is a homeomorphism, it suffices to show that it takes any basic open set $D(s)$ to an open set of P . Write $s = s_1 \wedge s_2$; then $s \notin \mathfrak{p}$ if and only if $s_1 \wedge 1, 1 \wedge s_2 \notin \mathfrak{p}$. We saw in Proposition 1.11 that if \mathfrak{p} maps to $(\mathfrak{p}_1, \mathfrak{p}_2)$ then $\mathfrak{p} = \mathfrak{p}_1 \wedge A_2 \cup A_1 \wedge \mathfrak{p}_2$, and that $s_1 \wedge 1 \notin \mathfrak{p}$ (resp., $1 \wedge s_2 \notin \mathfrak{p}$) is equivalent to $s_1 \notin \mathfrak{p}_1$ (resp., $s_2 \notin \mathfrak{p}_2$). This shows that f takes $D(s)$ to the open set $(D(s_1) \times D(s_2)) \cap P$, as required. \square

Example 3.1.1. The product $X \times Y$ is just the pullback when S is the terminal monoid scheme $\mathrm{MSpec}(S^0)$.

Remark 3.1.2. Let X and Y be monoid schemes of finite type, over a common S . Then the pullback $X \times_S Y$ has finite type. Indeed, it has a finite cover by affine opens of the form $\mathrm{MSpec}(A_1 \wedge_C A_2)$, and in each case $A_1 \wedge_C A_2$ is finitely generated because A_1 and A_2 are.

Example 3.2. Proposition 3.1 shows that given two closed subschemes Z_1, Z_2 of X , the pullback $Z_1 \times_X Z_2$ is a subscheme whose underlying topological space is the intersection of the two subspaces of X . More generally, given any family of closed immersions $Z_i \hookrightarrow X$, we can form the inverse limit $\lim Z_i \hookrightarrow X$ by patching the inverse limits on each affine open $\mathrm{MSpec}(A)$, because the colimit of a family of surjections $A \twoheadrightarrow B_i$ exists and is a surjection.

Separated morphisms. An important hypothesis in many theorems about monoid schemes, often overlooked in the literature, is that they be separated.

Definition 3.3. A morphism $f : X \rightarrow S$ of monoid schemes is *separated* if the diagonal map $\Delta : X \rightarrow X \times_S X$ is a closed immersion. We say that X is separated if it is separated over $\mathrm{MSpec}(S^0)$ where we recall $S^0 = \{0, 1\}$.

Being separated is local on the base: if S has an open cover $\{U\}$ then f is separated if and only if each $f^{-1}(U) \rightarrow U$ is separated.

Lemma 3.4. *If $A \rightarrow B$ is a morphism of monoids then $\mathrm{MSpec}(B) \rightarrow \mathrm{MSpec}(A)$ is a separated morphism of monoid schemes.*

In particular, closed immersions are separated.

Proof. By Proposition 3.1, the diagonal map Δ corresponds to the multiplication map $B \wedge_A B \rightarrow B$, which is surjective. By Lemma 2.6, Δ is a closed immersion. \square

Remark 3.4.1. Example 1.10 shows that $X \times \mathbb{A}^1 \rightarrow X$ is separated and universally closed for every monoid scheme X . This shows that “separated and universally closed” does not provide a good notion of proper morphism of monoid schemes; we will discuss an appropriate definition in Section 8.

Example 3.5. Here is an example of a monoid scheme which is non-separated. Let A and B each be the free abelian monoid with two generators, F_2 (see Example 1.2). Let U be the open subset of each of $\text{MSpec}(A)$ and $\text{MSpec}(B)$ given by removing the unique closed point (associated to the maximal ideal in each monoid). Then we may glue $\text{MSpec}(A)$ and $\text{MSpec}(B)$ along U to form a monoid scheme X of finite type. As a poset, X has five elements, two of which are maximal and the rest are in U .

The k -realization of X (defined in 5.3 below) is the non-separated scheme given by the affine plane with the origin doubled.

Lemma 3.6. *A map $f : (X, \mathcal{A}) \rightarrow (S, \mathcal{B})$ of monoid schemes is separated if and only if for every x_1, x_2 in X such that $f(x_1) = f(x_2)$ and such that $\text{MSpec}(\mathcal{A}_{x_1})$ and $\text{MSpec}(\mathcal{A}_{x_2})$ are open, either there is no lower bound for $\{x_1, x_2\}$ in the poset X or else there is a unique maximal lower bound $x_0 = x_1 \cap x_2$, and $\mathcal{A}_{x_1} \wedge_{\mathcal{B}_{f(x_1)}} \mathcal{A}_{x_2} \rightarrow \mathcal{A}_{x_0}$ is onto.*

Proof. By Proposition 3.1 and Lemma 2.4, an affine open subset of $X \times_S X$ has the form $U = (U_1 \times U_2) \cap (X \times_S X)$, where the maximal point (x_1, x_2) of U determines the affine open subsets $U_i = \text{MSpec}(\mathcal{A}_{x_i})$ of X . Since $\Delta^{-1}(U) = U_1 \cap U_2$, Proposition 3.1 implies that $X \rightarrow \Delta(X)$ is a homeomorphism and that the poset underlying $U_1 \cap U_2$ is the subset $\{z \in X \mid z \leq x_1, z \leq x_2\}$ of lower bounds for $\{x_1, x_2\}$. If $U_1 \cap U_2 = \emptyset$, $\{x_1, x_2\}$ has no lower bound.

By Lemma 2.4, $U_1 \cap U_2$ is nonempty affine if and only if it has a unique maximal element. Thus Δ is a closed immersion if and only if, in the above situation, whenever $U_1 \cap U_2$ is nonempty it is affine (and hence has a unique maximal lower bound x_0), and $\mathcal{A}_{x_1} \wedge_C \mathcal{A}_{x_2} \rightarrow \mathcal{A}_{x_0}$ is onto, where $s = f(x_1) = f(x_2)$ and $C = \mathcal{B}_s$. \square

Corollary 3.7. *If X is a monoid scheme of finite type with stalk functor A , then X is separated if and only if whenever two points x_1, x_2 of X have a lower bound they have a greatest lower bound $x_1 \cap x_2$, and $A(x_1) \wedge A(x_2) \rightarrow A(x_1 \cap x_2)$ is onto.*

Proof. Combine Lemma 3.6 and Proposition 2.9. \square

Corollary 3.8. *The intersection of two affine open subschemes of a separated monoid scheme is affine.*

Proof. Suppose X is a separated monoid scheme, with U_1, U_2 affine and open in X . Let x_1, x_2 be the unique closed points of U_1, U_2 . If x_1 and x_2 do not have a common lower bound in X , then $U_1 \cap U_2 = \emptyset$. Otherwise, by Lemma 3.6, they have a greatest lower bound, which is the unique maximal point of $U_0 = U_1 \cap U_2$. By Lemma 2.4, U_0 is affine. \square

4. TORIC MONOID SCHEMES

As observed by Kato [16] and Deitmar [6], the fan associated to a toric variety produces a monoid scheme. In this section we clarify this correspondence, using the following definition.

Definition 4.1. A *toric monoid scheme* is a separated, connected, torsionfree, normal monoid scheme of finite type.

Recall that a *fan* consists of a free abelian group N of finite rank (written additively) together with a finite collection Δ of strongly convex rational polyhedral cones σ in $N_{\mathbb{R}}$ (hereafter referred to as just *cones*), satisfying the conditions that (1) every face of a member of Δ is also a member of Δ and (2) the intersection of any two members of Δ is a face of each. Note that Δ is a finite poset under containment.

Construction 4.2. Given a fan (N, Δ) , set $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. We define a contravariant functor A from Δ to monoids (written additively) by

$$A(\sigma) = (\sigma^{\vee} \cap M)_{*}, \quad \sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid m(\sigma) \geq 0\}.$$

Each such monoid is torsion-free, normal and finitely generated (Gordon's Lemma). It is not hard to see that the faces τ of σ correspond to prime ideals $P_{\sigma}(\tau)$ of $A(\sigma)$ and that $A(\tau) = A(\sigma)_{P(\tau)}$. By Proposition 2.9 and Corollary 3.7, A is the stalk functor of a toric monoid scheme $X(N, \Delta)$, which by abuse of notation we write as

$$X(\Delta) = (\Delta, A).$$

Thus any fan Δ determines a toric monoid scheme in the sense of Definition 4.1.

A morphism of fans, from (N, Δ) to (N', Δ') , is given by a group homomorphism $\phi : N \rightarrow N'$ such that the image of each cone in Δ under the induced map $N_{\mathbb{R}} \rightarrow N'_{\mathbb{R}}$ is contained in a cone in Δ' . Such a map of fans induces a poset map $\Delta \rightarrow \Delta'$, sending σ to the smallest cone σ' in Δ' that contains $\phi(\sigma)$, and precomposition with ϕ yields a natural transformation $((\sigma')^{\vee} \cap M')_{*} \rightarrow (\sigma^{\vee} \cap M)_{*}$ of stalk functors. According to Proposition 2.9, this data determines a morphism of monoid schemes:

$$X(\phi) : X(\Delta) \rightarrow X(\Delta').$$

If $\phi_1 \neq \phi_2$ then $X(\phi_1) \neq X(\phi_2)$, as $\phi_1^* \neq \phi_2^*$ on some $A(\sigma)$. Thus we have a faithful functor X from fans to toric monoid schemes.

If (X, A) is a toric monoid scheme and $x \in X$, we will write M_x for the group completion of the unpointed monoid $A(x) \setminus \{0\}$. Each M_x is a torsionfree abelian group of finite rank, and these groups are all isomorphic, because X has a unique minimal point η by Lemma 2.3, and $M_x \rightarrow M_{\eta} = A(\eta)$ is an isomorphism for all x .

Theorem 4.3. *The faithful functor $\Delta \mapsto X(\Delta)$ from fans to toric monoid schemes, defined by Construction 4.2 has the following properties.*

- (1) *Every toric monoid scheme (X, A) is isomorphic to $X(N, \Delta)$, where:*
 - a) *The lattice N is the \mathbb{Z} -linear dual of $M = M_{\eta}$, where η is the unique minimal point of X .*
 - b) *The poset Δ of cones in $N_{\mathbb{R}}$ is isomorphic to X . For each $x \in X$, the cone σ_x in $N_{\mathbb{R}}$ is the dual cone of the convex hull of $A(x) \setminus \{0\}$ in $M_{\mathbb{R}}$.*
- (2) *For fans (N, Δ) and (N', Δ') , a morphism $f : X(\Delta) \rightarrow X(\Delta')$ of monoid schemes is given by a (necessarily unique) morphism of fans if and only if f maps the generic point η of $X(\Delta)$ to the generic point η' of $X(\Delta')$. In this case, the map of fans $(N, \Delta) \rightarrow (N', \Delta')$ is given by the \mathbb{Z} -linear dual of the group homomorphism*

$$f_{\eta}^{\#} : M = (A(\eta') \setminus \{0\}) \rightarrow (A(\eta) \setminus \{0\}) = M.$$

Proof. Throughout this proof, for a cancellative monoid A , we write A° for the unpointed monoid $A \setminus \{0\}$, written additively, and we identify each $A^\circ(x)$ with a submonoid of M . For $x \in X$, let $\sigma_x^\vee \subset M_{\mathbb{R}}$ denote the convex hull of $A^\circ(x)$ in $M_{\mathbb{R}}$. It is a rational polyhedral cone because it is spanned by a finite set $\{a_i\}$ of generators of $A^\circ(x)$; the cone σ_x is thus also a rational polyhedral cone, and it is strongly convex since $A^\circ(x)^+ = M$.

To see that $A(x) = (\sigma_x^\vee \cap M)_*$, let $b = \sum q_i a_i$ be an element of $\sigma_x^\vee \cap M$, written as a positive \mathbb{Q} -linear combination of the a_i . Clearing denominators, nb is a positive \mathbb{Z} -linear combination of the a_i for some positive integer n and hence is in $A^\circ(x)$. Because $A(x)$ is normal, b is in $A^\circ(x)$, as required.

If τ is a face of σ_x , it is defined by the vanishing of some $m \in \sigma_x^\vee$. Clearing denominators, we may assume $m \in A^\circ(x)$. By definition, τ is the set of linear functionals on $M_{\mathbb{R}}$ that are non-negative on $A^\circ(x)[-m]$. By Lemma 1.1, $A(x)[-m]$ coincides with $A(y)$ for some $y \leq x$, and thus the face τ is the element σ_y of Δ .

If $x, y \in X$, we claim that the intersection $\sigma_x \cap \sigma_y$ is a cone in Δ . Since X is cancellative and separated, x and y have a unique greatest common lower bound, written $x \cap y$, and the additive version $A^\circ(x) \oplus A^\circ(y) = A^\circ(x \cap y)$ of $A(x) \wedge A(y) = A(x \cap y)$ holds. (See Corollary 3.7.) A linear functional on $M_{\mathbb{R}}$ is non-negative on $A^\circ(x) \oplus A^\circ(y)$ if and only if it is non-negative on $A^\circ(x)$ and $A^\circ(y)$, and thus we have the required identity:

$$\sigma_x \cap \sigma_y = \sigma_{x \cap y}.$$

Moreover, $\sigma_{x \cap y}$ is a face of both σ_x and σ_y , because by Lemmas 1.3 and 1.5 there are m_1, m_2 such that $A^\circ(\sigma_{x \cap y}) = A^\circ(\sigma_x)[-m_1] = A^\circ(\sigma_y)[-m_2]$. This proves that Δ is a fan.

By Example 4.2, the fan (N, Δ) determines a monoid scheme (Δ, B) . The bijection $\sigma : X \rightarrow \Delta$ ($x \mapsto \sigma_x$) is order preserving, because if $x < y$ in X , then $A^\circ(y) \subset A^\circ(x) \subseteq M$. By construction, we have a natural isomorphism $A(x) = (\sigma_x^\vee \cap M)_* \cong B(x)$. This proves that σ determines an isomorphism of monoid schemes, completing the proof of property 1).

Construction 4.2 shows that the condition in property 2) is necessary, since a morphism of fans sends the zero cone to the zero cone. Conversely, if $f(\eta) = \eta'$, then $f_\eta^\#$ induces a monoid map $A'(\eta') = M'_* \rightarrow M_* = A(\eta)$; since any such map sends units to units, it induces a group homomorphism $M' \rightarrow M$. Let $\phi : N \rightarrow N'$ be the \mathbb{Z} -linear dual of this map. Since for each $x \in X$, the map $f_x^\#$ is the restriction of $f_\eta^\#$, it follows that $f = X(\phi)$, as desired. \square

Remark 4.3.1. There are differing assertions in the literature related to Theorem 4.3. Using a different definition of ‘toric variety’ it is claimed in [6, Thm. 4.1] that any connected cancellative monoid scheme of finite type yields a toric variety, but not every such ‘toric variety’ is associated to a fan. For example, MSpec of the cusp monoid $C = \{t^2, t^3, \dots\}_*$ yields the cusp. In [17, 4.2], the hypothesis that A has no torsion is added and the assertion is changed to a toric variety associated to a fan, but the cusp monoid is also a counterexample to the assertion in *loc. cit.*

We conclude this section with a description of separated normal monoid schemes. If X is connected and cancellative, with minimal prime η , then M_η is a finitely generated abelian group. Therefore there is a non-canonical isomorphism $M_\eta \cong M \times T$, where M is a free abelian group and T is a finite torsion group.

Proposition 4.4. *Any separated, connected, normal monoid scheme of finite type decomposes as a cartesian product of monoid schemes*

$$X \cong (X, A) \times \mathrm{MSpec}(T_*),$$

where (X, A) is a toric monoid scheme and T is a finite abelian group.

Proof. If $\mathrm{MSpec}(A)$ is an affine open of X then A is a submonoid of $A_\eta = (M \times T)_*$; since A is normal, T_* is a submonoid of A . Every element of $A_\eta \setminus \{0\}$ can be written uniquely as a product mt with $m \in M$ and $t \in T$; since $t \in A$, if $mt \in A$ then $m \in A \cap M$. Thus if we set $B = A \cap M_*$ there is a decomposition $A \cong B \wedge T_*$. In other words, $\mathrm{MSpec}(A) \cong \mathrm{MSpec}(B) \times \mathrm{MSpec}(T_*)$

Since every localization of A has the form $A_{\mathfrak{p}} = B' \wedge T_*$, the affine open subsets of $\mathrm{MSpec}(A)$ are all of the form $\mathrm{MSpec}(B') \times \mathrm{MSpec}(T_*)$. Gluing these together gives the decomposition of X . \square

Note that the factorization in Proposition 4.4 is not unique; it depends upon the choice of isomorphism $A_\eta \cong (M \times T)_*$.

Corollary 4.5. *If $f : X \rightarrow X'$ is a morphism between separated and connected normal monoid schemes of finite type, inducing an isomorphism $f^* : \mathcal{A}'_{\eta'} \rightarrow \mathcal{A}_\eta$ of group completions, then f is isomorphic to the product of a morphism $X(\Delta) \rightarrow X(\Delta')$ of toric monoid schemes and an isomorphism $\mathrm{MSpec}(T_*) \rightarrow \mathrm{MSpec}(T'_*)$.*

Proof. By assumption, f maps the generic point η of X to the generic point η' of X' . Choosing a decomposition $\mathcal{A}_\eta \cong (M \times T)_*$, we have an implicitly defined decomposition $\mathcal{A}'_{\eta'} \cong (M' \times T')_*$. Then for each $x \in X$ the decompositions $\mathcal{A}_x \cong B_x \wedge T_*$, $\mathcal{A}'_{f(x)} \cong B'_{f(x)} \wedge T'_*$ of Proposition 4.4 satisfy $f^*(B'_{f(x)}) \subseteq B_x \subseteq M_*$. Therefore the map $\mathcal{A}'_{f(x)} \rightarrow \mathcal{A}_x$ factors as a product of $f^*(B'_{f(x)}) \subseteq B_x$ and $T_* \cong T'_*$, for each x . The result follows. \square

Remark 4.5.1. Not every morphism $(X, A) \times \mathrm{MSpec}(T_*) \rightarrow (X', A') \times \mathrm{MSpec}(T'_*)$ between connected normal monoid schemes of finite type will factor as a cartesian product of maps $(X, A) \rightarrow (X', A')$ and $\mathrm{MSpec}(T_*) \rightarrow \mathrm{MSpec}(T'_*)$. For example, this fails for the canonical $\mathrm{MSpec}((\mathbb{Z}/n)_*) \rightarrow \mathrm{MSpec}(\mathbb{Z}_*)$. However, such a map determines both a toric map $(X, A) \rightarrow (X', A')$ and a map $\mathrm{MSpec}(T_*) \rightarrow \mathrm{MSpec}(T'_*)$.

5. REALIZATIONS OF MONOID SCHEMES

In this section we fix a commutative ring k . If A is a monoid, the ring $k[A]$ gives rise to a scheme $\mathrm{Spec}(k[A])$, which is called the *k-realization* of $\mathrm{MSpec}(A)$. The affine spaces $\mathbb{A}_k^n = \mathrm{Spec}(k[t_1, \dots, t_n])$ of 1.2 are useful examples. The *k-realization* is a faithful functor from monoids to affine *k*-schemes; a monoid morphism $A \rightarrow B$ naturally gives rise to a morphism $\mathrm{Spec}(k[B]) \rightarrow \mathrm{Spec}(k[A])$.

Lemma 5.1. *If S is multiplicatively closed in A , $S^{-1}k[A] \cong k[S^{-1}A]$.*

Proof. The natural map $S^{-1}k[A] \rightarrow k[S^{-1}A]$ is onto, because each basis element $\frac{a}{s}$ of $k[S^{-1}A]$ is the image of a/s . To see that the map is injective, it suffices to show that if $f \in k[A]$ vanishes in $k[S^{-1}A]$ then $f = 0$ in $S^{-1}k[A]$. We may assume that $f = \sum \gamma_i a_i$ ($\gamma_i \in k$) and that all of the $a_i \in A$ map to the same nonzero element of $S^{-1}A$. In this case $\sum \gamma_i = 0$ and there is an $s \in S$ so that $sa_i = b$ is independent of i . But then $a_i = b/s$ for all i and $f = (\sum \gamma_i)(b/s) = 0$. \square

Remark 5.2. Let A be a monoid. Any affine open monoid subscheme of $\mathrm{MSpec}(A)$ has the form $\mathrm{MSpec}(A_{\mathfrak{p}})$ for some prime ideal \mathfrak{p} of A , by Lemma 2.4, and $A_{\mathfrak{p}} = A[1/s]$ by Lemma 1.3. Hence $\mathrm{Spec}(k[A_{\mathfrak{p}}]) \rightarrow \mathrm{Spec}(k[A])$ is an open immersion, by Lemma 5.1.

For the next definition, let us say that a point x in a monoid scheme X is *nice* if $U = \mathrm{MSpec}(\mathcal{A}_x)$ is open in X . Every closed point is nice by Lemma 2.4, but the points of Example 1.4 are not nice. If X is of finite type, then every point is nice by Lemma 1.5. The nice points $x \in X$ are a cofinal subset of the poset underlying X by Lemmas 1.3 and 2.4, because the closed points in any open subscheme are nice. If $x < y$ are two nice points then $\mathrm{Spec}(k[\mathcal{A}_x]) \rightarrow \mathrm{Spec}(k[\mathcal{A}_y])$ is an open immersion by Lemma 5.1. The criterion for separatedness in Lemma 3.6 uses nice points.

Definition 5.3. Given a commutative ring k and a separated monoid scheme (X, \mathcal{A}) , we define its *k-realization* X_k to be the scheme obtained by gluing the $U_k = \mathrm{Spec}(k[\mathcal{A}_x])$ together. That is,

$$X_k = \varinjlim_{x \in X} \mathrm{Spec}(k[\mathcal{A}_x]).$$

This colimit makes sense by [EGA 0_I, (4.1.7)] or [13, Ex. II.2.12], since it may taken over the nice points of X and the intersection of two affine open subschemes of a separated monoid scheme is affine open by Corollary 3.8.

If X is not separated, when the notion of gluing is more subtle; the usual technique is to use a variant of [EGA 0_I, (4.5)]. In our case, it is easiest to define the realization X_k using Theorem 5.8 below, which is based on [EGA 0_I, (4.5)].

The k -realization functor from monoid schemes to k -schemes is faithful, because it is so locally: $\mathrm{MSpec}(A)_k = \mathrm{Spec}(k[A])$. (This is clear if X is separated, and follows from Theorem 5.8 below if it is not separated.) It is not full because k -schemes such as \mathbb{A}_k^1 have many more endomorphisms than their monoidal counterparts.

The realization functor loses information, because distinct monoid schemes can have isomorphic realizations. This is a well known phenomenon even for toric varieties, where the additional data of a (faithful) torus action is needed to recover the fan.

Remark 5.3.1. Observe that $X_k = X_{\mathbb{Z}} \otimes_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} k$ for any monoid scheme X and commutative ring k . Those preferring the notion of a field with one element (\mathbb{F}_1) might prefer writing X_k as $X \times_{\mathrm{Spec} \mathbb{F}_1} \mathrm{Spec} k$ or just $X \times_{\mathbb{F}_1} k$.

Example 5.4. For a fan Δ and any field k , the variety $X(\Delta)_k$ is the usual toric k -variety associated to Δ . This is clear from Construction 4.2.

Example 5.5. Let T be a finite abelian group. The k -realization of $\mathrm{MSpec}(T_*)$ is the group scheme $\mathrm{Spec}(k[T])$. If $|T|$ is a unit (or nonzerodivisor) in k then $k[T]$ is reduced, but this fails if k is a field of characteristic $p > 0$ and T has p -torsion.

Lemma 5.6. *Let k be a field and A a cancellative monoid. Set $X = \mathrm{MSpec}(A)$ and $U = \mathrm{MSpec}(A^+)$.*

- (1) *If A^+ is torsionfree then $k[A]$ is a domain (i.e., X_k is integral).*
- (2) *Suppose that $\mathrm{char}(k) = 0$, or that $\mathrm{char}(k) = p > 0$ and A^+ has no p -torsion. Then $k[A^+]$ is normal and its subalgebra $k[A]$ is reduced. That is, U_k is normal and X_k is reduced.*

- (3) *Suppose that $\text{char}(k) = p > 0$ and A^+ has p -torsion. Then $k[A]$ is not reduced; $k[A]_{\text{red}} = k[B]$, where the monoid B is the quotient of A by the congruence relation that $a_1 \sim a_2$ if and only if $a_1^{p^e} = a_2^{p^e}$ for some $e \geq 0$.*

Proof. Since A is the union of its finitely generated submonoids A_i , and $k[A] = \cup k[A_i]$, we may assume that A is finitely generated. As noted before 4.4, we can write $A^+ = (M \times T)_*$ where M is a free abelian group and T is a finite torsion group. Since A is a submonoid of A^+ , $k[A]$ is a subalgebra of $k[A^+]$. If T is trivial, $k[A]$ is a subring of $k[M]$, which is manifestly a domain. If $\text{char}(k) = 0$ or if $\text{char}(k) = p$ and $p \nmid |T|$ then $k[T]$ is a product of fields and $k[A]$ is a subring of $k[A^+] = k[T][M]$, which is manifestly normal. Hence $k[A^+]$ and its subalgebra $k[A]$ are reduced.

Finally, suppose that $\text{char}(k) = p$ and that the p -torsion subgroup T_p of T is non-trivial. Since $k[T_p]_{\text{red}} = k$ and $k[A^+/T_p]$ is reduced by (2), we have $k[A^+]_{\text{red}} = k[A^+/T_p]$. If B is the image of $A \rightarrow A^+/T_p$ then $k[A]_{\text{red}}$ is the image $k[B]$ of $k[A] \rightarrow k[A^+/T_p]$. It is not hard to see that the congruence relation on A such that $B = A/\sim$ is given by the displayed formula. \square

Remark 5.6.1. If (X, A) is a cancellative monoid scheme of finite type, and k is a field of characteristic $p > 0$, Lemma 5.6(3) implies that $(X_k)_{\text{red}}$ is the k -realization of (X, B) , where $B = A/\sim$ is the quotient stalk functor of A defined as in 5.6(3).

We prove next that the k -realization functor (from monoid schemes to schemes) commutes with pullback. (See Corollary 5.10.) This is not trivial, since k -realization is not a right adjoint. We will show, however, that k -realization represents the Zariski sheafification of a pullback-preserving functor on affine k -schemes.

We saw in (1.8) that the restriction of the k -realization functor to the category of affine monoid schemes does commute with pullback, because it has a left adjoint (defined on the category of affine k -schemes) sending $\text{Spec}(R)$ to $\text{MSpec}(R, \times)$, where (R, \times) is the multiplicative monoid whose underlying pointed set is R . Thus if $X = \text{MSpec}(A)$ is an affine monoid, then the adjunction $\text{Hom}(\text{Spec}(R), X_k) \cong \text{Hom}_{\text{MSch}}(\text{MSpec}(R, \times), X)$ means that X_k represents the functor sending $\text{Spec } R$ to $\text{Hom}_{\text{MSch}}(\text{MSpec}(R, \times), X)$.

Definition 5.7. Let X be a monoid scheme and k a ring. Define a contravariant functor F_X from the category of affine k -schemes to sets to be the Zariski sheafification of the presheaf

$$\text{Spec } R \mapsto \text{Hom}_{\text{MSch}}(\text{MSpec}(R, \times), X).$$

If X is affine, the presheaf is already a sheaf since it is represented by X_k .

Recall from [7, VI-14] that a contravariant functor F from affine k -schemes to sets is represented by a unique k -scheme X if and only if F is a Zariski sheaf and F admits a covering by open subfunctors F_α , each of which is represented by an affine scheme U_α . If so, the representing scheme X is obtained by gluing the U_α together. Here, a subfunctor $F_\alpha \subseteq F$ is *open* if for every k -algebra R and every morphism $\text{Hom}(-, \text{Spec } R) \rightarrow F$, i.e., for every element of $F(\text{Spec } R)$, the pullback functor $F_\alpha \times_F \text{Hom}(-, \text{Spec } R)$ is represented by an open subscheme of $\text{Spec } R$. A collection of subfunctors $\{F_\alpha\}$ of F *covers* F if for every field extension L of k , we have $F(\text{Spec } L) = \bigcup_\alpha F_\alpha(\text{Spec } L)$.

Theorem 5.8. *The functor F_X is represented by a scheme X_k . If X is separated, X_k is the k -realization of X defined in 5.3.*

Proof. Suppose that $U = \text{MSpec}(A)$ is any affine monoid subscheme of X . Since sheafification preserves monomorphisms such as $\text{Hom}(-, U) \subseteq \text{Hom}(-, X)$, F_U is a subfunctor of F_X . If Λ is a local k -algebra and $L = \text{Spec}(\Lambda)$ then

$$(5.8a) \quad F_X(L) = \text{Hom}_{\text{MSch}}(\text{MSpec}(\Lambda, \times), X).$$

Since $\text{MSpec}(\Lambda, \times)$ has a unique point, each map $\text{MSpec}(\Lambda, \times) \rightarrow X$ factors through an affine open submonoid U . Therefore F_X is covered by the collection of subfunctors F_U , as U ranges over all affine open monoid subschemes of X . We will show that the F_U are open subfunctors of F_X ; we have seen that each F_U is represented by the affine scheme U_k . By [7, VI-14], this will prove that F_X is representable by the k -scheme which is obtained by gluing the affine schemes U_k . Since this is the definition of the k -realization X_k (see Definition 5.3), this will prove Theorem 5.8.

Fix an affine open monoid subscheme $U = \text{MSpec}(A)$. To prove that F_U is open, fix a k -algebra R and consider a morphism $\text{Hom}(-, \text{Spec } R) \rightarrow F_X$ and its corresponding element $\phi \in F_X(\text{Spec } R)$. We have to show that the pullback $G = F_U \times_{F_X} \text{Hom}(-, \text{Spec } R)$ is represented by an open subscheme V of $\text{Spec}(R)$. Since F_X is a sheaf, $\text{Spec}(R)$ has an affine open covering $\{\text{Spec } R[1/s] \mid s \in \mathcal{S}\}$ such that the restriction of ϕ to $F_X(\text{Spec } R[1/s])$ is represented by a morphism $\phi_s : \text{MSpec}(R[1/s], \times) \rightarrow X$ of monoid schemes. By Observation 5.8.1 below, there are continuous maps

$$\text{Spec}(R[1/s]) \hookrightarrow \text{MSpec}(R[1/s], \times) \xrightarrow{\phi_s} X.$$

Let V'_s denote the inverse image of U under ϕ_s and let V_s denote the open subspace $V'_s \cap \text{Spec}(R[1/s])$; we regard V_s as an open subscheme of $\text{Spec}(R[1/s])$ and hence of $\text{Spec}(R)$. We claim that G is represented by the open subscheme $V = \cup V_s$ of $\text{Spec } R$. To prove our claim, it suffices to consider a local k -scheme $L = \text{Spec}(\Lambda)$ and prove that $G(L) = \text{Hom}(L, V)$ as subsets of $\text{Hom}(L, \text{Spec } R)$. Since L is local, we have $F_U(L) = \text{Hom}(A, (\Lambda, \times))$, and (5.8a) holds for X . Thus $G(L)$ is the set of all $f : L \rightarrow \text{Spec } R$ such that

$$\text{MSpec}(\Lambda, \times) \xrightarrow{f^\times} \text{MSpec}(R, \times) \xrightarrow{\phi} X$$

maps the closed point \mathfrak{m} of L into U . If the image of f lies in V , \mathfrak{m} lands in some V_s and hence f^\times maps the closed point (\mathfrak{m}, \times) of $\text{MSpec}(\Lambda, \times)$ into V'_s . It follows that $\phi f^\times(\mathfrak{m}, \times) \in U$, i.e. $f \in G(L)$. Thus $\text{Hom}(L, V) \subseteq G(L)$.

Conversely, if $f : L \rightarrow \text{Spec}(R)$ is in $G(L)$ then f factors through some $f_s : L \rightarrow \text{Spec}(R[1/s])$ and $\phi_s f_s^\times$ maps the closed point (\mathfrak{m}, \times) of $\text{MSpec}(\Lambda, \times)$ to a point in the subset U of X , so $f_s(\mathfrak{m}) \in V_s$. But since L is local, this implies that $f_s(L) \subseteq V_s$. The desired equality $G(L) = \text{Hom}(L, V)$ follows. \square

Observation 5.8.1. Let R be any commutative ring, and (R, \times) its underlying multiplicative monoid. If \mathfrak{p} is a prime ideal of the ring R , then (\mathfrak{p}, \times) is a prime ideal of the monoid (R, \times) . The resulting inclusion $\text{Spec}(R) \hookrightarrow \text{MSpec}(R, \times)$ is continuous because if $s \in R$ the open subspace $D(s)$ of $\text{MSpec}(R)$ intersects $\text{Spec}(R)$ in the open subspace $\{\mathfrak{p} \subset R \mid s \notin \mathfrak{p}\}$. If R is local, the maximal ideal \mathfrak{m} of R maps to the maximal prime (\mathfrak{m}, \times) of $\text{MSpec}(R, \times)$.

Definition 5.9. Given a commutative ring k and a non-separated monoid scheme (X, \mathcal{A}) , we define its k -realization X_k to be the scheme representing F_X . As pointed out in Theorem 5.8, this agrees with the definition of X_k for separated monoid schemes.

Corollary 5.10. *The k -realization functor $X \mapsto X_k$ preserves arbitrary limits (when they exist). In particular, it preserves pullbacks.*

Proof. Suppose that $\{X_i, i \in I\}$ is a diagram of monoid schemes and that its limit X exists in the category of monoid schemes. It suffices to prove the canonical map

$$F_X \rightarrow F = \varprojlim F_{X_i}$$

is an isomorphism of sheaves on the category of affine k -schemes. Recall that the limit of a diagram of sheaves exists and coincides with the limit as presheaves. That is, we have $F(\text{Spec } R) = \varprojlim F_{X_i}(\text{Spec } R)$. When R is local, we have $F_X(\text{Spec } R) = \text{Hom}(\text{MSpec}(R, \times), X)$ and also

$$F(\text{Spec } R) = \varprojlim \text{Hom}(\text{MSpec}(R, \times), X_i) \cong \text{Hom}(\text{MSpec}(R, \times), X),$$

where the second isomorphism holds since $X = \varprojlim_i X_i$. Since the sheaf map $F_X \rightarrow F$ is an isomorphism on all local rings, it is an isomorphism of sheaves. \square

Proposition 5.11. *If $(Y, \mathcal{B}) \xrightarrow{f} (X, \mathcal{A})$ is a closed immersion of monoid schemes then $f_k : Y_k \rightarrow X_k$ is a closed immersion of schemes for all rings k .*

Proof. If $V \subseteq X$ is an affine open subscheme, then by Lemma 2.4 there exists $x \in X$ such that $V = \text{MSpec}(\mathcal{A}_x)$. If $V \cap Y = \emptyset$ then $V_k \cap Y_k = (V \cap Y)_k = \emptyset$. Otherwise $V \cap Y = \text{MSpec}(\mathcal{B}_y)$ for some y , and $\mathcal{A}_x \rightarrow \mathcal{B}_y$ is onto, by Definition 2.5. Since k -realization preserves pullbacks by Corollary 5.10, we have $f_k^{-1}(V_k) = f^{-1}(V)_k = \text{Spec}(k[\mathcal{B}_y])$ and the restriction $f_k^{-1}(V_k) \rightarrow V_k = \text{Spec } k[\mathcal{A}_x]$ of f is induced by the surjection $k[\mathcal{A}_x] \rightarrow k[\mathcal{B}_y]$. This proves that the restriction $Y_k \cap V_k \rightarrow V_k$ of f_k is a closed immersion. Since V is an arbitrary affine open subscheme of X , this proves that $Y_k \rightarrow X_k$ is a closed immersion. \square

A partial converse of this proposition is true.

Lemma 5.12. *Suppose $i : Y \rightarrow X$ is a morphism of monoid schemes such that the underlying map of topological spaces induces a homeomorphism onto its image. For any ring k , if $i_k : Y_k \rightarrow X_k$ is a closed immersion, then i is a closed immersion of monoid schemes.*

Proof. It suffices to prove that if $X = \text{MSpec}(A)$ is affine, then Y is also affine and the associated map of monoids is surjective. Let \mathcal{B} be the sheaf of monoids for the scheme Y and set $B = \Gamma(Y, \mathcal{B})$. The map $Y \rightarrow X$ factors as

$$Y \rightarrow \text{MSpec } B \rightarrow \text{MSpec } A.$$

Upon taking k -realizations we have $Y_k = \text{Spec}(R)$ and the map induced by $Y_k \rightarrow X_k$ is a surjection: $k[A] \twoheadrightarrow R$. Since this surjection factors through the map $k[A] \rightarrow k[B]$, which is induced by a map of monoids $A \rightarrow B$, we see that $k[B] \twoheadrightarrow R$ is surjection as well. Let $Y = \cup_j W_j$ be a covering by open affine subschemes, with $W_j = \text{MSpec } B_j$. Then the map $B \rightarrow \prod_j B_j$ is injective and hence so is the map $k[B] \rightarrow \prod_j k[B_j]$. Since the latter map factors as $k[B] \rightarrow R \rightarrow \prod_j k[B_j]$, it follows that $k[B] \xrightarrow{\cong} R$ is an isomorphism. That is, the k -realization of

$$Y \rightarrow \text{MSpec}(B)$$

is an isomorphism. Moreover, since $k[A] \twoheadrightarrow k[B]$ is onto, so is the map $A \rightarrow B$, and hence $\text{MSpec}(B) \rightarrow X$ is a closed immersion. In particular, the map of underlying topological spaces is a homeomorphism onto its image. It follows from this (and

our assumption) that the map of topological spaces underlying $Y \rightarrow \mathrm{MSpec}(B)$ is a homeomorphism onto its image.

We may thus assume that the k -realization $Y_k \rightarrow X_k = \mathrm{Spec}(k[A])$ is an isomorphism. We next claim that $Y \rightarrow X$ is a surjection on points, and hence (by our assumption that Y is homeomorphic to its image) a homeomorphism on underlying topological spaces. To see this, fix a point $\mathfrak{p} \in X$ and consider the monoid map $i_{\mathfrak{p}} : A \rightarrow S^0 = \{0, 1\}$ sending \mathfrak{p} to 0 and $A \setminus \mathfrak{p}$ to 1. Let Y' denote the pullback of $Y \rightarrow X$ along the map $\mathrm{MSpec} S^0 \xrightarrow{i_{\mathfrak{p}}} X$. By Corollary 5.10, the map $Y'_k \rightarrow (\mathrm{MSpec} S^0)_k = \mathrm{Spec} k$ is an isomorphism, so in particular Y' is non-empty. By Proposition 3.1, it follows that $Y \rightarrow X$ is onto.

Since X has a unique maximal point, so does Y . By Lemma 2.4, Y is affine. Since $Y_k \cong \mathrm{Spec}(k[A])$ we conclude that $Y \cong X$. \square

Proposition 5.13. *For any ring k and morphism of monoid schemes $f : Y \rightarrow X$, the map f is a separated morphism of monoid schemes if and only if its k -realization $f_k : Y_k \rightarrow X_k$ is a separated morphism of schemes.*

Proof. One direction is immediate from Corollary 5.10 and Proposition 5.11.

Assume f_k is separated. Since the underlying topological space of $Y \times_X Y$ is given by the pullback in the category of topological spaces, it follows that $Y \xrightarrow{\Delta} Y \times_X Y$ is a homeomorphism onto its image. (Observe that $Y \rightarrow \Delta(Y)$ and $\Delta(Y) \xrightarrow{\pi_1} Y$ are continuous, and both compositions are the identity, where $\Delta(Y) \subset Y \times_X Y$ is given the subspace topology.) Since Δ_k is a closed immersion, Lemma 5.12 applies to finish the proof. \square

6. NORMAL AND SMOOTH MONOID SCHEMES

Throughout this section, k denotes a field of characteristic $p \geq 0$. The normalization A_{nor} of a cancellative monoid A is defined in Definition 1.6; since $(A_{\mathfrak{p}})_{\mathrm{nor}} = (A_{\mathrm{nor}})_{\mathfrak{p}_{\mathrm{nor}}}$, it makes sense to talk about the normalization of any cancellative monoid scheme.

The k -realization of X cannot be normal unless X_k is reduced. Lemma 5.6 shows that $k[A]$ is reduced unless $p > 0$ and A^+ has p -torsion, in which case $k[A]_{\mathrm{red}}$ is $k[B]$, where B is a particular quotient of A , described there.

Proposition 6.1. *Let $X = (X, A)$ be a cancellative monoid scheme of finite type such that its k -realization X_k is a reduced scheme. Then*

- (1) *the normalization of X_k is the k -realization of (X, A_{nor}) .*
- (2) *if X is normal, connected and separated, there is a decomposition*

$$X_k = X'_k \times_k \mathrm{Spec} k[T]$$

where X'_k is a toric k -variety and $k[T]$ is a finite product of fields.

As in Remark 4.5.1, the decomposition in Proposition 6.1(2) is not natural in X .

Proof. Part (2) is immediate from Proposition 4.4 and Corollary 5.10.

Since the normalization of a reduced scheme is the scheme constructed by patching together the normalizations of an affine cover, we may assume that X is affine, i.e., $X = \mathrm{MSpec}(A)$. Since $k[A_{\mathrm{nor}}]$ is integral over $k[A]$, we may assume that $A = A_{\mathrm{nor}}$. In this situation, where A is a normal monoid of finite type, Proposition 4.4 states that $A \cong A' \wedge T_*$ where A' is torsionfree and T is a finite abelian group. Since X_k is reduced, we know from Lemma 5.6(3) and Example 5.5 that T has no

p -torsion and $k[T]$ is a finite product of fields. Since $k[A] = k[T][A']$, we are reduced to the case in which A is normal and torsionfree, i.e., $X = \text{MSpec}(A)$ is an affine toric monoid scheme. By Theorem 4.3, X is associated to a fan Δ ; by Example 5.4, X_k is the toric variety associated to Δ , and in particular X_k is normal. \square

Remark 6.1.1. It is possible to give an elementary proof of this result using that if A is a torsionfree normal monoid then $k[A]$ is integrally closed; see [12, 12.6].

Finite morphisms. We will need to know that the normalization of a monoid scheme is a finite morphism, at least when X is of finite type.

We say that a morphism of monoid schemes $f : Y \rightarrow X$ is *affine* if X can be covered by affine open subschemes $U_i = \text{MSpec}(A_i)$ such that $f^{-1}(U_i)$ is affine. Equivalently, f is affine if $f^{-1}(U)$ is affine for every affine open subscheme $U \subset X$.

Definition 6.2. Let $f : Y \rightarrow X$ be a morphism of monoid schemes. We say that f is *finite* if it is affine and $\mathcal{A}_X(U) \rightarrow \mathcal{A}_Y(f^{-1}(U))$ is finite for every affine subscheme $U \subset X$. We say that f is *integral* if it is affine and $\mathcal{A}_X(U) \rightarrow \mathcal{A}_Y(f^{-1}(U))$ is integral for every affine subscheme $U \subset X$.

If X is cancellative, its normalization $X_{\text{nor}} \rightarrow X$ is an integral morphism. To see this, we may assume $X = \text{MSpec}(A)$ is affine so that $X_{\text{nor}} \rightarrow X$ is given by $A \hookrightarrow A_{\text{nor}}$, where the normalization A_{nor} is integral by Definition 1.6. We now show that if X is also of finite type, then $X_{\text{nor}} \rightarrow X$ is finite.

Proposition 6.3. *If X is a cancellative monoid scheme of finite type, the normalization $X_{\text{nor}} \rightarrow X$ is a finite morphism.*

Proof. It suffices to show that if A is a cancellative monoid of finite type then $A \rightarrow A_{\text{nor}}$ is finite. Since A_{nor} is integral over A it suffices by Lemma 1.7(i) to show that A_{nor} is of finite type. Because the group completion A^+ is finitely generated, it has the form $(M \times T)_*$ where T is a finite abelian group and M is free abelian. Since $A[T] = \bigcup At$ is finite over A , we may replace A by $A[T]$ to assume that $T \subset A$. As in the proof of Proposition 4.4, this implies that $A = B \wedge T_*$ where $B = A \cap M_*$ is a finitely generated submonoid of M . If β is the rational convex polyhedral cone of $M_{\mathbb{R}}$ spanned by the generators of B , B_{nor} is $(\beta \cap M)_*$. By Gordon's Lemma [8], B_{nor} is finitely generated. A fortiori, $A_{\text{nor}} = B_{\text{nor}} \wedge T_*$ is finitely generated. \square

Smoothness.

Definition 6.4. Let p be a prime. A separated monoid scheme of finite type is *p -smooth* if each stalk (equivalently, each maximal stalk) is the smash product $S \wedge T_*$, where S is the product of a (pointed) free abelian group and a (pointed) free abelian monoid, and T is a finite abelian group having no p -torsion. A separated monoid scheme is *0-smooth* if each stalk has the form $S \wedge T_*$ with T an arbitrary finite abelian group.

We will say that X is *smooth* if it is p -smooth for all p , i.e., if each stalk is the product of a free group of finite rank and a free monoid of finite rank.

A cone in a fan (N, Δ) is said to be *nonsingular* if it is spanned by part of a \mathbb{Z} -basis for the lattice N , in which case each monoid $\sigma^\vee \cap M$ is the product of a free abelian group and a free abelian monoid. A fan is said to be nonsingular if all its cones are nonsingular. Recall from [8, 2.1] that the toric variety associated to a fan is smooth if and only if each of its cones is nonsingular. Therefore the following result is an immediate corollary of Proposition 6.1 and Lemma 5.6.

Proposition 6.5. *Let $X = (X, A)$ be a separated cancellative monoid scheme of finite type. Its k -realization X_k is smooth over a field k of characteristic $p \geq 0$ if and only if X is p -smooth. If X is connected and p -smooth then, under the decomposition*

$$X = (X, A') \times \mathrm{MSpec}(T)$$

of Proposition 6.1, the fan underlying (X, A') is nonsingular.

Example 6.5.1. The hypothesis that X be cancellative is necessary. For example, consider the monoid $A = \langle t, e \mid e = e^2 = te \rangle$, which has $k[A] \cong k[x] \times k$. Thus $X = \mathrm{MSpec}(A)$ is not p -smooth but X_k is smooth for every k .

7. MProj AND BLOW-UPS

An \mathbb{N} -grading of a monoid A is a pointed set decomposition

$$A = \bigvee_{i=0}^{\infty} A_i$$

such that $A_i \cdot A_j \subseteq A_{i+j}$; \mathbb{Z} -gradings are defined similarly. For each nonzero a in A , let $|a|$ denote the unique i such that $a \in A_i$. For every multiplicative set S , the localization $S^{-1}A$ is \mathbb{Z} -graded by $|a/s| = |a| - |s|$. For example, if $s \in A_i$ is non-zero we have

$$A[\frac{1}{s}]_0 = \left\{ \frac{a}{s^n} \mid |a| = |s^n| = ni, n \geq 0 \right\} \cup \{0\}.$$

Let $A_{\geq 1}$ denote the ideal $\bigvee_{i \geq 1} A_i = \{a \mid |a| > 0\} \cup \{0\}$, so that $A/A_{\geq 1} \cong A_0$; the image of the corresponding map $\mathrm{MSpec}(A_0) \rightarrow \mathrm{MSpec}(A)$ consists of the prime ideals of A containing $A_{\geq 1}$.

Definition 7.1. If A is an \mathbb{N} -graded monoid, we define $\mathrm{MProj}(A) = (X, \mathcal{B})$ to be the following monoid scheme. The underlying topological space is $X = \mathrm{MSpec}(A) \setminus \mathrm{MSpec}(A_0)$ — i.e., the open subspace of those prime ideals of A that do not contain $A_{\geq 1}$. The stalks of \mathcal{B} on X are defined by sending $\mathfrak{p} \in \mathrm{MSpec}(A) \setminus \mathrm{MSpec}(A_0)$ to $\mathcal{B}_{\mathfrak{p}} = (A_{\mathfrak{p}})_0$, the degree zero part of $A_{\mathfrak{p}}$. One checks that if $\mathrm{MSpec}(A_{\mathfrak{p}}) \subset X$ is open, that is, if $A_{\mathfrak{p}} = A[1/s]$ for some $s \in A_{\geq 1}$, then the map $\mathrm{MSpec}(A_{\mathfrak{p}}) \rightarrow \mathrm{MSpec}(A_{\mathfrak{p}})_0$ is a homeomorphism. It follows that $\mathrm{MProj}(A)$ is covered by the affine open subschemes $D_+(s) = \mathrm{MSpec}(A[\frac{1}{s}]_0)$ where $s \in A_{\geq 1}$, and moreover, every affine open subscheme is of this form. Thus $\mathrm{MProj}(A)$ is a monoid scheme of finite type whenever A is a finitely generated monoid. The maps $A_0 \rightarrow (A_{\mathfrak{p}})_0$ induce a structure morphism $\mathrm{MProj}(A) \rightarrow \mathrm{MSpec}(A_0)$.

Remark 7.1.1. The k -realization of A is the graded ring $k[A]$, and $k[A[\frac{1}{s}]_0]$ is the degree 0 part of the ring $k[A][\frac{1}{s}]$, so the k -realization of $\mathrm{MProj}(A)$ is $\mathrm{Proj}(k[A])$.

Observation 7.1.2. The construction is natural in A for maps $A \rightarrow A'$ of graded monoids such that $A' = A \cdot A'_0$. For such maps there is a canonical morphism $\mathrm{MProj}(A') \rightarrow \mathrm{MProj}(A)$ induced by the restriction of $\mathrm{MSpec}(A') \rightarrow \mathrm{MSpec}(A)$. If $s \in A_{\geq 1}$, the affine open $\mathrm{MSpec}(A'[\frac{1}{s}]_0)$ maps to the affine open $\mathrm{MSpec}(A[\frac{1}{s}]_0)$.

If $S \subset A_0$ is multiplicatively closed, $S^{-1}A$ is graded and $\mathrm{MProj}(S^{-1}A)$ is $\mathrm{MProj}(A) \times_{\mathrm{MSpec}(A_0)} \mathrm{MSpec}(S^{-1}A_0)$. It follows that this construction may be sheafified: for any monoid scheme (X, \mathcal{A}_0) and any sheaf \mathcal{A} of graded monoids on X with $(\mathcal{A}_x)_0 = (\mathcal{A}_0)_x$ for all $x \in X$, there is a monoid scheme $\mathrm{MProj}(\mathcal{A})$ over X whose stalk at each x is $\mathrm{MProj}(\mathcal{A}_x)$. Moreover, if $f : (X', \mathcal{A}'_0) \rightarrow (X, \mathcal{A}_0)$ is a morphism of monoid schemes, equipped with sheaves \mathcal{A}' and \mathcal{A} of graded monoids as

above, any graded extension $f^{-1}\mathcal{A} \rightarrow \mathcal{A}'$ of $f^{-1}\mathcal{A}_0 \rightarrow \mathcal{A}'_0$ such that $\mathcal{A}' = f^{-1}\mathcal{A} \cdot \mathcal{A}'_0$ induces a canonical morphism $\text{MProj}(\mathcal{A}') \rightarrow \text{MProj}(\mathcal{A})$ over f .

Lemma 7.2. *If $f : A \rightarrow B$ is a surjective homomorphism of graded monoids, then the induced map $\text{MProj}(B) \rightarrow \text{MProj}(A)$ is a closed immersion.*

Proof. As noted above, any affine open subscheme $U \subset \text{MProj}(A)$ is of the form $U = \text{MSpec}(A[\frac{1}{s}]_0)$ for some $s \in A_{\geq 1}$. But $U \cap \text{MProj}(B) = \text{MSpec}(B[\frac{1}{f(s)}]_0)$ is affine, so we are in the case of Lemma 2.6. \square

Projective monoid schemes. For a monoid A and indeterminates T_0, \dots, T_n , let $A[T_0, \dots, T_n]$ denote the monoid freely generated by A and the T_i . It is a graded monoid, where each element of A has degree 0 and each T_i has degree 1, and we define \mathbb{P}_A^n to be $\text{MProj}(A[T_0, \dots, T_n])$. More generally, for any monoid scheme $X = (X, \mathcal{A})$, define \mathbb{P}_X^n to be $\text{MProj}(\mathcal{B})$ where \mathcal{B} is the sheaf of graded monoids on X defined by sending an open subset U to $\mathcal{A}(U)[T_0, \dots, T_n]$. In other words, \mathbb{P}_X^n is defined by patching together the monoid schemes of the form \mathbb{P}_A^n as $\text{MSpec}(A)$ ranges over affine open subschemes of X . If X has finite type, so does \mathbb{P}_X^n .

A morphism of monoid schemes $Y \rightarrow X$ is *projective* if, locally on X , it factors as a closed immersion $Y \rightarrow \mathbb{P}_X^n$ for some n followed by the projection $\mathbb{P}_X^n \rightarrow X$.

Lemma 7.3. *Projective morphisms are separated.*

Although this follows from Proposition 5.13, we give an elementary proof here.

Proof. Since closed immersions are separated by Lemma 3.4, it suffices to show that the morphisms $\mathbb{P}_X^n \rightarrow X$ are separated. We may assume that $X = \text{MSpec}(A)$, so that $\mathbb{P}_X^n = \text{MProj}(A[T_0, \dots, T_n])$. By Definition 7.1, points of \mathbb{P}_X^n correspond to prime ideals in $A[T_0, \dots, T_n]$ not containing $\{T_0, \dots, T_n\}$. By Lemma 1.9 and Example 1.2, every such prime ideal has the form $P_{S, \mathfrak{p}} = A \wedge \langle S \rangle \cup \mathfrak{p}[T_0, \dots, T_n]$ for a unique prime ideal \mathfrak{p} of A and a unique proper subset S of $\{T_0, \dots, T_n\}$, and the projection to $\text{MSpec}(A)$ sends this point to \mathfrak{p} . According to Lemma 3.6, it suffices to observe that for every $P_{S, \mathfrak{p}}$ and $P_{S', \mathfrak{p}}$ the prime $P_{S \cap S', \mathfrak{p}}$ is a unique lower bound. (The surjectivity condition of Lemma 3.6 is easy, and left to the reader.) \square

Example 7.3.1. If B is a finitely generated graded monoid, then $\text{MProj}(B) \rightarrow \text{MSpec}(B_0)$ is projective and hence separated by Lemma 7.3. Indeed, this is a particular case of Lemma 7.2, since B is a quotient of some $B_0[T_0, \dots, T_n]$.

Blow-ups.

Given a monoid A and an ideal I , we consider the graded monoid $A \vee I \vee I^2 \vee \dots$, where I^n has degree n . It is useful to introduce a variable t , and rewrite this as

$$A[It] = \bigvee_{n \geq 0} I^n t^n \subseteq A \wedge F_1.$$

If S is multiplicatively closed in A , then $S^{-1}(A[It]) \cong (S^{-1}A)[S^{-1}It]$. It follows that if \mathcal{I} is a quasi-coherent sheaf of ideals in a monoid scheme (X, \mathcal{A}) then there is a monoid scheme $\text{MProj}(\mathcal{A}[\mathcal{I}t])$ over (X, \mathcal{A}) obtained by patching the $\text{MProj}(A[It])$ in the evident manner.

Definition 7.4. The *blow-up* of $\text{MSpec}(A)$ along an equivariant closed subscheme $C = \text{MSpec}(A/I)$ is defined to be $\text{MProj}(A[It])$, together with the structure morphism $\text{MProj}(A[It]) \rightarrow \text{MSpec}(A)$. More generally, if $X = (X, \mathcal{A})$ is a monoid scheme and $C \subseteq X$ is a equivariant closed subscheme, given by a quasi-coherent sheaf of ideals \mathcal{I} , we define the blow-up of X along C to be the monoid scheme

$X_C = \text{MProj}(\mathcal{A}[\mathcal{I}t])$. Since $\text{MProj}(A[t]) \cong \text{MSpec}(A)$, it follows that for $U = X \setminus C$ we have $X_C \times_X U \cong U$.

This construction is natural in the pair (A, I) in the following sense. If $A \rightarrow B$ is a morphism of monoids, I is an ideal of A and $J = IB$, there is a canonical graded morphism $A[It] \rightarrow B[Jt]$ satisfying the hypotheses of 7.1.2. Hence there is a morphism $\text{MProj}(B[Jt]) \rightarrow \text{MProj}(A[It])$ of the blow-ups over $\text{MSpec}(B) \rightarrow \text{MSpec}(A)$. More generally, if $f : X' \rightarrow X$ is a morphism of monoid schemes, \mathcal{I} is a quasi-coherent sheaf of ideals on X and $\mathcal{J} = f^{-1}\mathcal{I} \cdot \mathcal{A}'$, then the morphism $f^{-1}\mathcal{A}[It] \rightarrow \mathcal{A}'[\mathcal{J}t]$ induces a canonical morphism $\text{MProj}(\mathcal{A}'[\mathcal{J}t]) \rightarrow \text{MProj}(\mathcal{A}[It])$ over f , described in 7.1.2.

Remark 7.4.1. The blow-up of X along a quasi-coherent sheaf of ideals \mathcal{I} is projective provided \mathcal{I} is given locally on X by finitely generated ideals, by 7.3.1. For example, if X has finite type then the blowup of X along any quasi-coherent sheaf of ideals is projective.

Example 7.5. Suppose N is a free abelian group with basis $\{v_1, \dots, v_n\}$, and $\{x_1, \dots, x_n\}$ is the dual basis of M . If σ is the cone in $N_{\mathbb{R}}$ generated by $\{v_1, \dots, v_d\}$, the corresponding affine monoid scheme is $X(\sigma) = \text{MSpec}(A)$, where A is generated by x_1, \dots, x_d and $\{x_i, x_i^{-1} \mid i > d\}$. Then the blow-up of $X(\sigma)$ along the ideal generated by x_1, \dots, x_d is the toric monoid scheme $X(\Delta)$, where Δ is the subdivision of the fan $\{\sigma\}$ given by insertion of the ray spanned by $v_0 = v_1 + \dots + v_d$. To see this, it suffices to copy the corresponding argument for toric varieties given in [8, p. 41].

Example 7.5.1. If C is an equivariant closed subscheme of X , defined by a quasi-coherent sheaf of ideals \mathcal{I} , and $f : X' \rightarrow X$ is a morphism, then by 3.1 the pullback $C' = C \times_X X'$ is defined by the quasi-coherent sheaf of ideals $\mathcal{J} = \mathcal{I}A'$. By 7.4 there is a canonical morphism over f , from the blow-up $X_{C'}$ to the blow-up X_C .

Lemma 7.6. *Let $f : X' \rightarrow X$ be a finite morphism of monoid schemes (6.2). Let C be an equivariant closed subscheme of X , X_C the blow-up along C , and $X_{C'}$ the blow-up of X' along the pullback $C' = C \times_X X'$. Then $\tilde{f} : X_{C'} \rightarrow X_C$ is a finite morphism.*

Proof. We may assume that X , and hence X' , is affine. Then f is induced by a map $A \rightarrow A'$, C is defined by an ideal $I \subset A$ and C' is defined by $J = IA'$. Moreover because f is assumed finite, there are elements $c_1, \dots, c_r \in B$ such that $B = \bigcup_i Ac_i$. If a_0, \dots, a_n generate I , and b_0, \dots, b_n are their images in B , then \tilde{f} restricts to maps $D_+(b_i) \rightarrow D_+(a_i)$ induced by the monoid maps $A_i = A[a_0/a_i, \dots, a_n/a_i] \rightarrow B_i = B[b_0/b_i, \dots, b_n/b_i]$. By inspection, $B_i = \bigcup_{j=1}^n A_i c_j$. \square

Proposition 7.7. *Let C be an equivariant closed subscheme of a monoid scheme X of finite type. Then for any field k the blow-up of X_k along C_k coincides with the k -realization of the blow-up of X along C .*

Proof. It suffices to consider the case $X = \text{MSpec}(A)$, $C = \text{MSpec}(A/I)$. In this case $S = k[A[It]]$ is the usual Rees ring $k[A][Jt]$, $J = k[I]$. Since the blowing-up of $X_k = \text{Spec}(k[A])$ along $C_k = \text{Spec}(k[A/I])$ is $\text{Proj}(S)$, we have the desired identification $\text{Proj}(S) = \text{Proj}(k[A[It]]) = \text{MProj}(A[It])_k$. \square

We conclude this section by observing that blow-ups of monoid schemes satisfy a universal property analogue to that for blow-ups of usual schemes. To state it, we need some notation. We define a *principal invertible* ideal of A to be an ideal I such that there is an $x \in I$ such that the map $A \xrightarrow{x} I$ ($a \mapsto ax$) is a bijection. If I is a principal invertible ideal of A then $\text{MProj}(A[It]) \cong \text{MSpec}(A)$.

A quasi-coherent sheaf of ideals of a monoid scheme X is said to be *invertible* if X can be covered by affine open subschemes U such that $\mathcal{I}(U)$ is a principal invertible ideal of $\mathcal{A}_X(U)$. If (X, \mathcal{A}) is a monoid scheme and $\mathcal{I} \subset \mathcal{A}$ is a quasi-coherent sheaf of ideals (see Definition 2.7), we say that a morphism $f : Y \rightarrow X$ *inverts* \mathcal{I} if $f^{-1}\mathcal{I} \cdot \mathcal{B}$ is an invertible sheaf on Y .

Proposition 7.8. *Let X be a monoid scheme of finite type, Z an equivariant closed subscheme defined by a quasi-coherent sheaf of ideals \mathcal{I} , and $\pi : \tilde{X} \rightarrow X$ the blow-up of X along Z . Then π inverts \mathcal{I} and is universal with this property in the sense that if Y is of finite type and $f : Y \rightarrow X$ inverts \mathcal{I} , then the dotted arrow in the diagram below exists and is unique.*

$$\begin{array}{ccc} Y & \cdots \cdots \cdots \rightarrow & \tilde{X} \\ & \searrow f & \downarrow \pi \\ & & X \end{array}$$

Proof. We may assume that $X = \text{MSpec}(A)$ for some finitely generated monoid A , that \mathcal{I} corresponds to an ideal I of A , and that $\tilde{X} = \text{MProj}(A[It])$. The map π inverts I because the restriction of $\pi^{-1}\mathcal{I}$ to $D_+(s)$ is generated by s for each $s \in I$. Let \mathcal{B} be the structure sheaf of Y , and write \mathcal{J} for the sheaf of ideals $f^{-1}\mathcal{I} \cdot \mathcal{B}$. By Example 7.5.1, there is a unique morphism from the blow-up $\tilde{Y} = \text{MProj}(\mathcal{B}[\mathcal{J}t])$ to \tilde{X} over f . By assumption, \mathcal{J} is an invertible sheaf. Hence $\tilde{Y} \rightarrow Y$ is an isomorphism, because locally \mathcal{J} is a principal invertible ideal J of B and $\text{MProj}(B[Jt]) \cong \text{MSpec}(B)$. \square

8. PROPER MORPHISMS

A monoid V is called a *valuation monoid* if V is cancellative and for every non-zero element $\alpha \in V^+$, either α or $\frac{1}{\alpha}$ belongs to V . For example, if R is a valuation ring, then the underlying multiplicative monoid (R, \times) is a valuation monoid. Also, the free pointed monoid on one generator is a valuation monoid. Given a valuation monoid V , the monoid $V \wedge M_*$ is also a valuation monoid for any abelian group M . For example, the monoid $\langle y_1^{\pm 1}, \dots, y_n^{\pm 1}, x \rangle$ is a valuation monoid.

Given a valuation monoid V , the units $U(V)$ are a subgroup of $V^+ \setminus 0$ and the quotient group $(V^+ \setminus 0)/U(V)$ is a totally ordered abelian group with the total ordering defined by $x \geq y$ if and only if $\frac{x}{y}$ belongs to the image of $V \setminus 0$. To conform to usual custom, we convert the group law for $(V^+ \setminus 0)/U(V)$ into $+$. We also adjoin a base point, written ∞ , to obtain the totally ordered pointed (additive) monoid

$$\Gamma := ((V^+ \setminus 0)/U(V))_*.$$

The total ordering is such that $\gamma < \infty$ for all $\gamma \neq \infty$. We call Γ the *value monoid* of the valuation monoid V , and the canonical surjection

$$(8.1) \quad \text{ord} : V^+ \rightarrow \Gamma$$

is called the *valuation map* of V . The monoid V is then identified with the set of $x \in V^+$ such that $\text{ord}(x) \geq 0$ (where, recall 0 is the identity of Γ), and the maximal ideal \mathfrak{m} of V is $\{x \mid \text{ord}(x) > 0\}$ (since $\text{ord}(x) = 0$ just in case x is a unit of V).

Note that (8.1) satisfies $\text{ord}(x) \leq \infty$, $\text{ord}(xy) = \text{ord}(x) + \text{ord}(y)$ and $\text{ord}(x) = \infty$ if and only if $x = 0$. Conversely, given an abelian group M and a surjective morphism $\text{ord} : M_* \rightarrow \Gamma$ onto a totally ordered monoid $(\Gamma, +, 0, \infty)$ that satisfies these conditions, the set $C = \{a \in M \mid \text{ord}(a) \geq 0\}$ is a valuation monoid whose pointed group completion is M_* and whose associated valuation map is ord .

Lemma 8.2. *A valuation monoid V has no finite extensions contained in V^+ .*

Proof. Suppose that $V \subseteq B \subseteq V^+$ with B finite over V . By Lemma 1.7(ii), B is integral over V . For every nonzero $b \in B$ there is an $n \geq 1$ so that $b^n \in V$ and hence $n \text{ord}(b) \geq 0$, which implies that $\text{ord}(b) \geq 0$ and thus $b \in V$. \square

Example 8.3. A *discrete valuation monoid* is a valuation monoid whose value monoid is isomorphic to $\mathbb{Z} \cup \{\infty\}$ with its canonical ordering. In this case, a lifting of the generator $1 \in \mathbb{Z}$ to an element π in the discrete valuation monoid V is a generator of the maximal ideal of V and every non-zero element of V^+ may be written uniquely as $u\pi^n$ for $n \in \mathbb{Z}$ and $u \in U(V)$. Let's call such an element a *uniformizing parameter*.

Observe that if R is a discrete valuation ring, then (R, \times) is a discrete valuation monoid and the notion of a uniformizing parameter has its usual meaning.

If V is a discrete valuation monoid with valuation map $\text{ord} : V^+ \setminus 0 \rightarrow (\mathbb{Z} \cup \{\infty\})$, then a choice of a splitting of ord identifies $V^+ \setminus 0$ with $M \times \langle t^{\pm 1} \rangle$ for some abelian group M and, under this identification, ord is the evident projection. Thus, every discrete valuation monoid V is isomorphic to $U(V)_* \wedge \langle t \rangle$, where $\langle t \rangle$ is the free abelian monoid on one generator and $U(V)$ is the group of units of V . Any element of the form $u \wedge t$ with $u \in U(M)$ is a uniformizing parameter.

Remark 8.3.1. It is well-known that a valuation ring is Noetherian if and only if it is a discrete valuation ring; see [24, §VI.10, Thm. 16] for a proof. The same argument shows that a valuation monoid is finitely generated if and only if it is a discrete valuation monoid with a finitely generated group of units.

Definition 8.4. A map $f : Y \rightarrow X$ of monoid schemes satisfies the *valuative criterion for properness* if for every valuation monoid V and commutative square

$$(8.4a) \quad \begin{array}{ccc} \text{MSpec}(V^+) & \longrightarrow & Y \\ \downarrow & \nearrow \text{dotted} & \downarrow f \\ \text{MSpec}(V) & \longrightarrow & X \end{array}$$

there is a unique map $\text{MSpec}(V) \rightarrow Y$ causing both triangles to commute.

We say f satisfies the *valuative criterion of separatedness* if each such square has at most one completion.

A map $Y \rightarrow X$ of monoid schemes of finite type is said to be *proper* if it satisfies the valuative criterion for properness.

Remark 8.4.1. We are not certain what the correct definition of “proper” is for monoid schemes not of finite type. (Recall from Remark 3.4.1 that “separated and universally closed” is clearly not the correct definition.)

Given any morphism $f : \text{MSpec}(V) \rightarrow X$, any affine open $U \subset X$ containing $f(\mathfrak{m})$ (where \mathfrak{m} is the unique closed point of $\text{MSpec}(V)$) will contain the image of $\text{MSpec}(V)$. Hence the valuative criterion of properness and separatedness are local on the base: if $Y|_U \rightarrow U$ satisfies one of these criteria for every U in a covering of X , then so does $Y \rightarrow X$.

It is immediate from Definition 8.4 that the class of maps satisfying the valuative criterion of properness (resp., separatedness) is closed under composition and pullback.

Proposition 8.5. *A finite morphism between monoid schemes satisfies the valuative criterion of properness.*

Proof. Suppose $Y \rightarrow X$ is finite and consider a commutative square (8.4a) with V a valuation monoid. We may assume $Y \rightarrow X$ is a map of affine schemes, say given by a map of monoids $A \rightarrow B$. Then the square (8.4a) is associated to the square

$$\begin{array}{ccc} V^+ & \longleftarrow & B \\ \uparrow & & \uparrow \\ V & \longleftarrow & A \end{array}$$

of monoids. The image of B in V^+ is finite over V , but V is closed under finite extensions in V^+ , by Lemma 8.2. It follows that the map $B \rightarrow V^+$ actually lands in V , which gives the diagonal map we seek. \square

Corollary 8.6. *Closed immersions satisfy the valuative criterion of properness.*

Construction 8.7. To prove Theorem 8.9 below, we need a technical construction: Let V be a valuation monoid with group completion V^+ and value monoid $(\Gamma, +)$. Recall that totally ordered groups are necessarily torsion-free, and hence, for any field k , the ring $k[\Gamma]$ is an integral domain by Lemma 5.6.

For an element

$$\alpha = \sum_{\gamma} a_{\gamma} \gamma$$

of $k[\Gamma]$ (where for this ring we have rewritten Γ using \cdot instead of $+$ notation), define

$$\text{ord}(\alpha) = \min\{\gamma \in \Gamma \mid a_{\gamma} \neq 0\}.$$

(For $\alpha = 0$, set $\text{ord}(0) = \infty$.) It is easily verified that $\text{ord} : (k[\Gamma], \times) \rightarrow \Gamma$ is a monoid map such that $\text{ord}(\alpha) = \infty$ if and only if $\alpha = 0$.

It follows that we get an induced map of pointed group completions

$$\text{ord} : (k(\Gamma), \times) \rightarrow \Gamma$$

where $k(\Gamma)$ denotes the field of fractions of $k[\Gamma]$. Moreover, the composition

$$V^+ \rightarrow (k(\Gamma), \times) \xrightarrow{\text{ord}} \Gamma$$

coincides with the original valuation map $\text{ord} : V^+ \rightarrow \Gamma$.

Finally, the pair $(k(\Gamma), \text{ord})$ is a valuation in the usual ring-theoretic sense. To prove this, it remains to show

$$\text{ord}(\alpha + \beta) \geq \min\{\text{ord}(\alpha), \text{ord}(\beta)\} \text{ for all } \alpha, \beta \in k(\Gamma).$$

One easily reduces to the case when $\alpha, \beta \in k[\Gamma]$, where it is obvious from the definition of ord .

Proposition 8.8. *Given a valuation monoid V , with pointed group completion V^+ and value monoid Γ , let ord be the valuation map on the field $k(\Gamma)$ given in Construction 8.7, and let $R \subset k(\Gamma)$ denote the associated valuation ring. Then the square of affine monoid schemes*

$$\begin{array}{ccc} \text{MSpec}(k(\Gamma), \times) & \longrightarrow & \text{MSpec}(V^+) \\ \downarrow & & \downarrow \\ \text{MSpec}(R, \times) & \longrightarrow & \text{MSpec}(V) \end{array}$$

is a pushout square in the category of monoid schemes.

Proof. For any monoid scheme T , suppose morphisms $f : \text{MSpec}(V^+) \rightarrow T$ and $g : \text{MSpec}(R, \times) \rightarrow T$ are given causing the evident square to commute. Let $t \in T$ be the image of the unique closed point of $\text{MSpec}(R, \times)$ under g , and let $U \subset T$ be any affine open subscheme of T containing t . Then g factors through U . Since $\text{MSpec}(k(\Gamma), \times) \rightarrow \text{MSpec}(V^+)$ is a bijection on underlying sets (each is a one-point set), the unique point of $\text{MSpec}(V^+)$ also lands in U and hence f too factors through U . We may thus assume $T = U$ is affine. That is, it suffices to prove

$$\begin{array}{ccc} V & \longrightarrow & (R, \times) \\ \downarrow & & \downarrow \\ V^+ & \longrightarrow & (k(\Gamma), \times) \end{array}$$

is a pullback square in the category of pointed monoids. But this is evident since

$$\begin{array}{ccc} V^+ & \longrightarrow & (k(\Gamma), \times) \\ \downarrow \text{ord} & & \downarrow \text{ord} \\ \Gamma & \xrightarrow{=} & \Gamma \end{array}$$

commutes, $V = \{\alpha \in V^+ \mid \text{ord}(\alpha) \geq 0\}$ and $R = \{\beta \in k(\Gamma) \mid \text{ord}(\beta) \geq 0\}$. \square

Recall that a map of (classical) k -schemes $Y_k \rightarrow X_k$, where k is a field, is said to satisfy the *valuative criterion of properness* (resp., *separatedness*) if every solid arrow square

$$\begin{array}{ccc} \text{Spec}(F) & \longrightarrow & Y_k \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \text{Spec}(R) & \longrightarrow & X_k \end{array}$$

has a unique (resp., at most one) completion making both triangles commute, whenever R is a valuation ring (which is necessarily a k -algebra) and F is its field of fractions.

Theorem 8.9. *Let $f : X \rightarrow Y$ be a morphism of monoid schemes and let k be a field. The morphism $f_k : X_k \rightarrow Y_k$ satisfies the valuative criterion of properness (resp., separatedness) if and only if f satisfies the valuative criterion of properness (resp., separatedness).*

Proof. By Theorem 5.8, for any local k -algebra R , there is a natural adjunction isomorphism

$$\mathrm{Hom}_k(\mathrm{Spec}(R), X_k) \cong F_X(\mathrm{Spec} R) = \mathrm{Hom}_{\mathrm{MSch}}(\mathrm{MSpec}(R, \times), X).$$

Now suppose R is a valuation ring with field of fractions F . Then $V = (R, \times)$ is a valuation monoid with $V^+ = (F, \times)$. Since R and F are local, a commutative square of the form

$$\begin{array}{ccc} \mathrm{Spec} F & \longrightarrow & Y_k \\ \downarrow & & \downarrow \\ \mathrm{Spec} R & \longrightarrow & X_k \end{array}$$

corresponds via adjunction to a commutative square of monoid schemes given by the solid arrows in the diagram

$$(8.10) \quad \begin{array}{ccc} \mathrm{MSpec}(V^+) & \longrightarrow & Y \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathrm{MSpec}(V) & \longrightarrow & X. \end{array}$$

If $Y \rightarrow X$ satisfies the valuative criterion of properness (resp. separatedness), there exists a unique (resp., at most one) morphism of monoid schemes $\mathrm{MSpec}(V) \rightarrow Y$ represented by the dotted arrow above that causes both triangles to commute. Again by adjunction, this gives a unique map $\mathrm{Spec}(R) \rightarrow Y_k$ causing both triangles to commute in the first square.

Conversely, say a square (8.10) is given. By Construction 8.7, there is a valuation ring R with field of fractions $F = k(\Gamma)$ and morphisms $\mathrm{MSpec}(R, \times) \rightarrow \mathrm{MSpec} V$ and $\mathrm{MSpec}(F, \times) \rightarrow \mathrm{MSpec} V^+$ fitting into a commutative diagram

$$(8.11) \quad \begin{array}{ccccc} \mathrm{MSpec}(F, \times) & \longrightarrow & \mathrm{MSpec}(V^+) & \longrightarrow & Y \\ \downarrow & & \downarrow & \nearrow \text{dotted} & \downarrow \\ \mathrm{MSpec}(R, \times) & \longrightarrow & \mathrm{MSpec}(V) & \longrightarrow & X. \end{array}$$

By Proposition 8.8, the left-hand square is a pushout square in the category of monoid schemes. Using adjunction as above, if $Y_k \rightarrow X_k$ satisfies the valuative criterion of properness (resp., separatedness), there exists a unique (resp., at most one) map represented by the dotted arrow in (8.11) that causes the outer two triangles to commute. Since the left-hand square is a pushout, it follows immediately that there exists a unique (resp., at most one) arrow $\mathrm{MSpec}(V) \rightarrow Y$ causing both triangles in (8.11) to commute. \square

Corollary 8.12. *For any field k , a morphism between monoid schemes of finite type $Y \rightarrow X$ is proper if and only if $Y_k \rightarrow X_k$ is proper.*

Proof. Merely observe that Y_k and X_k are Noetherian, and apply the valuative criterion of properness theorem [13, II.4.7]. \square

Remark 8.12.1. Say $f : Y \rightarrow X$ satisfies the valuative criterion of properness. If Y_k is quasi-compact, EGA II(7.2.1) implies that f_k is proper. However, this condition appears complicated if Y is not of finite type.

Corollary 8.13. *A morphism between monoid schemes of finite type is proper if and only if it satisfies the valuative criterion of properness of Definition 8.4 for all discrete valuation monoids.*

Proof. If $f : X \rightarrow Y$ satisfies Definition 8.4 for all discrete valuation monoids, then, for any field k , its k -realization $f_k : X_k \rightarrow Y_k$ satisfies the valuative criterion of properness for all DVRs. This follows, using adjunction, from the fact that $\text{MSpec}(R, \times)$ is a discrete valuation monoid if R is a DVR. Since X_k and Y_k are Noetherian and f_k has finite type, it follows that f_k is proper (see [13, Ex. II.4.11]). The result now follows from Corollary 8.12. \square

Corollary 8.14. *A projective morphism $Y \rightarrow X$ between monoid schemes of finite type is proper. In particular, if X is a monoid scheme of finite type and X_C is the blow-up along an equivariant closed subscheme C , then the map $X_C \rightarrow X$ is proper.*

Proof. Using Proposition 5.11 and Remark 7.1.1, we see that $Y_k \rightarrow X_k$ is a projective morphism of k -schemes and hence is proper. For the second assertion, recall that $X_C \rightarrow X$ is projective and X_C has finite type. \square

Remark 8.15. In fact, a projective morphism of arbitrary monoid schemes satisfies the valuative criterion of properness. We omit the (non-trivial) proof of this fact.

Recall from 4.1 that a monoid scheme of finite type is toric if it is separated, connected, torsionfree and normal. By Theorem 4.3, there is a faithful functor from fans to toric monoid schemes.

Corollary 8.16. *Let $\phi : (N', \Delta') \rightarrow (N, \Delta)$ be a morphism of fans. Then the associated morphism of toric monoid schemes $X' \rightarrow X$ is proper if and only if ϕ has the property that for each $\sigma \in \Delta$, $\phi_{\mathbb{R}}^{-1}(\sigma)$ is the union of cones in Δ' .*

Proof. This follows from the well-known fact that $X'_k \rightarrow X_k$ is proper if and only if ϕ has the stated property (see [8, p. 39]). \square

Corollary 8.17. *Every proper map between monoid schemes of finite type is separated.*

Proof. By Theorem 8.9 and the Valuative Criterion of Separatedness Theorem for Noetherian schemes, the k -realization of a proper map between monoid schemes of finite type is separated. Now use Proposition 5.13. \square

9. PARTIALLY CANCELLATIVE TORSION FREE MONOID SCHEMES

A monoid A is *pctf* if it is isomorphic to a monoid of the form B/I where B is a cancellative torsion free monoid (i.e., a cancellative monoid whose group completion is torsion free) and I is an ideal. A monoid scheme is *pctf* if all of its stalks are.

Proposition 9.1. *We have:*

- (1) *If a pctf monoid is finitely generated, then it is isomorphic to A/I where A is a finitely generated torsion free cancellative monoid.*
- (2) *All submonoids and localizations of a pctf monoid are pctf. In particular, for a monoid A , $\text{MSpec}(A)$ is pctf if and only if A is pctf.*
- (3) *If A is a pctf monoid and \mathfrak{p} is a prime ideal, then A/\mathfrak{p} is a cancellative torsionfree monoid.*
- (4) *An open subscheme of a pctf monoid scheme is pctf.*
- (5) *An equivariant closed subscheme of a pctf monoid scheme is pctf.*

Proof. Say $A = B/I$ with B cancellative and torsion free. Pick elements b_1, \dots, b_m in B that map to a generating set of A and let B' be the submonoid of B they generate. Then $A = B'/(I \cap B')$, proving the first assertion.

For the second, say $A = C/I$ with C cancellative and torsion free. If B is a submonoid of A , let B' denote the inverse image of B in C and set $I' = I \cap B'$. Then $B = B'/I'$, and so B is pctf. The assertion concerning localizations holds since $S^{-1}(C/I) \cong S^{-1}C/S^{-1}I$. The remaining assertion of part (2) is clear.

If $A = B/I$ then $A/\mathfrak{p} = B/\mathfrak{p}'$ for some prime ideal of B , so (3) follows from the elementary observation that if A is cancellative and torsionfree then so is A/\mathfrak{p} .

Assertion (4) is local and follows from (2); hence (5) is local, and is then easy. \square

Proposition 9.2. *The blow-up of a pctf monoid scheme along an equivariant closed subscheme is pctf.*

Proof. Let $Y \rightarrow X$ be the blow-up of a pctf monoid scheme X along an equivariant closed subscheme. Since the question is local on X , we may assume that X is affine, say $X = \text{MSpec}(A)$ with A pctf. Then Y is $\text{MProj}(A[It])$ for an ideal I . For each $s \in I$, we get an affine open subset of Y given by the monoid $\{\frac{f}{s^n} \mid f \in I^n, n \geq 0\}$. This is a submonoid of $A[\frac{1}{s}]$ and hence is pctf. The collection of such open subsets as s varies over all elements of I form an open cover of Y . Thus Y is pctf. \square

Proposition 9.3. *Let $X = (X, \mathcal{A})$ and $Y = (Y, \mathcal{B})$ be monoid schemes and let $f : Y \rightarrow X$ be a morphism. There is a unique closed subscheme Z of X which is minimal with respect to the property that f factors through $Z \subset X$.*

If $U \subset X$ is an affine open subscheme of X , then $Z \cap U$ is the affine scheme $\text{MSpec}(C)$, where C is the image of $\mathcal{B}(U) \rightarrow \mathcal{A}(U \times_X Y)$. In particular, if X is of finite type then so is Z .

We call Z the scheme-theoretic closure of f . If f is an open immersion, we abuse notation and call Z the closure of Y , and write it as \bar{Y} .

Proof. If f factors through two different closed subschemes W_1 and W_2 of X , then it factors through $W_1 \times_X W_2$, which is (canonically isomorphic to) a closed subscheme of X (see Example 3.2). So, we define Z to be the inverse limit taken over the partially ordered set of closed subschemes W of X such that f factors through W .

For the local description of Z , we may assume that $X = U = \text{MSpec}(B)$ is affine. Any closed subscheme of X has the form $W = \text{MSpec}(D)$ with $B \rightarrow D$ a surjection of monoids. Then f factors through W if and only if $B \rightarrow \mathcal{A}(Y)$ factors through D , that is, if and only if $B \rightarrow C$ factors as $B \rightarrow D \rightarrow C$; in other words, if and only if $Z \subseteq W$. \square

Proposition 9.4. *Let Y be a monoid scheme and suppose $U \subset Y$ is an open subscheme that is pctf. Then the scheme-theoretic closure \bar{U} of U in Y is pctf. Moreover, if Y is separated, then \bar{U} is separated.*

Proof. The first assertion is local on Y and so we may assume $Y = \text{MSpec}(B)$ for a monoid B and $U = \text{MSpec}(S^{-1}B)$ for a multiplicative subset S . Then \bar{U} is the affine scheme associated to the image \bar{B} of $B \rightarrow S^{-1}B$. The monoid $S^{-1}B$ is pctf by assumption and 9.1(4), and hence so is \bar{B} by 9.1(2).

The second assertion is just the observation that a closed subscheme of a separated scheme is also separated by Lemma 3.4. \square

10. BIRATIONAL MORPHISMS

A morphism $p : Y \rightarrow X$ of monoid schemes is *birational* if there is an open dense subscheme U of X such that $p^{-1}(U)$ is dense in Y and p induces an isomorphism from $p^{-1}(U)$ to U .

Proposition 10.1 (Birational maps). *Let $p : (Y, B) \rightarrow (X, A)$ be a map between monoid schemes of finite type, Then p is birational if and only if the following conditions hold:*

- (1) p maps the generic points of Y bijectively onto the generic points of X
- (2) A point $y \in Y$ is generic if (and only if) $p(y) \in X$ is generic
- (3) for each generic point $y \in Y$ the induced map $A(p(y)) \rightarrow B(y)$ on stalks is an isomorphism.

Proof. If p is birational and U is as in the definition above, then U contains all of the generic points of X and $p^{-1}(U)$ contains all the generic points of Y as well as every point of y that maps to a generic point of X . The conditions are then clearly satisfied.

Conversely, take U to be the (dense open) set of generic points of X . By hypothesis, $p^{-1}(U)$ is the set of generic points of Y and the map $p : p^{-1}(U) \rightarrow U$ is bijective. Hence $p^{-1}(U)$ is open and dense. Since the map $p^{-1}(U) \rightarrow U$ is bijective and induces an isomorphism on all stalks, it is an isomorphism. \square

Corollary 10.2. *If $p : X' \rightarrow X$ is a proper map of toric monoid schemes that is birational, then p is given by a map of fans $\phi : (N', \Delta') \rightarrow (N, \Delta)$ such that $\phi : N' \xrightarrow{\cong} N$ and the image of Δ' under the isomorphism $\phi_{\mathbb{R}}$ is a subdivision of Δ .*

Proof. From 4.3(2), p comes from a morphism of fans such that $\phi : N' \xrightarrow{\cong} N$, and such a morphism is a subdivision by Corollary 8.16. \square

Example 10.3. If X is a monoid scheme of finite type, let X_{η} denote the equivariant closure of a generic point η (in the sense of 2.8). Then each X_{η} has a unique generic point, namely η . If X is pctf, then each X_{η} is cancellative and torsionfree by 9.1(3), and hence pctf. If X is reduced, the morphism $\coprod_{\eta} X_{\eta} \rightarrow X$ is birational.

Proposition 10.4. *If $Y \rightarrow X$ is a birational map and $X' \rightarrow X$ is a morphism such that X' is of finite type and every generic point of X' maps to a generic point of X , then the pullback $Y \times_X X' \rightarrow X'$ is birational.*

Proof. The poset underlying $Y \times_X X'$ is given by the pullback of the underlying posets (by 3.1). Since $Y \rightarrow X$ is birational, a point (y, x') in $Y \times_X X'$ is generic if and only if x' is a generic point of X' , and in this case y and x' map to the same point x of X , which is generic. Hence the map $Y \times_X X' \rightarrow X'$ is a bijection on sets of generic points. Writing A', A and B for the stalk functors of X', X and Y , the map on generic stalks is of the form $A'(x') \rightarrow A'(x') \wedge_{A(x)} B(y)$. This is an isomorphism, since the map $A(x) \rightarrow B(y)$ is an isomorphism. \square

Define the *height* of a point x in a monoid scheme X to be the dimension of \mathcal{A}_x ; i.e., it is the largest integer n such that there exists a strictly decreasing chain $x = x_n > \cdots > x_0$ in the poset underlying X . We write this as $ht(x)$ or $ht_X(x)$.

For example, if $X = X(N, \Delta)$ is the monoid scheme associated to a fan, then $ht(\sigma) = \dim(\sigma)$ for each cone $\sigma \in \Delta$. Here $\dim(\sigma)$ refers to the dimension of the real vector subspace of $N_{\mathbb{R}}$ spanned by σ .

Lemma 10.5. *Suppose $p : Y \rightarrow X$ is a proper, birational map of separated pctf schemes of finite type. Then for any $y \in Y$, we have $ht_Y(y) \leq ht_X(p(y))$.*

Proof. Suppose $ht_Y(y) = m$, so that we have a chain of points $y = y_m > \dots > y_0$ in Y . Clearly y_0 must be minimal, and thus generic. Let $\eta = p(y_0)$, and define X_η to be the equivariant closure of $\{\eta\}$ in X . As pointed out in Example 10.3, X_η is cancellative and torsionfree. The pullback $Y_\eta = X_\eta \times_X Y$ is an equivariant closed subscheme of Y containing y_0 as its unique generic point, and hence each y_i . By Proposition 9.1(5), Y_η is also pctf, and $Y_\eta \rightarrow X_\eta$ is birational by Proposition 10.4.

Let Y' denote the equivariant closure of y_0 in Y_η . By Example 10.3, $Y' \rightarrow Y_\eta$ is birational, Y' contains all the y_i and Y' is cancellative and torsionfree. Replacing X and Y by X_η and Y' , we may assume that both X and Y are connected, cancellative and torsionfree. Hence the normalization maps $X_{\text{nor}} \rightarrow X$ and $Y_{\text{nor}} \rightarrow Y$ exist and are homeomorphisms (by 1.6.1), and both X_{nor} and Y_{nor} are torsionfree. Since $Y \rightarrow X$ is birational, it induces a birational morphism $Y_{\text{nor}} \rightarrow X_{\text{nor}}$. The map $Y_{\text{nor}} \rightarrow Y$ is finite by 1.7 and hence proper by 8.5. Thus $Y_{\text{nor}} \rightarrow X$ and hence $Y_{\text{nor}} \rightarrow X_{\text{nor}}$ are proper. Thus we may assume that X and Y are separated, normal and torsionfree.

By Proposition 4.4 and Corollary 4.5, we have reduced to the case where $Y \rightarrow X$ is a proper birational map of toric monoid schemes, given by a map of fans $\phi : (N', \Delta') \rightarrow (N, \Delta)$. The birational hypothesis means that $\phi : N' \rightarrow N$ is an isomorphism. By Corollary 10.2, the proper hypothesis means that Δ' is a subdivision of Δ . Since $\phi(\sigma)$ is the smallest cone in Δ containing the image of σ under $\phi_{\mathbb{R}}$ and since height corresponds to dimension of cones, the result is now clear. \square

11. RESOLUTIONS OF SINGULARITIES FOR TORIC VARIETIES

The purpose of this section is to establish some properties for monoid schemes that are analogous to those known to hold for arbitrary varieties in characteristic zero. These properties will be used in Section 12 to prove that certain presheaves of spectra satisfy the analogue of “smooth *cdh* descent” for monoid schemes.

Theorem 11.1. *Let X be a separated cancellative pctf monoid scheme of finite type. Then there is a birational proper morphism $Y \rightarrow X$ such that Y is smooth.*

Proof. We may assume that X is connected. Since the normalization map is proper birational by Propositions 6.3 and 8.5, we may assume that X is normal. Since X is pctf it is torsionfree by Proposition 9.1(3). By Proposition 4.4, X is toric and $X \cong X(\Delta)$ for some fan Δ . By Corollary 10.2, there exists a subdivision Δ' of Δ such that $X(\Delta')$ is smooth and the morphism $X(\Delta') \rightarrow X(\Delta)$ is proper birational. \square

We will need the notion of the barycentric subdivision of a simplicial fan Δ in $N_{\mathbb{R}}$: For a simplicial cone σ in $N_{\mathbb{R}}$ of dimension d , let v_1, \dots, v_d be the minimal lattice points along the rays of σ . For each non-empty subset S of $\{1, \dots, d\}$, let $v_S = \sum_{i \in S} v_i$. The *barycentric subdivision* of σ , which we write as $\sigma^{(1)}$, is defined as the collection of 2^d cones given as the span of vectors of the form v_{S_1}, \dots, v_{S_e} , where $0 \leq e \leq d$ and $S_1 \subset \dots \subset S_e$ is a chain of proper subsets of $\{1, \dots, d\}$. It is

clear that if τ is a face of σ , then the set of cones in $\sigma^{(1)}$ that are contained in τ form the fan $\tau^{(1)}$. It follows that

$$\Delta^{(1)} := \left\{ \sigma^{(1)} \mid \sigma \in \Delta \right\}$$

is again a simplicial fan. We inductively define $\Delta^{(i)} = (\Delta^{(i-1)})^{(1)}$ for $i \geq 2$.

Lemma 11.2. *If Δ' is any subdivision of a simplicial fan Δ in $N_{\mathbb{R}}$, then for $i \gg 0$, the fan $\Delta^{(i)}$ is a subdivision of Δ' .*

Proof. It suffices to show that any vertex of Δ' is a vertex of some $\Delta^{(i)}$. Given a positive integer combination $v = \sum n_i v_i$ of the vertices in a cone, we may reorder the vertices to assume the n_i are in decreasing order. Then v is in the cone of $\Delta^{(1)}$ spanned by the v_{S_i} , where $S_i = \{1, \dots, i\}$, and (if $v \neq v_1$) we can write $v = \sum n'_i v_{S_i}$ with $\sum n'_i < \sum n_i$. The result follows by induction on $\sum n_i$. \square

Lemma 11.3. *If Δ is a smooth fan, then for all $i \geq 1$, the toric monoid scheme $X(\Delta^{(i)})$ is obtained from $X(\Delta)$ via a sequence of blow-ups along smooth centers.*

Proof. We may assume $i = 1$. The fan $\Delta^{(1)}$ is obtained from Δ via a series of steps of the following sort: starting with a smooth fan Δ , we form a subdivision Δ' by picking a cone σ , letting v_1, \dots, v_d be the minimal lattice points along its rays, and defining Δ' to be the subdivision of Δ given by insertion of the ray spanned by $v_1 + \dots + v_d$. By Example 7.5, $X(\Delta') \rightarrow X(\Delta)$ is the blow-up along the smooth, closed equivariant subscheme defined by $x_1 = \dots = x_d = 0$. \square

Theorem 11.4. *For a morphism $\pi : Y \rightarrow X$ between separated cancellative pcf monoid schemes of finite type, assume X is smooth and $\pi : Y \rightarrow X$ is proper and birational. Then there exists a sequence of blow-ups along smooth closed equivariant centers,*

$$X^n \rightarrow \dots \rightarrow X^1 \rightarrow X_0 = X,$$

such that $X^n \rightarrow X$ factors through $\pi : Y \rightarrow X$.

Proof. By Theorem 11.1, there is a proper birational morphism $Z \rightarrow Y$ with Z smooth. We may therefore assume that Y is smooth. We may also assume that X and Y are connected, so that they have unique generic points.

Thus, by Corollary 10.2, $Y \rightarrow X$ is given by a morphism $(N', \Delta') \rightarrow (N, \Delta)$ of fans that is an isomorphism of lattices and such that Δ' is a subdivision of Δ . Lemmas 11.2 and 11.3 complete the proof. \square

12. CD STRUCTURES ON MONOID SCHEMES.

Let $\mathcal{M}_{\text{pctf}}$ denote the category of monoid schemes of finite type that are separated and pcf. In this section, we will be concerned with cartesian squares of the form

$$(12.1) \quad \begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X. \end{array}$$

Definition 12.2. An *abstract blow-up* is a cartesian square of monoid schemes of finite type of the form (12.1) such that p is proper, e is an equivariant closed immersion, and p maps the open complement $Y \setminus D$ isomorphically onto $X \setminus C$. The square with $Y = \emptyset$ and $C = X_{\text{red}}$ is such a square.

Proposition 12.3. *If X is of finite type, C is an equivariant closed subscheme of X and $p : Y \rightarrow X$ is the blow-up of X along C , then the resulting cartesian square is an abstract blow-up. If X belongs to $\mathcal{M}_{\text{pctf}}$, so do Y , C and D .*

Proof. By Corollary 8.14, p is proper. As noted in Definition 7.4, p maps $Y \setminus D$ isomorphically to $X \setminus C$ (because $D = C \times_X Y$). The second assertion follows from Propositions 9.1 and 9.2. \square

Proposition 12.4. *Suppose an abstract blow-up square (12.1) is given with X in $\mathcal{M}_{\text{pctf}}$. Let \bar{Y} be the scheme-theoretic closure of $Y \setminus D$ in Y , and define $\bar{D} = C \times_X \bar{Y}$. Then*

$$\begin{array}{ccc} \bar{D} & \longrightarrow & \bar{Y} \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X \end{array}$$

is an abstract blow-up square in $\mathcal{M}_{\text{pctf}}$.

Proof. By Proposition 9.1, $X \setminus C$ and hence $Y \setminus D$ is pctf, and so by Proposition 9.4, \bar{Y} is pctf as well. Since equivariant closed subschemes of pctf schemes are pctf, C and \bar{D} also belong to $\mathcal{M}_{\text{pctf}}$. The map $\bar{Y} \rightarrow X$ is a composition of proper maps and hence is proper. Finally, $\bar{Y} \setminus \bar{D} = Y \setminus D$. \square

Recall from [20, 2.1] that a *cd* structure on a category \mathcal{C} is a collection of distinguished commutative squares in \mathcal{C} . If \mathcal{C} has an initial object \emptyset , any *cd* structure defines a topology: the smallest Grothendieck topology such that for each distinguished square (12.1) the sieve generated by $\{p, e\}$ is a covering sieve (and the empty sieve is a covering of the initial object). The coverings $\{p, e\}$ are called *elementary*.

Definition 12.5. The *blow-up cd structure* on $\mathcal{M}_{\text{pctf}}$ is given by the collection of all abstract blow-up squares with X, Y, C, D all belonging to $\mathcal{M}_{\text{pctf}}$. The *Zariski cd structure* on $\mathcal{M}_{\text{pctf}}$ is given by all cartesian squares associated to a covering of X by two open subschemes.

The *cdh* topology on $\mathcal{M}_{\text{pctf}}$ is the topology generated by the union of these two *cd* structures.

Following [20, 2.3, 2.4], we say that a *cd* structure is *complete* if \mathcal{C} has an initial object \emptyset and any pullback of an elementary covering contains a sieve which can be obtained by iterating elementary coverings. We say that a *cd* structure is *regular* (see [20, 2.10]) if each distinguished square (12.1) is a pullback, e is a monomorphism and the morphism of representable sheaves

$$(12.6) \quad \rho(D) \times_{\rho(C)} \rho(D) \amalg \rho(Y) \rightarrow \rho(Y) \times_{\rho(X)} \rho(Y)$$

is onto, where $\rho(T)$ denotes the sheafification of the presheaf represented by T .

Theorem 12.7. *The blow-up and Zariski cd structures on $\mathcal{M}_{\text{pctf}}$ are complete and regular.*

Proof. The completeness property for Zariski squares is clear since they are preserved by pullback, and the regularity property is even clearer. For the blow-up *cd* structure, consider an abstract blow-up square

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X. \end{array}$$

Let $X' \rightarrow X$ be any morphism in $\mathcal{M}_{\text{pctf}}$ and consider the square involving X', C', Y' and D' formed by pullback. The scheme Y' might not belong to $\mathcal{M}_{\text{pctf}}$, but the scheme-theoretic closure Y'' of $Y' \setminus D'$ in Y' does by Proposition 12.4. The resulting square involving C', X', Y'' and $D'' := C' \times_{X'} Y''$ is an abstract blow-up by the same result, and hence by [20, Lemma 2.4] the blow-up *cd* structure is complete.

For the regularity property, we need to show that (12.6) is onto. Every object admits a covering in this topology by affine, cancellative monoids, and it suffices to prove surjectivity of the map given by the underlying presheaves evaluated at such an affine cancellative U . That is, say $f : U \rightarrow Y, g : U \rightarrow Y$ are given with $p \circ f = p \circ g$. We need to prove either $f = g$ or they both factor through D and coincide as maps to C . Let u be the unique generic point of U . If either $f(u)$ or $g(u)$ lands in $Y \setminus D$, then they both must land there. Since $Y \setminus D \cong X \setminus C$, it follows that f and g coincide generically. But since U is cancellative, it follows $f = g$ on all of U . (To see this, one may work locally: If $h, l : A \rightarrow B$ are two maps of monoids with B cancellative and the compositions of h, l with the inclusion $B \hookrightarrow B^+$ coincide, then $h = l$.) Otherwise, we have that the generic point, and hence every point, of U is mapped by both f and g to points in the closed subset D of Y . Again using that U is cancellative, it follows that f, g factor through $D \hookrightarrow Y$. (This is also proven by working locally.) Finally, the compositions of these maps $f, g : U \rightarrow D$ with $D \rightarrow C$ coincide since $C \rightarrow X$ is a closed immersion. \square

We define the *standard density structure* on $\mathcal{M}_{\text{pctf}}$ as follows: The set $\mathcal{D}_i(X)$ consists of those open immersions $U \subset X$ such that every point in $X \setminus U$ has height at least i . It is clear that this satisfies the axioms required of a density structure of finite dimension in [20, 2.20].

A *cd* structure is said to be *bounded* for a given density structure if any distinguished square has a refinement which is reducing for the density structure in the sense of [20, 2.21].

Theorem 12.8. *The blow-up and Zariski *cd* structures on $\mathcal{M}_{\text{pctf}}$ are both bounded for the standard density structure.*

Proof. To see that the blow-up *cd* structure is bounded, we need to show that any abstract blow-up square (12.1) in $\mathcal{M}_{\text{pctf}}$ has a refinement that is reducing for \mathcal{D}_* . Consider the square obtained by replacing Y by the monoid scheme-theoretic closure of $Y \setminus D$ (in the sense of Proposition 9.3), and D by the pullback. This is also an abstract blow-up square, and it refines (12.1). This refinement has the features that $p^{-1}(X \setminus C)$ is dense in Y , Y maps birationally onto the scheme-theoretic closure of $X \setminus C$ in X , and D does not contain any generic points of Y .

To show that this square is reducing, we assume given $C_0 \in \mathcal{D}_i(C), Y_0 \in \mathcal{D}_i(Y)$ and $D_0 \in \mathcal{D}_{i-1}(D)$. Define X' to be the open subscheme $X \setminus Z$ of X , where $Z \subset X$ is the equivariant closure (in the sense of 2.8) of the union of the images of each

of $C \setminus C_0$, $D \setminus D_0$ and $Y \setminus Y_0$ in X . We need to show that X' belongs to $\mathcal{D}_i(X)$ and that the pullback of the original square (12.1) along $X' \hookrightarrow X$ gives an abstract blow-up square.

If $y \in Y$ is a point of height at least i , then $p(y)$ has height at least i in the scheme-theoretic closure of $X \setminus C$, by Lemma 10.5. Hence $p(y)$ has height at least i in X itself (since a closed immersion is an injection on underlying posets). If $d \in D$ has height at least $i - 1$, then its height in Y is at least i (since D contains no generic points of Y) and hence its image in X has height at least i too. Since C is an equivariant closed subscheme, if $c \in C$ has height at least i , it has height at least i in X .

Recall that $Z \subset X$ is the equivariant closure of the union of the images of each of $C \setminus C_0$, $D \setminus D_0$ and $Y \setminus Y_0$ in X . Each of these images consists of points of height at least i and hence every point in Z has height at least i in X by Remark 2.8.1. Therefore X' belongs to $\mathcal{D}_i(X)$ and the pullback of the above square along $X' \hookrightarrow X$ gives an abstract blow-up square that proves our original square is reducing.

The argument in the previous paragraphs applies *mutatis mutandis* to show that every Zariski square is reducing. \square

Given a Grothendieck topology, the category of contravariant functors \mathcal{F} from $\mathcal{M}_{\text{pctf}}$ to spectra (*presheaves* of spectra) has a closed model structure, in which a morphism $\phi : \mathcal{F} \rightarrow \mathcal{F}'$ is a cofibration when $\mathcal{F}(X) \rightarrow \mathcal{F}'(X)$ is a cofibration for every monoid scheme X in $\mathcal{M}_{\text{pctf}}$; ϕ is a weak equivalence if it induces isomorphisms between the sheaves of stable homotopy groups. We write $\mathbb{H}_{\text{cdh}}(-, \mathcal{F})$ for the fibrant replacement of \mathcal{F} using this model structure for the *cdh* topology, as in [5].

A presheaf of spectra \mathcal{F} on $\mathcal{M}_{\text{pctf}}$ satisfies the *Mayer-Vietoris property* for some family \mathcal{C} of cartesian squares if $\mathcal{F}(\emptyset) = *$ and the application of \mathcal{F} to each member of the family gives a homotopy cartesian square of spectra.

Proposition 12.9. *Let \mathcal{F} be a presheaf of spectra on $\mathcal{M}_{\text{pctf}}$. Then the canonical map $\mathcal{F}(X) \rightarrow \mathbb{H}_{\text{cdh}}(X, \mathcal{F})$ is a homotopy equivalence of spectra for all X if and only if it has the Mayer-Vietoris property for every abstract blow-up square and every Zariski square of *pctf* monoid schemes.*

Proof. By Theorems 12.7 and 12.8, the *cdh* *cd*-structure is complete, regular and bounded. Now the assertion follows from [5, Theorem 3.4]. \square

Given Proposition 12.9, the definition of *cdh*-descent given in [5, 3.5] becomes:

Definition 12.10. Let \mathcal{F} be a presheaf of spectra on $\mathcal{M}_{\text{pctf}}$. We say that \mathcal{F} satisfies *cdh descent* if the canonical map $\mathcal{F}(X) \rightarrow \mathbb{H}_{\text{cdh}}(X, \mathcal{F})$ is a homotopy equivalence of spectra for all X .

Remark 12.10.1. Writing $\mathbb{H}_{\text{zar}}(-, \mathcal{F})$ for the fibrant replacement with respect to the model structure for the Zariski topology, we obtain the notion of Zariski descent. The proof of Proposition 12.9 applies to show that \mathcal{F} satisfies Zariski descent if and only if it has the Mayer-Vietoris property for every Zariski square. It follows that *cdh*-descent implies Zariski descent.

It is useful to restrict to the full subcategory \mathcal{S} of smooth monoid schemes (see Definition 6.4). By Proposition 6.5, these are the cancellative, torsionfree, separated monoid schemes of finite type whose k -realizations are smooth for any fixed field k . (This condition is independent of k , by 6.5.)

Definition 12.11. We define the *smooth blow-up cd structure* on \mathcal{S} to consist of squares (12.1) such that X is smooth, e is the inclusion of an equivariant, smooth closed subscheme and Y is the blow-up of X along C . (These assumptions ensure, by (7.7), that Y and S are also smooth.)

The Zariski cd structure is given by all cartesian squares in \mathcal{S} associated to a covering of X by two open subschemes.

We define the $scdh$ topology on \mathcal{S} to be the Grothendieck topology associated to the union of the smooth blow-up cd -structure and the Zariski cd -structure on \mathcal{S} . For a presheaf of spectra on \mathcal{S} , we define $\mathbb{H}_{scdh}(-, \mathcal{F})$ just as \mathbb{H}_{cdh} was defined above. We say such a presheaf \mathcal{F} satisfies $scdh$ descent if the canonical fibrant replacement map

$$\mathcal{F}(X) \rightarrow \mathbb{H}_{scdh}(X, \mathcal{F})$$

is a weak homotopy equivalence for all $X \in \mathcal{S}$.

Proposition 12.12. *The smooth blow-up cd -structure and the Zariski cd structure on \mathcal{S} are regular, bounded, and complete. Consequently, a presheaf of spectra defined on \mathcal{S} satisfies $scdh$ descent if and only if it has the Mayer-Vietoris property for each smooth blow-up square and each Zariski square in \mathcal{S} .*

Proof. That the smooth blow-up cd -structure is complete can be proved exactly as Voevodsky did for smooth k -schemes in [21, Lemma 4.3], replacing resolution of singularities by our Theorem 11.4. Regularity is proved exactly as in Theorem 12.7 for the non-smooth case. The proof that the smooth blow-up cd -structure is bounded works exactly as in Theorem 12.8, keeping in mind that open subschemes of smooth monoid schemes are smooth. The proof that the Zariski cd -structure is complete, regular and bounded is again the same as in the non-smooth category. It follows that the $scdh$ topology is generated by a complete, regular, bounded cd -structure and so [5, Theorem 3.4] applies to prove the second assertion. \square

Proposition 12.13. *For any $X \in \mathcal{S}$ and any presheaf of spectra \mathcal{F} defined on \mathcal{M}_{pctf} , we have a homotopy equivalence*

$$\mathbb{H}_{cdh}(X, \mathcal{F}) \xrightarrow{\sim} \mathbb{H}_{scdh}(X, \mathcal{F}|_{\mathcal{S}}).$$

Proof. In this proof we write \mathcal{F}_{cdh} for the restriction of the presheaf $\mathbb{H}_{cdh}(-, \mathcal{F})$ to \mathcal{S} . By Proposition 12.9, \mathcal{F}_{cdh} satisfies the Mayer-Vietoris property for smooth blow-up and Zariski squares. Therefore \mathcal{F}_{cdh} satisfies $scdh$ descent (Definition 12.11).

By Theorems 11.1 and 11.4, every covering sieve for the cdh topology on \mathcal{M}_{pctf} has a refinement containing a sieve generated by a cover consisting of objects of \mathcal{S} . It follows that $\mathcal{F}|_{\mathcal{S}} \rightarrow \mathcal{F}_{cdh}$ is an $scdh$ -local weak equivalence. Therefore $\mathbb{H}_{scdh}(-, \mathcal{F}|_{\mathcal{S}}) \rightarrow \mathbb{H}_{scdh}(-, \mathcal{F}_{cdh})$ is an objectwise weak equivalence. Together, the two objectwise weak equivalences exhibited in the proof give the assertion. \square

13. WEAK cdh_k DESCENT

Throughout this section, we fix a field k .

Definition 13.1. Let X_k be a scheme of finite type over k and assume $Z_k \subset X_k$ is a closed subscheme. We say Z_k is *regularly embedded* in X_k if the sheaf of ideals defining Z_k is locally generated by a regular sequence — that is, if for all $x \in Z_k$, the kernel I_x of $\mathcal{O}_{X_k, x} \rightarrow \mathcal{O}_{Z_k, x}$ is generated by a $\mathcal{O}_{X_k, x}$ -regular sequence of elements.

Definition 13.2. A presheaf of spectra \mathcal{F} defined on $\mathcal{M}_{\text{pctf}}$ has *weak cdh_k descent* if \mathcal{F} has the Mayer-Vietoris property for each cartesian square

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X \end{array}$$

in $\mathcal{M}_{\text{pctf}}$ satisfying one of the following conditions:

- (1) It is member of the Zariski cd structure.
- (2) It is a finite abstract blow-up — i.e., it is a member of the abstract blow-up cd structure having the additional property that p is a finite morphism.
- (3) C is an equivariant closed subscheme, $Y \rightarrow X$ is the blow-up of X along C , and C_k is a regularly embedded closed subscheme of X_k .

Remark 13.2.1. Theorems 13.3 and 14.3 below suggest (but do not prove) that the definition of weak cdh_k descent is actually independent of the choice of k .

Since a smooth blow-up square is an example of a blow-up along a regularly embedded subscheme, Propositions 12.12 and 12.13 imply the following theorem.

Theorem 13.3. *If \mathcal{F} is a presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ that satisfies weak cdh_k descent, then \mathcal{F} satisfies $scdh$ descent. That is, the canonical map*

$$\mathcal{F}(X) \rightarrow \mathbb{H}_{cdh}(X, \mathcal{F})$$

is a weak equivalence for every smooth monoid scheme X .

The main goal of this paper, realized in the next section, is to establish a partial generalization of Theorem 13.3 to all schemes in $\mathcal{M}_{\text{pctf}}$. The goal of the rest of this section is to establish some technical properties needed in the next. We first introduce a slightly stronger notion than that of weak cdh_k descent.

Recall from [EGA IV, 6.10.1] that given a closed subscheme C_k of a k -scheme X_k , defined by an ideal sheaf \mathcal{I} , X_k is said to be *normally flat* along C_k if the restriction of each $\mathcal{I}^n/\mathcal{I}^{n+1}$ to C_k is flat.

Remark 13.3.1. Here is a monoid-theoretic condition on a sheaf I of ideals on a monoid scheme (X, A) which guarantees that, for all k , the k -realization of X is normally flat along the k -realization of the equivariant closed submonoid C defined by I : at each point x of C , under the natural action of the monoid A_x/I_x on each of the pointed sets $L_n = \mathcal{I}_x^n/\mathcal{I}_x^{n+1}$, each L_n is a bouquet of copies of A_x/I_x . We do not know if this condition is necessary.

We will say that a cartesian square of schemes in $\mathcal{M}_{\text{pctf}}$,

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X, \end{array}$$

is a *nice blow-up square* if C is an equivariant closed subscheme of X , Y is the blow-up of X along C and there exists a cartesian square in $\mathcal{M}_{\text{pctf}}$ of the form

$$(13.4) \quad \begin{array}{ccc} C & \xrightarrow{e} & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Z \end{array}$$

such that Z is cancellative, $X \rightarrow Z$ is the normalization of Z and B is an equivariant closed smooth subscheme of Z such that Z_k is normally flat along B_k .

Definition 13.5. A presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ satisfies *weak+nice cdh_k descent* provided it satisfies weak cdh_k descent and, in addition, it has the Mayer-Vietoris property for all nice blow-up squares in $\mathcal{M}_{\text{pctf}}$.

Proposition 13.6. *If \mathcal{F} is a presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ that satisfies cdh descent, then \mathcal{F} satisfies weak+nice cdh_k descent for any field k .*

Proof. This is immediate from Proposition 12.9, since each of the squares appearing in the definition of weak+nice cdh_k descent is a member of the cdh cd structure. \square

We will need the following technical result about local domains. Recall that if I is an ideal in a commutative ring R then an ideal $J \subseteq I$ is called a *reduction* of I if $JJ^{n-1} = I^n$ for some $n > 0$; a *minimal reduction* of I is a reduction which contains no other reduction of I .

Lemma 13.7. *Let R be a local domain essentially of finite type over an infinite field k , let \mathfrak{p} be a prime ideal, and assume R is normally flat along R/\mathfrak{p} . Let J be a minimal reduction of \mathfrak{p} that is generated by $h := ht(\mathfrak{p}) = ht(J)$ elements. (Given R and \mathfrak{p} with these properties, such a J exists by [15, 5.2, 5.3].) Let \tilde{R} be the normalization of R and assume \tilde{R} is Cohen-Macaulay.*

Then $J\tilde{R}$ is a reduction of $\mathfrak{p}\tilde{R}$ generated by h elements and $\text{Spec}(\tilde{R}/J\tilde{R})$ is regularly embedded in $\text{Spec}(\tilde{R})$.

Proof. We have that $J\mathfrak{p}^{n-1}\tilde{R} = \mathfrak{p}^n\tilde{R}$, and so the first assertion is clear.

Since $R \hookrightarrow \tilde{R}$ is an integral extension of domains, we have $h = ht(J) = ht(J\tilde{R})$. For any maximal ideal $\tilde{\mathfrak{m}}$ of \tilde{R} , we have that $J\tilde{R}_{\tilde{\mathfrak{m}}}$ is a height h ideal generated by h elements in the local ring $\tilde{R}_{\tilde{\mathfrak{m}}}$. Since $\tilde{R}_{\tilde{\mathfrak{m}}}$ is Cohen-Macaulay by assumption, these generators necessarily form a regular sequence. \square

The following is the evident analogue of the notion of weak cdh_k descent for presheaves of spectra on the category of k -schemes.

Definition 13.8. For a field k , let Sch/k be the category of separated schemes essentially of finite type over k . A presheaf of spectra defined on Sch/k satisfies *weak cdh descent* if it has the Mayer-Vietoris property for each cartesian square

$$\begin{array}{ccc} D & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ C & \xrightarrow{e} & X \end{array}$$

of schemes satisfying one of the following conditions:

- (1) e and p are open immersions whose images cover X .
- (2) It is a finite abstract blow-up — i.e., e is a closed immersion, p is finite, and p maps $Y \setminus D$ isomorphically onto $X \setminus C$.
- (3) e is a regular closed immersion and p is the blow-up of X along C .

Lemma 13.9. *Assume k is an infinite field and \mathcal{G}_k is a presheaf of spectra on Sch/k that satisfies weak cdh descent. Let \mathcal{G} be the presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ defined by $\mathcal{G}(X) := \mathcal{G}_k(X_k)$.*

Then \mathcal{G} satisfies weak+nice cdh_k descent on $\mathcal{M}_{\text{pctf}}$.

Proof. Since the k -realizations of the squares involved in the definition of weak cdh_k descent for $\mathcal{M}_{\text{pctf}}$ (Definition 13.2) are squares involved in the definition of weak cdh descent for Sch/k (Definition 13.8), it follows that \mathcal{G} satisfies weak cdh_k descent. Say X, Y, C, D, Z , and B are as in the definition of a nice blow-up square. We need to prove that the square

$$(13.10) \quad \begin{array}{ccc} \mathcal{G}_k(X_k) & \longrightarrow & \mathcal{G}_k(C_k) \\ \downarrow & & \downarrow \\ \mathcal{G}_k(Y_k) & \longrightarrow & \mathcal{G}_k(D_k) \end{array}$$

is homotopy cartesian.

Let R be any local ring of Z_k and let \mathfrak{p} be the prime ideal of R cutting out B_k locally. Let $V = \text{Spec}(\tilde{R}_{\tilde{\mathfrak{m}}})$ where \tilde{R} is the normalization of R and $\tilde{\mathfrak{m}}$ is any of the maximal ideals of \tilde{R} . Then, since X_k is the normalization of Z_k by Proposition 6.1, V is the spectrum of a local ring of X_k , and for various choices of R and $\tilde{\mathfrak{m}}$, every local ring of X_k arises in this manner.

By Corollary 5.10, $C_k = X_k \times_{Z_k} B_k$, so the closed subscheme $V \times_{X_k} C_k$ of V is cut out by $\mathfrak{q} = \mathfrak{p}\tilde{R}_{\tilde{\mathfrak{m}}}$. As X is the normalization of the separated cancellative, torsionfree monoid scheme Z , Proposition 6.1 implies that X_k is a toric variety. As all toric varieties are Cohen-Macaulay, so are X_k and V .

By Lemma 13.7, $\mathfrak{q} = \mathfrak{p}\tilde{R}_{\tilde{\mathfrak{m}}}$ admits a reduction $I \subset \mathfrak{q}$ such that $\text{Spec}(\tilde{R}_{\tilde{\mathfrak{m}}}/I) \hookrightarrow V$ is a regular embedding. Since $V \times_{X_k} Y_k$ is the blow-up of V_k along $V \times_{X_k} C_k$ (by Proposition 7.7), and the exceptional divisor is $V \times_{X_k} D_k$ (by 5.10), the proof of [15, 5.6] (with \mathcal{KH} replaced by \mathcal{G}) gives that

$$\begin{array}{ccc} \mathcal{G}_k(V) & \longrightarrow & \mathcal{G}_k(V \times_{X_k} C_k) \\ \downarrow & & \downarrow \\ \mathcal{G}_k(V \times_{X_k} Y_k) & \longrightarrow & \mathcal{G}_k(V \times_{X_k} D_k) \end{array}$$

is homotopy cartesian. Since \mathcal{G}_k satisfies the Mayer-Vietoris property for Zariski covers and the V occurring here is an arbitrary local scheme of X_k , the proof of [15, 5.7] (with \mathcal{KH} replaced by \mathcal{G}_k) shows that (13.10) is homotopy cartesian. \square

Example 13.11. Let \mathcal{KH} denote Weibel's homotopy algebraic K -theory [22]. We may view \mathcal{KH} as a presheaf of spectra on Sch/k . By abuse of notation, we also write \mathcal{KH} for the presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ defined by $\mathcal{KH}(X) = \mathcal{KH}(X_k)$.

By [19], [22, 4.9] and [18], \mathcal{KH} satisfies weak cdh descent on Sch/k (13.8); by Lemma 13.9, \mathcal{KH} satisfies weak+*nice* cdh_k -descent on $\mathcal{M}_{\text{pctf}}$.

14. MAIN THEOREM

In this section, we prove our main theorem (Theorem 14.3), which gives a condition for \mathcal{F} to satisfy cdh descent on $\mathcal{M}_{\text{pctf}}$. We will need the Bierstone-Milman Theorem, which we extract from the embedded version [1, Thm. 1.1].

Theorem 14.1. *Let X be a separated cancellative torsionfree monoid scheme of finite type, embedded as a closed subscheme (see Definition 2.5) in a smooth toric monoid scheme M (see Definition 4.1). For any field k , there is a sequence of*

blow-ups along smooth equivariant centers $Z_i \subset X_i$, $0 \leq i \leq n-1$,

$$Y = X_n \rightarrow \cdots \rightarrow X_0 = X$$

such that Y is smooth, and each $(X_i)_k$ is normally flat along $(Z_i)_k$.

Proof. Let \bar{k} denote the algebraic closure of k , and let T be the torus acting on $M_{\bar{k}}$. The Bierstone-Milman Theorem ([1, Thm.1.1]) tells us that we can find a sequence of blow-ups $M_n \rightarrow \cdots \rightarrow M_0 = M_{\bar{k}}$ of smooth toric \bar{k} -varieties, the blow-up of M_i being taken along a smooth T -invariant center N_i , with the following properties. Setting $X'_0 = X_{\bar{k}}$, we inductively define $Z'_i = N_i \cap X'_i$; then Z'_i is a smooth equivariant \bar{k} -variety, X'_i is normally flat along Z'_i , and X'_{i+1} is the strict transform of X'_i .

The \bar{k} -realization functor from fans to (normal) toric \bar{k} -varieties (and equivariant morphisms) is well known to be an equivalence. It follows that each of the N_i and M_i and the morphisms between them come from fans, and hence by Theorem 4.3 are \bar{k} -realizations of toric monoid schemes (which by abuse of notation, we will call N_i and M_i), and morphisms of such.

Inductively we define monoid schemes X_i and Z_i , starting from $X_0 = X$ and $Z_0 = N_0 \cap X$, to be the blow-up of the monoid scheme X_{i-1} along Z_{i-1} in the sense of 7.4. By Proposition 7.7 and Corollary 5.10, $Z'_i = (Z_i)_{\bar{k}}$ and $X'_i = (X_i)_{\bar{k}}$. In particular, $(X_n)_{\bar{k}} = Y$ is a smooth toric variety and therefore the monoid scheme X_n is smooth by Proposition 6.5. Finally, faithfully flat descent implies that $(X_i)_k$ is normally flat along $(Z_i)_k$ if and only if $(X_i)_{\bar{k}}$ is normally flat along $(Z_i)_{\bar{k}}$. \square

Theorem 14.2. *Suppose \mathcal{G} is a presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ satisfying weak+nice *cdh* _{k} descent. If $\mathcal{G}(X) \simeq *$ for all $X \in \mathcal{S}$ then $\mathcal{G}(X) \simeq *$ for all X in $\mathcal{M}_{\text{pctf}}$.*

Proof. We proceed by induction on the dimension of X . Given X , let x_1, \dots, x_l be its generic points, and let $Y_i = \overline{\{x_i\}}^{\text{eq}}$ be their equivariant closures (see Lemma 2.8). We have a cover $X = Y_1 \cup \cdots \cup Y_l$ by equivariant closed subschemes each of which is cancellative by Example 10.3. Moreover, each $Y_i \times_X Y_j$ is equivariant and closed, hence *pctf*. Since \mathcal{G} has the Mayer-Vietoris property for closed covers, and \mathcal{G} vanishes on the $Y_i \times_X Y_j$ for all $i \neq j$ by the induction hypothesis, we get

$$\mathcal{G}(X) = \prod_i \mathcal{G}(Y_i).$$

We may thus assume that X is cancellative. (This also establishes the base case $\dim(X) = 0$, since in that case the Y_i are in \mathcal{S} .)

Since \mathcal{G} satisfies Mayer-Vietoris for open covers, we may assume X is affine. In particular, we may assume X can be embedded in a smooth toric monoid scheme, for example, by choosing a surjection from a free abelian monoid onto A where $X = \text{MSpec}(A)$. This will allow us to apply the Bierstone-Milman Theorem 14.1 to obtain a sequence of blow-ups along smooth monoid schemes Z_i ,

$$Y = X_n \rightarrow \cdots \rightarrow X_0 = X.$$

We claim that $\mathcal{G}(X_i) \simeq \mathcal{G}(X_{i+1})$ for all i . Since $\mathcal{G}(Y) \simeq *$, this will finish the inductive step and hence the proof of the theorem. To simplify the notation, fix i and write Z for $Z_i \subset X_i$ and X_Z for X_{i+1} , the blow-up of X_i along Z , so that our goal is to prove that $\mathcal{G}(X_i) \rightarrow \mathcal{G}(X_Z)$ is a weak equivalence. Let \bar{X} denote the

normalization $(X_i)_{\text{nor}}$ of X_i and set $\tilde{Z} = Z \times_{X_i} \tilde{X}$. Write $\tilde{X}_{\tilde{Z}}$ for the blow-up of \tilde{X} along \tilde{Z} . By naturality of blow-ups (see 7.4), there is a commutative square

$$\begin{array}{ccc} \tilde{X}_{\tilde{Z}} & \longrightarrow & X_Z \\ \downarrow & & \downarrow \\ \tilde{X} & \longrightarrow & X_i \end{array}$$

(that need not be cartesian). Since the map $\tilde{X} \rightarrow X_i$ is finite, the map $\tilde{X}_{\tilde{Z}} \rightarrow X_Z$ is also finite, by Lemma 7.6. Applying \mathcal{G} gives a commutative square of spectra

$$\begin{array}{ccc} \mathcal{G}(\tilde{X}_{\tilde{Z}}) & \longleftarrow & \mathcal{G}(X_Z) \\ \uparrow & & \uparrow \\ \mathcal{G}(\tilde{X}) & \longleftarrow & \mathcal{G}(X_i). \end{array}$$

To prove that the right-hand vertical arrow is a weak equivalence, it suffices to prove the other three are.

The finite map $\tilde{X} \rightarrow X_i$ is an isomorphism on the generic points. Consider the equivariant closure $E \subset X_i$ of the finitely many height 1 points of X_i ; by Remark 2.8.1, every point in E has height ≥ 1 in X_i , so E is the complement of the generic point of X_i . Since E is pctf, $\mathcal{G}(E) \simeq *$ by our inductive assumption. Since the pullback $\tilde{E} := E \times_{X_i} \tilde{X}$ is an equivariant closed subscheme of \tilde{X} , it is pctf by Proposition 9.1, and hence $\mathcal{G}(\tilde{E}) \simeq *$ as well, by induction. Using the finite abstract blow-up square involving X_i , \tilde{X} , E and \tilde{E} , we have a weak equivalence

$$\mathcal{G}(X_i) \xrightarrow{\simeq} \mathcal{G}(\tilde{X}).$$

The map $\tilde{X}_{\tilde{Z}} \rightarrow X_Z$ is also finite and birational, and so the same argument shows

$$\mathcal{G}(X_Z) \xrightarrow{\simeq} \mathcal{G}(\tilde{X}_{\tilde{Z}}).$$

is a weak equivalence. Finally, observe that

$$\begin{array}{ccc} \tilde{Z} \times_{\tilde{X}} \tilde{X}_{\tilde{Z}} & \longrightarrow & \tilde{X}_{\tilde{Z}} \\ \downarrow & & \downarrow \\ \tilde{Z} & \longrightarrow & \tilde{X} \end{array}$$

is a nice blow-up square, because the bottom row may be compared with $Z \rightarrow X_i$ and $(X_i)_k$ is normally flat along Z_k . Because \mathcal{G} has descent for nice blow-up squares, and $\mathcal{G}(\tilde{Z}) \simeq \mathcal{G}(\tilde{Z} \times_{\tilde{X}} \tilde{X}_{\tilde{Z}}) \simeq *$ by the induction hypothesis, we get a weak equivalence

$$\mathcal{G}(\tilde{X}) \xrightarrow{\simeq} \mathcal{G}(\tilde{X}_{\tilde{Z}}).$$

It follows that $\mathcal{G}(X_i) \simeq \mathcal{G}(X_Z)$, as claimed. This completes the proof. \square

We now state and prove the main theorem of this paper, which gives a partial generalization of Theorem 13.3 to all objects in the category $\mathcal{M}_{\text{pctf}}$.

Theorem 14.3. *Let \mathcal{F}_k be a presheaf of spectra on Sch/k for some infinite field k , and define \mathcal{F} to be the presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ defined by $\mathcal{F}(X) = \mathcal{F}_k(X_k)$.*

If \mathcal{F}_k satisfies weak cdh descent on Sch/k , then \mathcal{F} satisfies cdh descent on $\mathcal{M}_{\text{pctf}}$.

Proof. Let \mathcal{G} be the homotopy fiber of $\mathcal{F} \rightarrow \mathbb{H}_{cdh}(-, \mathcal{F})$ — i.e., for all X in $\mathcal{M}_{\text{pctf}}$, $\mathcal{G}(X)$ is the homotopy fiber of $\mathcal{F}(X) \rightarrow \mathbb{H}_{cdh}(X, \mathcal{F})$. By Lemma 13.9 and Proposition 13.6, both \mathcal{F} and $\mathbb{H}_{cdh}(-, \mathcal{F})$ satisfy weak+nice cdh_k descent, and hence \mathcal{G} satisfies weak+nice cdh_k descent too. Theorem 13.3 gives that $\mathcal{G}(X) \simeq *$ for all $X \in \mathcal{S}$. Now we apply Theorem 14.2 to conclude $\mathcal{G}(X) \simeq *$ for all X in $\mathcal{M}_{\text{pctf}}$. \square

The following corollary is the Theorem announced in the introduction.

Corollary 14.4. *Assume k is an infinite field and let \mathcal{F}_k be a presheaf of spectra on Sch/k that satisfies the Mayer-Vietoris property for Zariski covers, finite abstract blow-up squares, and blow-ups along regularly embedded subschemes. Then \mathcal{F}_k satisfies the Mayer-Vietoris property for all abstract blow-up squares of toric k -varieties.*

Proof. By Definition 13.8, \mathcal{F}_k satisfies weak cdh descent on Sch/k . By Theorem 14.3, \mathcal{F} satisfies cdh descent in $\mathcal{M}_{\text{pctf}}$. Now use Proposition 12.9. \square

Corollary 14.5. *Let k be any field. The presheaf of spectra \mathcal{KH} on $\mathcal{M}_{\text{pctf}}$ (defined as $\mathcal{KH}(X) = \mathcal{KH}(X_k)$) satisfies cdh descent. Moreover, both natural maps*

$$\mathcal{KH}(X) \rightarrow \mathbb{H}_{cdh}(X, \mathcal{KH}) \leftarrow \mathbb{H}_{cdh}(X, \mathcal{K})$$

are weak equivalences for all X in $\mathcal{M}_{\text{pctf}}$

Proof. By a standard transfer argument, we may assume that the field k is infinite. By Example 13.11 and Theorem 14.3, \mathcal{KH} satisfies cdh descent on $\mathcal{M}_{\text{pctf}}$. For any X in $\mathcal{M}_{\text{pctf}}$, consider the commutative square of spectra:

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{KH}(X) \\ \downarrow & & \downarrow \\ \mathbb{H}_{cdh}(X, \mathcal{K}) & \longrightarrow & \mathbb{H}_{cdh}(X, \mathcal{KH}), \end{array}$$

where \mathcal{K} is algebraic K -theory, regarded as a presheaf of spectra on Sch/k and hence on $\mathcal{M}_{\text{pctf}}$. Since \mathcal{KH} satisfies cdh descent, the right-hand vertical map is a weak equivalence for all X . This is the first assertion of the corollary.

If X is smooth, then the top horizontal map is a weak equivalence by [22] (since X_k is smooth by 6.5). By fibrant replacement and Proposition 12.13, the bottom map is also a weak equivalence for all X in \mathcal{S} . By induction on $\dim(X)$ and Theorem 11.1, this implies that $\mathbb{H}_{cdh}(-, \mathcal{K}) \rightarrow \mathbb{H}_{cdh}(-, \mathcal{KH})$ is a local weak equivalence and, as observed (for any site) in [5, p. 561], this implies that $\mathbb{H}_{cdh}(X, \mathcal{K}) \rightarrow \mathbb{H}_{cdh}(X, \mathcal{KH})$ is a weak equivalence for all X in $\mathcal{M}_{\text{pctf}}$. \square

In order to apply Corollary 14.5 to the relation between K -theory and topological cyclic homology, we need to recall some terms. Fix a prime p and a perfect field k of characteristic p . To each scheme X essentially of finite type over k , there is a pro-spectrum $\{TC^\nu(X, p)\}_{\nu=0}^\infty$ and the cyclotomic trace is a compatible family of morphisms $tr^\nu : \mathcal{K}(X) \rightarrow TC^\nu(X, p)$. Define \mathcal{F}_k^ν to be the presheaf of spectra on Sch/k given as the homotopy fiber of $\mathcal{K}(X) \rightarrow TC^\nu(X, p)$. Then Geisser and Hesselholt observe in the proof of [9, Thm. B] that each \mathcal{F}_k^ν takes elementary Nisnevich squares and regular blow-up squares to homotopy cartesian squares of pro-spectra.

Following Geisser-Hesselholt [9], a strict map of pro-spectra $\{X^\nu\} \rightarrow \{Y^\nu\}$ is said to be a *weak equivalence* if for every q the induced map $\{\pi_q(X^\nu)\} \rightarrow \{\pi_q(Y^\nu)\}$ is an

isomorphism of pro-abelian groups. A square diagram of strict maps of pro-spectra is said to be *homotopy cartesian* if the canonical map from the upper left pro-spectrum to the level-wise homotopy limit of the other terms is a weak equivalence.

Given a class \mathcal{C} of squares we will say that a pro-presheaf of spectra satisfies the pro-analogue of \mathcal{C} -descent if it sends each square in \mathcal{C} to a homotopy cartesian square of pro-spectra.

Define $\{\mathcal{F}^\nu\}$ to be the pro-presheaf of spectra on $\mathcal{M}_{\text{pctf}}$ given as the family of homotopy fibers of the maps $\mathcal{K}(-) \rightarrow TC^\nu(-, p)$. That is, $\mathcal{F}^\nu(X) = \mathcal{F}_k^\nu(X_k)$ is the homotopy fiber of $\mathcal{K}(X_k) \rightarrow TC^\nu(X_k, p)$ for each X and ν .

Proposition 14.6. *Assume k is an infinite field of characteristic $p > 0$. Then $\{\mathcal{F}^\nu\}$ satisfies *cdh descent* on $\mathcal{M}_{\text{pctf}}$ in the sense that $\{\mathcal{F}^\nu\} \rightarrow \{\mathbb{H}(-, \mathcal{F}^\nu)\}$ is a weak equivalence of pro-spectra.*

Proof. Fix ν and let \mathcal{G}^ν be the homotopy fiber of $\mathcal{F}^\nu \rightarrow \mathbb{H}_{\text{cdh}}(-, \mathcal{F}^\nu)$. It suffices to prove that for each X and q the pro-abelian group $\{\pi_q \mathcal{G}^\nu(X)\}$ is pro-zero. We will do so by modifying the proof of Theorem 14.3.

For each ν , $\mathbb{H}_{\text{cdh}}(-, \mathcal{F}^\nu)$ satisfies weak+nice *cdh_k* descent by Proposition 13.6. By [10, Thm. 1] and [11, Thms. B, D], $\{\mathcal{F}_k^\nu\}$ sends finite abstract blow-up squares to homotopy cartesian squares of pro-spectra. Thus $\{\mathcal{F}_k^\nu\}$ satisfies the pro-analogue of weak *cdh* descent (Definition 13.8). In the proof of Lemma 13.9, the reduction ideals used are reduction ideals on affine neighborhoods of the maximal ideal \mathfrak{m} of R . By the argument used in the proof of [9, Thm. 1.1], the proof of our Lemma 13.9 now applies *mutatis mutandis* to show that the pro-presheaf of spectra \mathcal{F}^ν satisfies the Mayer-Vietoris property for nice blow-up squares. It now follows that $\{\mathcal{G}^\nu\}$ satisfies the pro-analogue of weak+nice *cdh_k* descent.

For each ν , \mathcal{F}^ν satisfies Zariski descent and also has the Mayer-Vietoris property for regular blow-ups, so \mathcal{F}^ν satisfies *scdh* descent by 12.12. By definition, this means that for each smooth X the spectrum $\mathcal{G}^\nu(X)$ is contractible. Now the proof of Theorem 14.2 applies *verbatim* to finish the proof. \square

Corollary 14.7. *Assume k is any field of characteristic $p > 0$. For any monoid scheme X in $\mathcal{M}_{\text{pctf}}$, the following square of pro-spectra is homotopy cartesian.*

$$\begin{array}{ccc} \mathcal{K}(X) & \longrightarrow & \mathcal{K}H(X) \\ \downarrow & & \downarrow \\ \{TC^\nu(X, p)\} & \longrightarrow & \{\mathbb{H}_{\text{cdh}}(X, TC^\nu(-, p))\}. \end{array}$$

Proof. By a standard transfer argument, we may assume that the field k is infinite. By Proposition 14.6, the homotopy fiber $\{\mathcal{F}^\nu(X)\}$ of the left vertical map is weakly equivalent to $\{\mathbb{H}_{\text{cdh}}(X, \mathcal{F}^\nu)\}$. By Corollary 14.5, this coincides up to weak equivalence with the homotopy fiber of the right vertical map. \square

REFERENCES

- [1] E. Bierstone and P. Milman, Desingularization of toric and binomial varieties *J. Alg. Geom.* 15 (2006), 443–486.
- [2] A. Connes and C. Consani, Schemes over \mathbb{F}_1 and zeta functions, *Compos. Math.* 146 (2010), 1383–1415.
- [3] A. Connes and C. Consani, On the notion of geometry over \mathbb{F}_1 , *J. Alg. Geom.* 20 (2011), 525–557.

- [4] A. Connes, C. Consani and M. Marcolli, Fun with \mathbb{F}_1 , *J. Number Theory* 129 (2009), 1532–1561.
- [5] G. Cortiñas, C. Haesemeyer, M. Schlichting and C. Weibel, Cyclic homology, cdh -cohomology and negative K -theory, *Ann. Math.* 167 (2008), 549–573.
- [6] A. Deitmar, \mathbb{F}_1 -schemes and toric varieties, *Beiträge Algebra Geom.* 49 (2008), 517–525.
- [7] D. Eisenbud and J. Harris, *The Geometry of Schemes*, Grad. Texts in Math. 197, Springer Verlag, 2000.
- [EGA 0_I] A. Grothendieck and J. Dieudonné, Chapitre 0, *Éléments de géométrie algébrique I*, Springer, 1971.
- [EGA IV] A. Grothendieck, *Éléments de géométrie algébrique IV (2^{ème} Partie)*, *Publ. Math. IHES* 24 (1964).
- [8] W. Fulton, *Introduction to Toric Varieties*, Annals of Math. Studies 131, Princeton Univ. Press, 1993.
- [9] T. Geisser and L. Hesselholt, On the vanishing of negative K -groups, *Math. Ann.* 348 (2010), 707–736.
- [10] T. Geisser and L. Hesselholt, Bi-relative algebraic K -theory and topological cyclic homology, *Invent. Math.* 166 (2006), 359–395.
- [11] T. Geisser and L. Hesselholt, On relative and bi-relative algebraic K -theory of rings of finite characteristic, *J. AMS* 24 (2011), 29–49.
- [12] R. Gilmer, *Commutative Semigroup Rings*, The University of Chicago Press, Chicago, 1984.
- [13] R. Hartshorne, *Algebraic Geometry*, Springer Verlag, 1977.
- [14] A. Hattori, Theory of multi-fans, *Osaka J. Math.* 40 (2003), 1–68.
- [15] C. Haesemeyer, Descent properties of homotopy K -theory, *Duke Math. J.* 125 (2004), 589–620.
- [16] K. Kato, Toric singularities, *Amer. J. Math.* 116 (1994), 1073–1099.
- [17] J. López Peña and O. Lorscheid, Mapping \mathbb{F}_1 -land: An overview of geometries over the field with one element, preprint (2009). Available at arXiv:0909.0069
- [18] R. W. Thomason, Les K -groupes d'un schéma éclaté et une formule d'intersection excédentaire, *Invent. Math.* 112 (1993), 195–215.
- [19] R. W. Thomason and T. Trobaugh, Higher algebraic K -theory of schemes and of derived categories. In *The Grothendieck Festschrift, Volume III*, volume 88 of *Progress in Math.*, pages 247–436. Birkhäuser, Boston, Basel, Berlin, 1990.
- [20] V. Voevodsky, Homotopy theory of simplicial sheaves in completely decomposable topologies, *J. Pure Appl. Algebra* 214 (2010), 1384–1398.
- [21] V. Voevodsky, Unstable motivic homotopy categories in Nisnevich and cdh -topologies, *J. Pure Appl. Algebra* 214 (2010), 1399–1406.
- [22] C. Weibel, Homotopy Algebraic K -theory, *Contemporary Math.* 83 (1989), 461–488.
- [23] C. Weibel, The negative K -theory of normal surfaces, *Duke Math. J.* 108 (2001), 1–35.
- [24] O. Zariski and P. Samuel, *Commutative algebra. Vol. II*. Reprint of the 1960 edition. Graduate Texts in Mathematics, Vol. 29. Springer-Verlag, New York-Heidelberg, 1975.

DEPT. MATEMÁTICA-INST. SANTALÓ, FCEYN, UNIVERSIDAD DE BUENOS AIRES, CIUDAD UNIVERSITARIA, (1428) BUENOS AIRES, ARGENTINA

E-mail address: `gcorti@dm.uba.ar`

DEPT. OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES CA 90095, USA

E-mail address: `chh@math.ucla.edu`

DEPT. OF MATHEMATICS, UNIVERSITY OF NEBRASKA - LINCOLN, LINCOLN, NE 68588, USA

E-mail address: `mwalker5@math.unl.edu`

DEPT. OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ 08901, USA

E-mail address: `weibel@math.rutgers.edu`